Sports Leagues and Competitive Societies

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Randomness in Competitions
EB, N. Hengartner, S. Redner, and F. Vazquez

Modeling and Control in Social Dynamics, Camden NJ, October 8, 2014

Talk, papers available from: http://cnls.lanl.gov/~ebn
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What is the most competitive sport?

- Soccer
- Baseball
- Hockey
- Basketball
- Football
What is the most competitive sport?

- Soccer
- Baseball
- Hockey
- Basketball
- Football

Can competitiveness be quantified?
How can competitiveness be quantified?
I. Modeling competitions
Parity of a sports league

- Teams ranked by win-loss record
- Win percentage
  \[ x = \frac{\text{Number of wins}}{\text{Number of games}} \]
- Standard deviation in win-percentage
  \[ \sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \]
- Cumulative distribution = Fraction of teams with winning percentage < x
  \[ F(x) \]

Major League Baseball
American League
2014 Season-end Standings

<table>
<thead>
<tr>
<th>Team</th>
<th>W</th>
<th>L</th>
<th>PCT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baltimore</td>
<td>96</td>
<td>66</td>
<td>.593</td>
</tr>
<tr>
<td>NY Yankees</td>
<td>84</td>
<td>78</td>
<td>.519</td>
</tr>
<tr>
<td>Toronto</td>
<td>83</td>
<td>79</td>
<td>.512</td>
</tr>
<tr>
<td>Tampa Bay</td>
<td>77</td>
<td>85</td>
<td>.475</td>
</tr>
<tr>
<td>Boston</td>
<td>71</td>
<td>91</td>
<td>.438</td>
</tr>
</tbody>
</table>

In baseball
\[ 0.400 < x < 0.600 \]
\[ \sigma = 0.08 \]
Data

- 300,000 Regular season games (all games ever played)
- 5 Major sports leagues in United States & England

<table>
<thead>
<tr>
<th>sport</th>
<th>league</th>
<th>full name</th>
<th>country</th>
<th>years</th>
<th>games</th>
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<tbody>
<tr>
<td>soccer</td>
<td>FA</td>
<td>Football Association</td>
<td>🇬🇧</td>
<td>1888-2005</td>
<td>43,350</td>
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<tr>
<td>baseball</td>
<td>MLB</td>
<td>Major League Baseball</td>
<td>🇺🇸</td>
<td>1901-2005</td>
<td>163,720</td>
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<tr>
<td>hockey</td>
<td>NHL</td>
<td>National Hockey League</td>
<td>🇨🇦</td>
<td>1917-2005</td>
<td>39,563</td>
</tr>
<tr>
<td>basketball</td>
<td>NBA</td>
<td>National Basketball Association</td>
<td>🇺🇸</td>
<td>1946-2005</td>
<td>43,254</td>
</tr>
<tr>
<td>football</td>
<td>NFL</td>
<td>National Football League</td>
<td>🇺🇸</td>
<td>1922-2004</td>
<td>11,770</td>
</tr>
</tbody>
</table>

Standard deviation in winning percentage

Distribution of winning percentage clearly distinguishes sports

• Baseball most competitive?
• Football least competitive?

Fort and Quirk, 1995
The competition model

- **Two, randomly selected, teams play**
- **Outcome of game depends on team record**
  - Weaker team wins with probability $q < 1/2 \quad \rightarrow \quad \begin{cases} q = 1/2 \\ q = 0 \end{cases}$ random
deterministic
  - Stronger team wins with probability $p > 1/2 \quad p + q = 1$
    
    $$(i, j) \rightarrow \begin{cases} (i + 1, j) \text{ probability } p \\ (i, j + 1) \text{ probability } 1 - p \end{cases} \quad i > j$$
  - When two equal teams play, winner picked randomly
- **Initially, all teams are equal (0 wins, 0 losses)**
- **Teams play once per unit time** $\langle x \rangle = \frac{1}{2}$
Rate equation approach

