

# Threshold Saturation via Spatial Coupling: Why Convolutional LDPC Ensembles Perform so well over the BEC

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**Abstract**— Convolutional LDPC ensembles, introduced by Felström and Zigangirov, have excellent thresholds and these thresholds are rapidly increasing as a function of the average degree. Several variations on the basic theme have been proposed to date, all of which share the good performance characteristics of convolutional LDPC ensembles. We describe the fundamental mechanism which explains why “convolutional-like” or “spatially coupled” codes perform so well. In essence, the spatial coupling of the individual code structure has the effect of increasing the belief-propagation (BP) threshold of the new ensemble to its maximum possible value, namely the maximum-a-posteriori (MAP) threshold of the underlying ensemble. For this reason we call this phenomenon “threshold saturation”. This gives an entirely new way of approaching capacity. One significant advantage of such a construction is that one can create capacity-approaching ensembles with an error correcting radius which is increasing in the blocklength. Our proof makes use of the area theorem of the BP-EXIT curve and the connection between the MAP and BP threshold recently pointed out by Méasson, Montanari, Richardson, and Urbanke. Although we prove the connection between the MAP and the BP threshold only for a very specific ensemble and only for the binary erasure channel, empirically the same statement holds for a wide class of ensembles and channels. More generally, we conjecture that for a large range of graphical systems a similar collapse of thresholds occurs once individual components are coupled sufficiently strongly. This might give rise to improved algorithms as well as to new techniques for analysis.

## I. INTRODUCTION

We consider the design of capacity-approaching codes based on the connection between the BP and MAP threshold of sparse graph codes. Recall that the BP threshold is the threshold of the “locally optimum” BP message-passing algorithm. As such it has low complexity. The MAP threshold, on the other hand, is the threshold of the “globally optimum” decoder. No decoder can do better, but the complexity of the MAP decoder is in general high. Surprisingly, for sparse graph codes there is a connection between these two thresholds which emerges in the limit of infinitely large blocklengths, see [1], [2]. We discuss a fundamental mechanism which ensures that these two thresholds coincide (or at least are very close). We call this phenomenon “threshold saturation via spatial coupling”. A prime example where this mechanism is at work are *convolutional low-density parity-check* (LDPC) ensembles.

These ensembles were introduced by Felström and Zigangirov [3].

Convolutional LDPC ensembles are constructed by *coupling* several standard  $(1, r)$ -regular LDPC ensembles together in a chain. Perhaps surprisingly, due to the coupling the threshold of the resulting ensemble is considerably improved. Indeed, if we start with an  $(3, 6)$ -regular ensemble then on the BEC the threshold is improved from  $\epsilon^{\text{BP}}(1 = 3, r = 6) \approx 0.4294$  to roughly 0.4881 (the capacity for this case is  $\frac{1}{2}$ ). The latter number is indeed the MAP threshold  $\epsilon^{\text{MAP}}(1, r)$  of the underlying  $(3, 6)$ -regular ensemble. In a nutshell, the convolutional structure allows one to convert the MAP threshold of the underlying ensemble into the BP threshold of the convolutional structure. This opens up an entirely new way of constructing capacity-approaching ensembles. It further shows that it is possible to construct ensembles that have large BP thresholds *and* low error floors, whereas it is a folk theorem that for *standard* constructions, improvements in the BP threshold go hand in hand with increases in the error floor.

