Bounds on the Quantum Satisfiability Threshold

Cristopher Moore
Center for Quantum Information and Control
Computer Science, UNM
Santa Fe Institute

joint work with Sergey Bravyi (IBM) and Alexander Russell (Connecticut)
Quantum k-SAT [Bravyi]

- Classical SAT: each clause forbids one out of 8 truth values. Think of this as forbidding a basis vector:

\[ (x_1 \lor \overline{x_2} \lor x_3) \iff \langle 010| x \rangle = 0 \]

- Quantum SAT: forbid an arbitrary vector in \( \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \),

\[ \langle v| x \rangle = 0 \]

- For each clause \( c \), we have \( \Pi_c |\psi\rangle = |\psi\rangle \) where

\[ \Pi_c = (1 - |v\rangle\langle v|) \otimes 1_{n-3} \]
A local Hamiltonian

- Alternately, ask whether there is a zero-energy state $|\psi\rangle$ of a local, disordered Hamiltonian:

$$H = \sum_c |v\rangle\langle v| \otimes 1$$

- What is its ground state energy? QMA$_1$-complete [Bravyi]

- When are its ground states entangled?
Forbidden and satisfying subspaces

\begin{itemize}
  \item The satisfying subspace is \( V_{\text{sat}} = V_{\text{forbidden}} \perp \)
  \item With probability 1, \( \text{rank} V_{\text{forbidden}} = 8 \), so \( \text{rank} V_{\text{sat}} = 32 - 8 = 24 \)
\end{itemize}
Generic clause vectors

These ranks take generic values with probability 1.

Coincidences can only decrease $\text{rank } V_{\text{forbidden}}$, and increase $\text{rank } V_{\text{sat}}$.

For a given hypergraph, if any choice of clause vectors make it unsatisfiable, it is generically unsatisfiable [Laumann et al.]

$V_{\text{forbidden}} = \text{span} \left\{ \begin{array}{c} |v\rangle \otimes |00\rangle \\ |v\rangle \otimes |01\rangle \\ |v\rangle \otimes |10\rangle \\ |v\rangle \otimes |11\rangle \\ |00\rangle \otimes |w\rangle \\ |01\rangle \otimes |w\rangle \\ |10\rangle \otimes |w\rangle \\ |11\rangle \otimes |w\rangle \end{array} \right\}$
Random quantum k-SAT formulas

• Two sources of randomness:

  • A random hypergraph with \( n \) vertices and \( m \) hyperedges (clauses), where

  \[ m = \alpha n \]

  • Random clause vectors, chosen uniformly from unit-length vectors in \( \mathbb{C}_2^\otimes k \)

• Threshold conjecture:

  \[ \lim_{n \to \infty} \Pr[H(n, m = \alpha n) \text{ is generically satisfiable}] = \begin{cases} 
1 & \alpha < \alpha_c \\
0 & \alpha > \alpha_c 
\end{cases} \]
A classical upper bound

• Compute the expected number of satisfying assignments. For k-SAT,

\[ \mathbb{E}[X] = 2^n \left( 1 - 2^{-k} \right)^m = \left( 2(1 - 2^{-k})^\alpha \right)^n \]

• This is an upper bound on the probability of satisfiability:

\[ \Pr[X > 0] \leq \mathbb{E}[X] \]

• This becomes exponentially small when \( \alpha \) is large enough:

\[ \alpha_c \leq \log_{1/(1-2^{-k})} 2 \approx 2^k \ln 2 \]

• This is asymptotically tight [Achlioptas&Moore, Achlioptas&Peres]
A simple quantum upper bound

• Number of solutions is analogous to $\text{rank } V_{\text{sat}}$

• Expectation of a clause projector:

$$\mathbb{E}_v \Pi_c = (1 - \mathbb{E}_v |v\rangle\langle v|) \otimes 1 = (1 - 2^{-k})1$$

• Since the clauses are independent, if $\Pi_{\phi} = \prod_c \Pi_c$ then

$$\text{rank } V_{\text{sat}} \leq \mathbb{E}\{v\} \text{tr } \Pi_{\phi}^\dagger \Pi_{\phi} = 2^n (1 - 2^{-k})^m$$

