Average-case complexity of Maximum Weighted Independent Sets

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Outline

• Average-case analysis of computational complexity. Independent Sets

• A ‘corrected’ BP algorithm: the cavity expansion

• Results: sufficient condition, hardness results.

• Conclusion
Combinatorial Optimization with Random Costs

• **Goal**: Study relation between randomness and computational complexity

• **Problems of interest**: combinatorial optimization on graph - here: Maximum Weighted Independent Set

• Rather than random graph, random costs

• Identify relations between **graph structure**, **cost distribution**, and **complexity**

• Techniques used: ‘message-passing’ algorithm, correlation decay analysis.
Max Weight Independent Sets

- Graph $(V,E)$, weights $\mathbf{W} \in \mathbb{R}^{|V|}_+$
- Independent Set $U$: $\forall u, v \in U, (u, v) \notin E$
- Max-Weight Independent Set (MWIS): given weights $\mathbf{W}$, find $U$ which maximizes $\sum_{v \in U} W_v$
- Our setting: weights are random i.i.d variables from a joint distribution $F$
- Arbitrary graph of bounded degree $\Delta$
- Similar models in Gamarnik, Nowicki, Swircz [05], Sanghavi, Shah, Willsky [08]
Hardness facts

- NP-hard, even for $\Delta = 3$
- Poly-time approx algorithm of ratio $\alpha$: finds an IS $\tilde{U}$ such that
  $$\frac{W(U)}{W(\tilde{U})} < \alpha$$
- Poly-time Approximation Scheme: for all $\alpha > 1$, there exists a approx. algorithm of ratio $\alpha$
- Hastad [99] NP-hard to approximate within $n^\beta$, $\beta < 1$
- Trevisan [01] NP-hard to approximate within
  $$\frac{\Delta}{2^O(\sqrt{\log \Delta})}$$
Theorem: Assume $P(W > t) = \exp(-t), \Delta \leq 3$

The problem can be approximated in polynomial time: for any $\epsilon > 0$, in $O(|V|2^{\epsilon^2})$, there exists an algo. which finds an I.S. $I$ such that

$$P\left(\frac{W(I^*)}{W(I)} > 1 + \epsilon\right) < \epsilon$$

* Linear in $|V|$ (with parallel computation, constant computation time)

* Case $\Delta \leq 3$ exceptional?

* Case of Exponential weights exceptional?
  ~ Only distribution which works?
  ~ MWIS always easy with random weights?
Message passing for MWIS

• Graphical model formulation of MWIS:

\[ p(x) = \frac{1}{Z} \prod_{i,j \in E} 1\{x_i + x_j \leq 1\} \prod_{i \in V} \exp(w_i x_i) \]

• Max-product (BP):

\[
\begin{align*}
\mu_{i \rightarrow j}(0) &= \max \left\{ \prod_{k \in N_i, k \neq i} \mu_{k \rightarrow i}(0), e^{w_i} \prod_{k \in N_i, k \neq i} \mu_{k \rightarrow i}(1) \right\} \\
\mu_{i \rightarrow j}(1) &= \prod_{k \in N_i, k \neq i} \mu_{k \rightarrow i}(0)
\end{align*}
\]

set \[ M_{i \rightarrow j} = \log\left(\frac{\mu_{i \rightarrow j}(0)}{\mu_{i \rightarrow j}(1)}\right) \]

then: \[ M_{i \rightarrow j} = \max(0, W_i - \sum_{k \in N_i, k \neq j} M_{k \rightarrow j}) \]
LP relaxation for MWIS - connection with BP

• IP formulation of MWIS:
  \[ \max_{x} \sum_{i} W_i x_i \]
  s.t. \( \forall (i, j) \in E, x_i + x_j \leq 1 \)
  \( \forall i, x_i \in \{0, 1\} \)

• LP relaxation:
  \[ \max_{x} \sum_{i} W_i x_i \]
  s.t. \( \forall (i, j) \in E, x_i + x_j \leq 1 \)
  \( \forall i, 0 \leq x_i \leq 1 \)

• LP is tight at variable i if \( x_i \in \{0, 1\} \)

• Fact [Sanghavi, Shah, Willsky]: If BP converges at variable i, then the LP is tight at i

• Converse: if the LP is not tight, then BP does not converge

IP solution: one node, opt. cost: 1

LP solution: \((1/2, 1/2, 1/2)\), opt. cost: 3/2 > 1 \quad : \text{LP not tight}
The Cavity Expansion: a corrected BP