The potential of convolutional LDPC codes has long been recognized. Our contribution lies therefore not in the introduction of a new coding scheme, but in clarifying the basic mechanism that make convolutional-like ensembles perform so well. There is a considerable literature on convolutional-like LDPC ensembles. Variations on the constructions as well as some analysis can be found in Engdahl and Zigangirov [4], Engdahl, Lentmaier, and Zigangirov [5], Lentmaier, Truhachev, and Zigangirov [6], Papaleo, Iyengar, Siegel, Wolf, and Corazza [7], as well as Tanner, D. Sridhara, A. Sridhara, Fuja, and Costello [8]. In [9], Sridharan, Lentmaier, Costello and Zigangirov consider density evolution (DE) for convolutional LDPC ensembles and determine thresholds for the BEC. The equivalent observations for general channels were reported by Lentmaier, Sridharan, Zigangirov and Costello in [10]. A protograph representation of convolutional LDPC ensembles was introduced by Lentmaier, Fettweis, Zigangirov and Costello [11]. In a recent paper [12], Lentmaier and Fettweis independently formulated the equality of the BP threshold of convolutional LDPC ensembles and the MAP threshold of the underlying ensemble as a conjecture. They attribute this numerical observation to G. Liva.

## II. CONVOLUTIONAL-LIKE LDPC ENSEMBLES

The principle that underlies the good performance of convolutional-like LDPC ensembles is very broad and there are many degrees of freedom in constructing such ensembles. In the sequel we introduce two basic variants. The  $(1, r, L)$ -ensemble is very close to the ensemble discussed in [11]. Experimentally it has a very good performance. We conjecture that it is capable of achieving capacity. We also introduce the ensemble  $(1, r, L, w)$ . Experimentally it shows a worse trade-off between rate, threshold, and blocklength. But it is easier to analyze and we will prove that it is capacity achieving. One can think of  $w$  as a “smoothing parameter” and we investigate the behavior of this ensemble when  $w$  tends to infinity.

### A. The $(1, r, L)$ Ensemble

The left-hand side of Figure 1 shows the protograph of a standard  $(3, 6)$ -regular ensemble. There are two variable nodes and there is one check node. Let  $M$  denote the number of variable nodes at each position. For our example,  $M = 100$  means that we have 50 copies of the protograph so that we have 100 variable nodes at each position. Next, consider a collection of  $(2L + 1)$  such protographs as shown on the right-hand side of Figure 1. These protographs are non-interacting

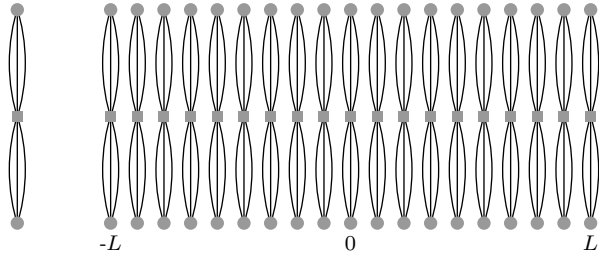


Fig. 1. Left figure shows the protograph of a standard  $(3, 6)$ -regular ensemble. The figure on the right illustrates a chain of  $(2L + 1)$  protographs of the standard  $(3, 6)$ -regular ensemble for  $L = 9$ . These protographs do not interact.

and so each component behaves just like a standard  $(3, 6)$ -regular component. In particular, the BP threshold of each protograph is just the standard threshold, call it  $\epsilon^{\text{BP}}(1 = 3, r = 6)$ . Slightly more general; start with an  $(1, r = k1)$ -regular ensemble where  $1$  is odd so that  $\hat{1} = (1 - 1)/2 \in \mathbb{N}$ .

An interesting phenomenon occurs if we couple these components. To achieve this coupling, connect each protograph to  $\hat{1}$  protographs “to the left” and to  $\hat{1}$  protographs “to the right.” This is shown in Figure 2 for the case  $(1 = 3, r = 6)$  and  $L = 9$ . An extra  $\hat{1}$  check nodes are added on each side to connect the “overhanging” edges at the boundary. There are two main effects resulting from this coupling:

(i) *Rate Reduction*: Recall that the design rate of the underlying standard  $(1, r = k1)$ -regular ensemble is  $1 - \frac{1}{r} = \frac{k-1}{k}$ . Let us determine the design rate of the corresponding  $(1, r = k1, L)$  ensemble. By design rate we mean here the rate that we get if we assume that every involved check node imposes a linearly independent constraint.