• So, the quantum bound is at most the classical one: $\alpha^q_c \leq \alpha_c$
Quantum SAT is more restrictive

- 2-SAT problem on a star of degree $d$
  - Classical: at least $2^{\lfloor d/2 \rfloor} + 2^{\lceil d/2 \rceil}$ solutions
  - Quantum: only $n + 1 = d + 2$
Quantum SAT is more restrictive

- Remember that any choice of forbidden vectors gives an upper bound

- Forbid singlets: $|v\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)$

- $\langle v|\psi\rangle = 0$ if and only if $|\psi\rangle$ is symmetric under transpositions

- If the graph is connected, $|\psi\rangle$ must be symmetric under all permutations
Entangled states

• This 2-SAT formula is satisfiable:

• Is this one?

• Classical: of course! Use the new variable to satisfy the new clause.

• Quantum: no! In entangled states, single variables don’t have values. Similarly, single variables can’t satisfy entangled clauses.
Better upper bounds

- For any gadget $H$ on $t$ vertices,
  \[
  \mathbb{E}[\Pi_H] = \frac{\text{rank } V_{\text{sat}}(H)}{2^t} \cdot 1
  \]
- Any time we add a gadget, we reduce the generic rank. With probability 1,
  \[
  \text{rank } V_{\text{sat}}(G \cup H) \leq \frac{\text{rank } V_{\text{sat}}(H) \cdot \text{rank } V_{\text{sat}}(G)}{2^t}
  \]
- Partition a random hypergraph into gadgets:
  \[
  \text{rank } V_{\text{sat}} \leq 2^n \prod_i \frac{\text{rank } V_{\text{sat}}(H_i)}{2^t}
  \]
The Sunflower

- Partition the hypergraph into \( n_d \) sunflowers of degree \( d \):

\[ \text{This gives } \quad \text{rank } V_{\text{sat}} \leq 2^n \prod_{d=1}^{\infty} \left( \left( \frac{3}{4} \right)^d \left( \frac{d}{6} + 1 \right) \right)^{n_d} \]
Sunflower partitions

- Naive: at each step, choose a random vertex, declare it and its clauses to be a sunflower, and remove them.

- Continuous time: give each vertex an index $t \in [0, 1]$, and remove in decreasing order.

- The degree of a sunflower of index $t$ is the number of clauses whose variables all have index $< t$. Poisson distribution with mean $k \alpha t^{k-1}$.

- Setting $\frac{\ln \text{rank } V_{\text{sat}}}{n} = 0$ gives $\alpha_c^q \leq 3.894$.

- Greedier partition: taking high-degree vertices first gives $\alpha_c^q \leq 3.689$ (analyze with system of differential equations).
The Nosegay

- Bigger gadgets: more conflict, smaller rank

- At each step, choose a random clause, and take it and its neighbors

- Gives $\alpha_c^q \leq 3.594$, far below the classical $\alpha_c \approx 4.267$
When k is large

• Asymptotically, we have

\[ \alpha_c \leq 2^k b \]

• where \( b \approx 0.573 < \ln 2 \) is the root of \( \ln 2 - 2b + \ln(b + 1) = 0 \)

• Classically,

\[ \alpha_c = (1 - o(1)) \leq 2^k \ln 2 \]

so the quantum threshold is a constant smaller.
Open questions

• Classical: counting satisfying assignments of a 3-SAT formula is \( \#P\)-complete. Quantum analog: computing \( \text{rank} V_{\text{sat}} \). What is its complexity? Might not be in \( \#P \): entanglement again.

• Similarly, is generic satisfiability of a hypergraph in NP? Is it NP-hard?

• Is there a satisfiable-but-entangled phase, in which random formulas are satisfiable, but all satisfying states are highly entangled?

• Assuming there is a transition, does \( \alpha^q_c \) grow as \( 2^k \)? Does it even grow without bound as \( k \) increases? Best lower bounds so far are less than 1!

• What is the adversarial classical threshold, where the hypergraph is random, but the adversary chooses which literals to negate?
Shameless Plug

Oxford University Press, 2010

THE NATURE of COMPUTATION

Cristopher Moore
Stephan Mertens
Acknowledgements

- Also, NSF and DTO