- **Probability distribution functions**
  \[ g_k = \text{fraction of teams with } k \text{ wins} \]
  \[ G_k = \sum_{j=0}^{k-1} g_j = \text{fraction of teams with less than } k \text{ wins} \]
  \[ H_k = 1 - G_{k+1} = \sum_{j=k+1}^{\infty} g_j \]

- **Evolution of the probability distribution**
  \[ \frac{dg_k}{dt} = (1 - q)(g_{k-1}G_{k-1} - g_kG_k) + q(g_{k-1}H_{k-1} - g_kH_k) + \frac{1}{2}(g_{k-1}^2 - g_k^2) \]
  better team wins  worse team wins  equal teams play

- **Closed equations for the cumulative distribution**
  \[ \frac{dG_k}{dt} = q(G_{k-1} - G_k) + (1/2 - q) (G_{k-1}^2 - G_k^2) \]
  Boundary Conditions  \[ G_0 = 0 \quad G_\infty = 1 \]
  Initial Conditions  \[ G_k(t = 0) = 1 \]

Nonlinear Difference-Differential Equations
An exact solution

• Stronger always wins ($q=0$)

\[
\frac{dG_k}{dt} = G_k(G_k - G_{k-1})
\]

• Transformation into a ratio

\[
G_k = \frac{P_k}{P_{k+1}}
\]

• Nonlinear equations reduce to linear recursion

\[
\frac{dP_k}{dt} = P_{k-1}
\]

• Exact solution

\[
G_k = \frac{1 + t + \frac{1}{2!} t^2 + \cdots + \frac{1}{k!} t^k}{1 + t + \frac{1}{2!} t^2 + \cdots + \frac{1}{(k+1)!} t^{k+1}}
\]
Long-time asymptotics

- **Long-time limit**
  \[ G_k \to \frac{k + 1}{t} \]

- **Scaling form**
  \[ G_k \to F\left(\frac{k}{t}\right) \]

- **Scaling function**
  \[ F(x) = x \]

Seek similarity solutions
Use winning percentage as scaling variable
Scaling analysis

- Rate equation

\[ \frac{dG_k}{dt} = q(G_{k-1} - G_k) + \left( \frac{1}{2} - q \right) \left( G_{k-1}^2 - G_k^2 \right) \]

- Treat number of wins as continuous

Inviscid Burgers equation

\[ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0 \]

\[ \frac{\partial G}{\partial t} + \left[ q + (1 - 2q)G \right] \frac{\partial G}{\partial k} = 0 \]

- Stationary distribution of winning percentage

\[ G_k(t) \to F(x) \quad x = \frac{k}{t} \]

- Scaling equation

\[ \left[ (x - q) - (1 - 2q)F(x) \right] \frac{dF}{dx} = 0 \]
Scaling solution

- Stationary distribution of winning percentage

\[ F(x) = \begin{cases} 
0 & 0 < x < q \\
\frac{x - q}{1 - 2q} & q < x < 1 - q \\
1 & 1 - q < x.
\end{cases} \]

- Distribution of winning percentage is uniform

\[ f(x) = F'(x) = \begin{cases} 
0 & 0 < x < q \\
\frac{1}{1 - 2q} & q < x < 1 - q \\
0 & 1 - q < x.
\end{cases} \]

- Variance in winning percentage

\[ \sigma = \frac{1/2 - q}{\sqrt{3}} \]

\[ \rightarrow \begin{cases} 
q = 1/2 & \text{perfect parity} \\
q = 0 & \text{maximum disparity}
\end{cases} \]
Approach to scaling

Numerical integration of the rate equations, $q=1/4$

- Winning percentage distribution approaches scaling solution
- Correction to scaling is very large for realistic number of games
- Large variance may be due to small number of games

\[
\sigma(t) = \frac{1/2 - q}{\sqrt{3}} + f(t)
\]

Variance inadequate to characterize competitiveness!
The distribution of win percentage