The variable nodes are indexed from  $-L$  to  $L$  so that in total there are  $(2L + 1)M$  variable nodes. The check nodes are indexed from  $-(L + \hat{1})$  to  $(L + \hat{1})$ , so that in total there are

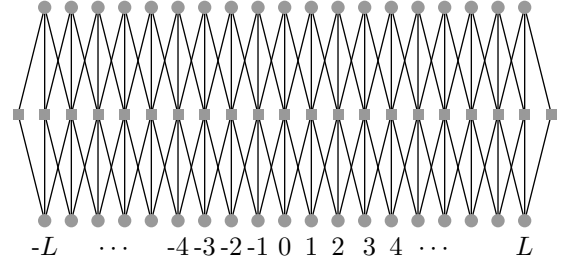


Fig. 2. A coupled chain of protographs with  $L = 9$  and  $(1 = 3, r = 6)$ .

$(2(L + \hat{1}) + 1)M/k$  check nodes. We see that, due to boundary effects, the design rate is reduced to

$$R(1, r = k1, L) = \frac{k-1}{k} - \frac{2\hat{1}}{k(2L+1)}.$$

(ii) *Threshold Increase*: The threshold changes dramatically from  $\epsilon^{\text{BP}}(1, r)$  to something close to  $\epsilon^{\text{MAP}}(1, r)$  (the MAP threshold of the underlying standard  $(1, r)$ -regular ensemble). This phenomenon (which we call “threshold saturation”) is much less intuitive and it is the aim of this paper to explain why this happens.

So far we have considered as starting points  $(1, r = k1)$ -regular ensembles. Let us now give a general definition of the  $(1, r, L)$ -ensemble which works for all parameters  $(1, r)$  so that  $1$  is odd. Rather than starting from a protograph, place variable nodes at positions  $[-L, L]$ . At each position there are  $M$  such variable nodes. Place  $\frac{1}{r}M$  check nodes at each position  $[-L - \hat{1}, L + \hat{1}]$ . Connect exactly one of the  $1$  edges of each variable node at position  $i$  to a check node at position  $i - \hat{1}, \dots, i + \hat{1}$ .

The next lemma asserts that the minimum stopping set distance of most codes is at least a fixed fraction of  $M$ . For the proof technique we follow the lead [13], [14].

*Lemma 1 (Stopping Set Distance of  $(1, r, L)$ -Ensemble)*:

Consider the  $(1, r, L)$ -ensemble with  $1 = 2\hat{1} + 1$ ,  $\hat{1} \geq 1$ , and  $r \geq 1$ . Define

$$p(x) = \sum_{i \neq 1} \binom{r}{i} x^i, \quad a(x) = \left( \sum_{i \neq 1} \binom{r}{i} i x^i \right) / \left( \sum_{i \neq 1} \binom{r}{i} x^i \right),$$

$$b(x) = -(1-1)h_2(a(x)/r) + \frac{1}{r} \log_2(p(x)) - a(x) \frac{1}{r} \log_2(x),$$

$$\omega(x) = a(x)/r, \quad h_2(x) = -x \log_2(x) - (1-x) \log_2(1-x).$$

Let  $\hat{x}$  denote the unique strictly positive solution of the equation  $b(x) = 0$  and let  $\hat{\omega}(1, r) = \omega(\hat{x})$ . Then, for any  $\delta > 0$ ,

$$\lim_{M \rightarrow \infty} \mathbb{P}\{d_{\text{ss}}(\mathcal{C})/M < (1 - \delta)\hat{\omega}(1, r)\} = 0,$$

where  $d_{\text{ss}}(\mathcal{C})$  denotes the minimum stopping set distance of the code  $\mathcal{C}$ .

*Discussion*: The quantity  $\hat{\omega}(1, r)$  is the relative weight (normalized to the blocklength) at which the exponent of the expected stopping set distribution of the underlying standard  $(1, r)$ -regular ensemble becomes positive. It is perhaps not too surprising that the same quantity also appears in our context. The lemma asserts that the minimum stopping set

distance grows linearly in  $M$ . But the stated quantity does *not* scale with  $L$ . We leave it as an interesting open problem to determine whether the stated quantity is only a lower bound or indeed equal to the typical relative stopping set distance.