- We try to compute exactly $B_G(i) = W(I_G^*) - W(I_G^*\{i\})$
  if $>0$, then $i \in I_G^*$, otherwise $i \notin I_G^*$ (w.p.1)

$$W(I_G^*) = \max(W_i + W(I_G^*\{i,j,k,l\}), W(I_G^*\{i\}))$$

$$B_G(i) = W(I_G^*\{i\}) - W(I_G^*\{i\})$$
The Cavity Expansion: a corrected BP

So: $B_G(i) = \max \left(0, W_i - (W(I^*_G \{i\}) - (W(I^*_G \{i,j,k,l\}))\right)$
The Cavity Expansion: a corrected BP

\[
W(I^*_G\{i\}) - W(I^*_G\{i,j,k,l\}) = \\
W(I^*_G\{i\}) - W(I^*_G\{i,j\}) + \\
W(I^*_G\{i,j\}) - W(I^*_G\{i,j,k\}) + \\
W(I^*_G\{i,j,k\}) - W(I^*_G\{i,j,k,l\})
\]
The Cavity Expansion: a corrected BP

\[ W(F_G\{i\}) - W(F_G\{i,j,k,l\}) = \]

\[ W(F_G\{i\}) - W(F_G\{i,j\}) + \]
\[ ( = B_G\{i\}(j) ) \]

\[ W(F_G\{i,j\}) - W(F_G\{i,j,k\}) + \]
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\[ W(F_G\{i,j,k\}) - W(F_G\{i,j,k,l\}) + \]
\[ ( = B_G\{i,j,k\}(l) ) \]
Cavity Expansion: Summary

• Cavity Expansion (for IS):
  \[ B_G(i) = \max(0, W_i - B_G\{i\}(j) - B_G\{i,j\}(k) - B_G\{i,j,k\}(l)) \]

• BP (for IS):
  \[ M_G(i) = \max(0, W_i - M_G(j) - M_G(k) - M_G(l)) \]

• Generalization for \textbf{arbitrary} optimization

• Similar approaches (for counting): Weitz (06), Bayati, Gamarnik, Katz, Nair, Tetali (07), Jung and Shah (07)

• CE always \textbf{converges}, and is \textbf{correct} at termination

• \textbf{caveat}: running time \( O(\Delta^{|V|}) \)

• \textbf{Fix}: interrupt after a fixed number of iterations
Correlation Decay analysis

- Let $B_G^r(i)$ be the r-step approx of $B_G(i)$
- **Definition:** System exhibits correlation decay if
  \[ |B_G^r(i) - B_G(i)| \to 0 \]
  exponentially fast (in $r$)

- Implies: whether $u$ is in the MWIS is asymptotically independent of the graph beyond a certain boundary
Correlation Decay analysis

- Let $B^r_G(i)$ be the r-step approx of $B_G(i)$
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• Recall $I^* = \{i : B_G(i) > 0\}$

• Candidate solution:
  $I^r = \{i : B^r_G(i) > 0\}$
Proof sketch of near-optimality

• Introduce ‘Lyapunov’ function $L_G(i) = \mathbb{E}[\exp(-B_G(i))]$
• From CE and expo weights assumption, find a recursion on the $L_G(i): L_G(i) = 1 - 1/2(L_G\{i\}(j)L_G\{i,j\}(k))$
• This implies a non-expansion of the recursion of $L_G$
• Prune a small fraction $\delta$ of the nodes
• This implies a contraction of factor $(1 - \delta)$
• After $r$ steps, error is $(1 - \delta)^r + \delta$
• minimize delta as a function of $r$ => correlation decay
• Final steps: prove that if $B^r_G(i) \approx B_G(i)$, then $I^r \approx I^*$
Generalization

- Phase-type distribution: absorption time in a Markov Process with exponential transit times
- Dense in the space of all distributions
- Different Lyapunov function to analyze recursions
- For any phase-type distribution $F$, can compute $\alpha(F)$ such that if $\alpha(F)\Delta < 1$, corr. decay occurs.
- Not many distributions work with $\Delta \geq 2$

**Theorem**: assume $P(W > t) = \frac{1}{\Delta} \sum \exp(-\rho^i t)$, $\rho > 17$, $\Delta \leq \tilde{\Delta}$

Then corr. decay occurs, average optimization easy
Negative result

$$\mathbb{P}(W > t) = \exp(-t) \quad \Delta \leq \Delta^*$$

Unless $P=NP$, the problem cannot be solved in polynomial time

Proof Intuition:

How good of a MIS is the random MWIS?

$$\frac{I^*_\text{MIS}}{E[I^*_\text{MWIS}]} \leq O(\log \Delta)$$

But MIS is inapproximable within

$$\frac{\Delta}{2^{O(\sqrt{\log \Delta})}}$$
Conclusion

• New algorithm for optimization in sparse graphs
• Long range-independence implies existence of efficient and distributed algo
• Open Q:
  – Relation between long-range dependence and hardness?
  – Pseudo-random cost and long-range independence?
  – Polytope interpretation (average integrality gap?)