### B. The $(1, r, L, w)$ Ensemble

In order to simplify the analysis we modify the ensemble  $(1, r, L)$  by adding a randomization of the edge connections. For the remainder of this paper we always assume that  $r \geq 1$ , so that the ensemble has a non-trivial design rate.

We assume that the variable nodes are at positions  $[-L, L]$ ,  $L \in \mathbb{N}$ . At each position there are  $M$  variable nodes,  $M \in \mathbb{N}$ . Conceptually we think of the check nodes to be located at all integer positions from  $[-\infty, \infty]$ . Only some of these positions actually interact with the variable nodes. At each position there are  $\frac{1}{r}M$  check nodes. It remains to describe how the connections are chosen.

Rather than assuming that a variable at position  $i$  has exactly one connection to a check node at position  $[i - \hat{1}, \dots, i + \hat{1}]$ , we assume that each of the  $1$  connections of a variable node at position  $i$  is uniformly and independently chosen from the range  $[i, \dots, i + w - 1]$ , where  $w$  is a “smoothing” parameter. In the same way, we assume that each of the  $r$  connections of a check node at position  $i$  is independently chosen from the range  $[i - w + 1, \dots, i]$ . We no longer require that  $1$  is odd.

As discussed beforehand, we will always consider the limit in which  $M$  first tends to infinity and then the number of iterations tend to infinity. Therefore, for any fixed number of rounds of DE the probability model is exactly the independent model described above<sup>1</sup>.

*Lemma 2 (Design Rate):* The design rate of the ensemble  $(1, r, L, w)$ , with  $w \leq 2L$ , is given by

$$R(1, r, L, w) = \left(1 - \frac{1}{r}\right) - \frac{1}{r} \frac{w + 1 - 2 \sum_{i=0}^w \left(\frac{i}{w}\right)^r}{2L + 1}.$$

There are many variations on the theme that show the same qualitative behavior. For real applications, these variations are vital to achieve the best trade-offs. However, the main aim of this paper is to explain why coupled LDPC codes perform so well rather than optimizing the ensemble. Therefore, despite the practical importance of these variations, we focus on the ensemble  $(1, r, L, w)$ . It is the simplest to analyze.

## III. MAIN STATEMENT AND INTERPRETATION

### A. The $(1, r, L)$ Ensemble

Consider the EBP EXIT curve of the  $(1, r, L)$  ensemble.<sup>2</sup> It is shown in Figure 3 for various values of  $L$ . Note that these EBP EXIT curves show a dramatically different behavior

<sup>1</sup>We refer the reader to [15] for the exact description of the  $(1, r, L, w)$  ensemble.

<sup>2</sup>For a general introduction into EBP EXIT curves of LDPC ensembles see [16]. To compute this curve we proceed as follows. We fix a desired entropy (defined to be the average of the erasure message of a variable node in each section) value,  $\chi$ . We initialize the constellation with the constant  $\chi$ . We then repeatedly perform one step of DE, where in each step we fix  $\epsilon$  in such a way that the entropy of the resulting constellation is equal to  $\chi$ . This procedure is the equivalent of the procedure introduced in [17, Section VIII] to compute the EBP EXIT curve for general binary-input memoryless output-symmetric channels.

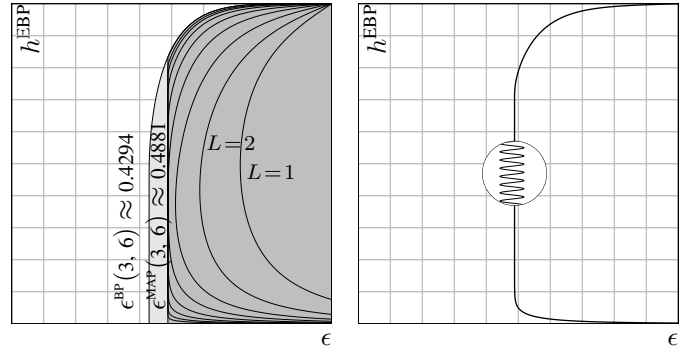


Fig. 3. The figure on the left depicts the EBP EXIT curves of the ensemble  $(3, 6, L)$  for  $L = 1, 2, 4, 8, 16, 32, 64$ , and  $128$ . The BP/MAP thresholds are  $\epsilon^{\text{BP/MAP}}(3, 6, 1) = 0.714309/0.820987$ ,  $\epsilon^{\text{BP/MAP}}(3, 6, 2) = 0.587842/0.668951$ ,  $\epsilon^{\text{BP/MAP}}(3, 6, 4) = 0.512034/0.574158$ ,  $\epsilon^{\text{BP/MAP}}(3, 6, 8) = 0.488757/0.527014$ ,  $\epsilon^{\text{BP/MAP}}(3, 6, 16) = 0.488151/0.505833$ ,  $\epsilon^{\text{BP/MAP}}(3, 6, 32) = 0.488151/0.496366$ ,  $\epsilon^{\text{BP/MAP}}(3, 6, 64) = 0.488151/0.492001$ ,  $\epsilon^{\text{BP/MAP}}(3, 6, 128) = 0.488151/0.489924$ . The light/dark gray areas mark the interior of the BP/MAP EXIT function of the underlying  $(3, 6)$ -regular ensemble, respectively. The figure on the right shows the wiggles in the EBP EXIT curve for the  $(3, 6, 32)$  ensemble. The circle shows a magnified portion of the curve. The horizontal magnification is  $10^7$ , the vertical one is  $1$ .

compared to the EBP EXIT curve of the underlying ensemble. These curves appear to be “to the right” of the threshold  $\epsilon^{\text{MAP}}(3, 6) \approx 0.48815$ . For small values of  $L$  one might be led to believe that this is true since the design rate of such an ensemble is considerably smaller than  $1 - 1/r$ . But even for large values of  $L$ , where the rate of the ensemble is close to  $1 - 1/r$ , this dramatic increase in the threshold is still true. Empirically we see that, for  $L$  increasing, the EBP EXIT curve approaches the MAP EXIT curve of the underlying  $(3, 6)$ -regular ensemble from the right. In particular, for  $\epsilon \approx \epsilon^{\text{MAP}}(1, r)$  the EBP EXIT curve drops essentially vertically until it hits zero. We will see that this is a fundamental property of this construction.

### B. Discussion

A look at Figure 3 might convey the impression that the transition of the EBP EXIT function is completely flat and that the threshold of the ensemble  $(1, r, L)$  is exactly equal to the MAP threshold of the underlying  $(1, r)$ -regular ensemble when  $L$  tends to infinity.

Unfortunately, the actual behavior is more subtle. Figure 3, on the right, shows the EBP EXIT curve for  $L = 32$  with a small section of the transition greatly magnified. As one can see from this magnification, the curve is not flat but exhibits small “wiggles” in  $\epsilon$  around  $\epsilon^{\text{MAP}}(1, r)$ . These wiggles do not vanish as  $L$  tends to infinity but their width remains constant. As we will see later, area considerations imply that, in the limit as  $L$  diverges to infinity, the BP threshold is slightly below  $\epsilon^{\text{MAP}}(1, r)$ .

Empirically, these wiggles are very small (e.g., they are of width  $10^{-7}$  for the  $(3, 6, L)$  ensemble), and further, these wiggles tend to 0 when  $1$  is increased. Unfortunately this is hard to prove. We therefore study the ensemble  $(1, r, L, w)$ . The wiggles for this ensemble are in fact larger, but they can be made arbitrarily small by letting  $w$  tend to infinity.

As pointed out in the introduction, numerical experiments indicate that there is a large class of convolutional-like LDPC ensembles that all have the property that their BP threshold is “close” to the MAP threshold of the underlying ensemble. Unfortunately, no general theorem is known to date that states when this is the case. The following theorem gives a particular instance of what we believe is a general principle. The bounds stated in the theorem are loose and can likely be improved.

### C. Main Statement

*Theorem 3 (BP Threshold of the  $(\mathbf{1}, \mathbf{r}, L, w)$  Ensemble):*

Consider transmission over the BEC( $\epsilon$ ) using random elements from the ensemble  $(\mathbf{1}, \mathbf{r}, L, w)$  with  $\mathbf{1} \geq 3$ . Let  $\epsilon^{\text{BP}}(\mathbf{1}, \mathbf{r}, L, w)$  denote the BP threshold and let  $R(\mathbf{1}, \mathbf{r}, L, w)$  denote the design rate of this ensemble. Then, in the limit as  $M$  tends to infinity, and for  $w > w(\mathbf{1}, \mathbf{r})$ ,

$$\epsilon^{\text{MAP}}(\mathbf{1}, \mathbf{r}, L, w) \leq \epsilon^{\text{MAP}}(\mathbf{1}, \mathbf{r}) + \frac{w}{2L(1 - (1 - x^{\text{MAP}})^{\mathbf{r}-1})^{\mathbf{1}}},$$

$$\epsilon^{\text{BP}}(\mathbf{1}, \mathbf{r}, L, w) \geq \left( \epsilon^{\text{MAP}}(\mathbf{1}, \mathbf{r}) - w^{-\frac{1}{8}} \frac{8\mathbf{1}\mathbf{r} + \frac{4\mathbf{r}\mathbf{1}^2}{(1-4w^{-\frac{1}{8}})^{\mathbf{r}}}}{(1-2^{-\frac{1}{2}})^2} \right) (1-4w^{-\frac{1}{8}})^{\mathbf{r}\mathbf{1}}$$

In the limit as  $M$ ,  $L$  and  $w$  (in that order) tend to infinity,

$$\lim_{w \rightarrow \infty} \lim_{L \rightarrow \infty} R(\mathbf{1}, \mathbf{r}, L, w) = 1 - \frac{1}{\mathbf{r}},$$

$$\lim_{w \rightarrow \infty} \lim_{L \rightarrow \infty} \epsilon^{\text{BP}}(\mathbf{1}, \mathbf{r}, L, w) = \lim_{L \rightarrow \infty} \epsilon^{\text{MAP}}(\mathbf{1}, \mathbf{r}, L, w) = \epsilon^{\text{MAP}}(\mathbf{1}, \mathbf{r}).$$

*Discussion:*

- (i) The constant  $w(\mathbf{1}, \mathbf{r})$  is strictly positive, and it depends only on  $\mathbf{1}, \mathbf{r}$ . An explicit expression can be found in [15], from which one can see that the lower bound is positive. Above,  $x^{\text{MAP}}$  is the stable fixed-point of DE at  $\epsilon^{\text{MAP}}(\mathbf{1}, \mathbf{r})$  for the  $(\mathbf{1}, \mathbf{r})$ -regular ensemble. The lower bound on  $\epsilon^{\text{BP}}(\mathbf{1}, \mathbf{r}, L, w)$  is the main result of the paper. It shows that, up to a term which tends to zero when  $w$  tends to infinity, the threshold of the chain is equal to the MAP threshold of the underlying ensemble. The statement in the theorem is weak. The convergence speed w.r.t.  $w$  is most likely exponential. We prove only a convergence speed of  $w^{-\frac{1}{8}}$ . We pose it as an open problem to improve this estimate.
- (ii) The MAP threshold  $\epsilon^{\text{MAP}}(\mathbf{1}, \mathbf{r})$  of the underlying ensemble quickly approaches the Shannon limit (see [16]). We therefore see that convolutional-like ensembles provide a way of approaching capacity with low complexity. E.g., for a rate equal to one-half, we get  $\epsilon^{\text{MAP}}(3, 6) = 0.48815$ ,  $\epsilon^{\text{MAP}}(4, 8) = 0.49774$ ,  $\epsilon^{\text{MAP}}(5, 10) = 0.499486$ ,  $\epsilon^{\text{MAP}}(6, 12) = 0.499876$ ,  $\epsilon^{\text{MAP}}(7, 14) = 0.499969$ .

### D. Proof Outline for the Lower Bound

The proof of the lower bound on  $\epsilon^{\text{BP}}(\mathbf{1}, \mathbf{r}, L, w)$  in Theorem 3 is long. Due to space limitations, we only provide a short outline. A detailed proof can be found in [15].

(i) *Existence of Fixed-Point (FP):* “The” key to the proof is to show the existence of a unimodal FP  $(\epsilon^*, \underline{x}^*)$  which takes on an essentially constant value in the “middle” ( $\approx x_s(\epsilon^*)$ )<sup>3</sup>, has

<sup>3</sup> $x_s(\epsilon^*)$  is the stable FP of DE at  $\epsilon^*$  for standard  $(\mathbf{1}, \mathbf{r})$ -regular ensemble.

a fast “transition”, and has arbitrarily small values towards the boundary. Figure 4 shows a typical such example. We will see

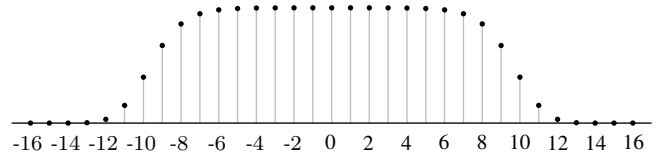


Fig. 4. Unimodal FP of the  $(\mathbf{1} = 3, \mathbf{r} = 6, L = 16, w = 3)$  ensemble with small values towards the boundary, a fast transition, and essentially constant values in the middle.

shortly that the associated channel parameter,  $\epsilon^*$ , of such a FP is necessarily very close to  $\epsilon^{\text{MAP}}(\mathbf{1}, \mathbf{r})$ .

(ii) *Construction of EXIT Curve:* Once we have established the existence of such a special FP we construct from it a whole *FP family*. The elements in this family of FPs look essentially identical. I.e., they differ only in their “width” and their channel parameter is very close to  $\epsilon^*$ . This width changes continuously, initially being equal to roughly  $2L + 1$  until it reaches zero. This family “explains” how the overall constellation collapses once the channel parameter has reached a value close to  $\epsilon^{\text{MAP}}(\mathbf{1}, \mathbf{r})$ : starting from the two boundaries, the whole constellation “moves in” like a wave until the two wave ends meet in the middle. The EBP EXIT curve is a projection of this wave (by computing the EXIT value<sup>4</sup> of each member of the family). If we look at the EBP EXIT curve, this phenomenon corresponds to the very steep vertical transition close to  $\epsilon^{\text{MAP}}(\mathbf{1}, \mathbf{r})$ .

When we construct the above family of FPs it is mathematically convenient to allow the channel parameter  $\epsilon$  to depend on the position. Let us describe this in more detail.

We start with a special FP as depicted in Figure 4. From this we construct a smooth family  $(\underline{\epsilon}(\alpha), \underline{x}(\alpha))$ , parameterized by  $\alpha$ ,  $\alpha \in [0, 1]$ , where  $\underline{x}(1) = \underline{\mathbf{1}}$  and where  $\underline{x}(0) = \underline{\mathbf{0}}$ . The components of the vector  $\underline{\epsilon}(\alpha)$  are essentially constants (for  $\alpha$  fixed). The possible exceptions are components towards the boundary. We allow those components to take on larger (than in the middle) values. From the family  $(\underline{\epsilon}(\alpha), \underline{x}(\alpha))$  we then derive an EBP EXIT curve. This smooth family of FPs essentially traces out (starting from  $\alpha = 1$ ) the EBP-EXIT curve for  $(\mathbf{1}, \mathbf{r})$ -regular ensemble till all the channel components are just greater than  $\epsilon^*$ . After this point, we show that for a “majority” portion of  $\alpha$ , the components of  $\underline{\epsilon}(\alpha)$  are essentially equal to  $\epsilon^*$ . We then measure the area enclosed by this curve. We show that this area is close to the design rate and conclude that  $\epsilon^* \approx \epsilon^{\text{MAP}}(\mathbf{1}, \mathbf{r})$ .

(iii) *Operational Meaning of EXIT Curve:* We finally show that the EBP EXIT curve constructed in step (ii) has an operational meaning. More precisely, we show that if we pick  $\epsilon < \epsilon^*$  then forward DE converges to the trivial FP.

## IV. DISCUSSION AND POSSIBLE EXTENSIONS

### A. New Paradigm for Code Design

The explanation of why convolutional-like LDPC ensembles perform so well given in this papers gives rise to a new

<sup>4</sup>EXIT value of each FP is the average of the EXIT value of each section.

paradigm in code design. In known graphical designs one encounters typically a trade-off between the threshold and the error floor behavior. E.g., for standard irregular graphs an optimization of the threshold tends to push up the number of degree-two variable nodes. The same quantity, on the other tends to favor the existence of low weight (pseudo)codewords.

For convolutional-like LDPC ensembles the important operational quantity is the MAP threshold of the underlying ensemble. As, e.g., regular LDPC ensembles show, it is simple to improve the MAP threshold *and* to improve the error-floor performance – just increase the minimum variable-node degree. An empirical study of the threshold saturation phenomenon for transmission over an additive white Gaussian noise channel can be found in [18].

### B. Scaling Behavior

In our design there are three parameters that tend to infinity. The number of variables nodes at each position, called  $M$ , the length of the constellation  $L$ , and the length of the smoothing window  $w$ . Assume we fix  $w$  and we are content with achieving a threshold slightly below the MAP threshold. How should we scale  $M$  with respect to  $L$  so that we achieve the best performance? This question is of considerable practical importance. Recall that the total length of the code is of order  $L \cdot M$ . We would therefore like to keep this product small. Further, the rate loss is of order  $1/L$  (so  $L$  should be large) and  $M$  should be chosen large so as to approach the performance predicted by DE. Finally, how does the number of required iterations scale as a function of  $L$ ?

Also, in the proof we assumed that we fix  $L$  and let  $M$  tend to infinity so that we can use DE techniques. We have seen that in this limit the boundary conditions of the constellation dictate the performance of the system regardless of the size of  $L$  (as long as  $L$  is fixed and  $M$  tends to infinity). Is the same behavior still true if we let  $L$  tend to infinity as a function of  $M$ ? At what scaling does the behavior change?

### C. Extension to General Coupled Graphical Systems

Codes based on graphs are just one instance of graphical systems that have distinct thresholds for “local” algorithms (BP threshold) and for “optimal” algorithms (MAP threshold). To be sure, coding is somewhat special – it is conjectured that the so-called replica-symmetric solution always determines the threshold under MAP processing for codes based on graphs. Nevertheless, it is interesting to investigate to what extent the coupling of general graphical systems shows a similar behavior. Is there a general class of graphical models in which the same phenomena occurs? If so, can this phenomenon either be used to analyze systems or to devise better algorithms?

### D. Why are Coupled Codes so Good?

For (uncoupled) LDPC ensembles it was shown in [2] that a suitable modified BP decoder, called the Maxwell decoder, is capable of decoding up to the MAP threshold. All the BP requires is some “push”. For the Maxwell decoder this push is provided in the form of guesses. For the coupled case it

is the spatial arrangement which provides this push. The key lies in the existence of the special FP as discussed in point (i) of Section III-D. As this FP indicates, the information can flow along the chain of coupled graphs starting from the boundary, where everything is known, towards the inside of the constellation. Neighbors are helping neighbors – how nice.

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