- Treat $q$ as a fitting parameter, time=number of games
- Allows to estimate $q_{model}$ for different leagues
The upset frequency

- Upset frequency as a measure of predictability

\[ q = \frac{\text{Number of upsets}}{\text{Number of games}} \]

- Addresses the variability in the number of games
- Measure directly from game-by-game results
  - Ties: count as 1/2 of an upset (small effect)
  - Ignore games by teams with equal records
  - Ignore games by teams with no record
The upset frequency

<table>
<thead>
<tr>
<th>League</th>
<th>q</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>FA</td>
<td>0.452</td>
<td>0.459</td>
</tr>
<tr>
<td>MLB</td>
<td>0.441</td>
<td>0.413</td>
</tr>
<tr>
<td>NHL</td>
<td>0.414</td>
<td>0.383</td>
</tr>
<tr>
<td>NBA</td>
<td>0.365</td>
<td>0.316</td>
</tr>
<tr>
<td>NFL</td>
<td>0.364</td>
<td>0.309</td>
</tr>
</tbody>
</table>

Soccer, baseball most competitive
Basketball, football least competitive

$q$ differentiates the different sport leagues!
• Parity, predictability mirror each other
• Football, baseball increasing competitiveness
• Soccer decreasing competitiveness (past 60 years)

\[ \sigma = \frac{1/2 - q}{\sqrt{3}} \]

I. Discussion

- **Model limitation:** it does not incorporate
  - Game location: home field advantage
  - Game score
  - Upset frequency dependent on relative team strength
  - Unbalanced schedule

- **Model advantages:**
  - Simple, involves only 1 parameter
  - Enables quantitative analysis
I. Conclusions

- Parity characterized by variance in winning percentage
  - Parity measure requires standings data
  - Parity measure depends on season length
- Predictability characterized by upset frequency
  - Predictability measure requires game results data
  - Predictability measure independent of season length
- Two-team competition model allows quantitative modeling of sports competitions
2. Tournaments
(post-season)
Single-elimination Tournaments

2006 NCAA Division I Men's Basketball Championship

Hampton vs. Monmouth
Winner plays Villanova in First Round

First Round

1 Connecticut
14 Albany

15 Kentucky
8 UAB

6 Washington
12 Utah St.

4 Illinois
11 Air Force

9 Michigan St.
6 George Mason

North Carolina

14 Murray St.
7 Wichita St.

13 Seton Hall
2 Tennessee

11 Winthrop

Second Round

Indiana

1 Villanova
16

8 Arizona
9 Wisconsin

5 Nevada
12 Montana

4 Boston College
12 Pacific

6 Oklahoma
11 Wis.-Milwaukee

3 Florida
1 South Ala.

7 Georgetown
15 Northern Iowa

Regional Sites:

Atlanta

2006 FINAL FOUR
INDIANAPOLIS

Indianapolis

April 1

Minneapolis

April 1

Final Four Sites:

Indianapolis

Staples Center

Staples Center

NCAA

First Round:

Duke 1
Southern U. 16
G. Washington 8
NC-Wilmington 6
Syracuse 5
Texas A&M 12
LSU 4
Iowa 11
West Virginia 6
Southern Illinois 14
Louisville 2

Second Round:

Hampton vs. Monmouth
Winner plays Villanova in First Round

Third Round:

1 Connecticut vs. 15 Kentucky
8 UAB vs. 6 Washington
4 Illinois vs. 11 Air Force
9 Michigan St. vs. 6 George Mason
North Carolina vs. Murray St.
Wichita St. vs. Seton Hall
Tennessee vs. Winthrop

Final Four:

1 Villanova vs. 8 Arizona
9 Wisconsin vs. 5 Nevada
4 Boston College vs. 12 Pacific
6 Oklahoma vs. 11 Wis.-Milwaukee

National Champion:

1 Villanova

*** ALL TIMES ARE LOCAL***

On March 12, the basketball committee will select two teams to play the opening-round game March 14 in Dayton. The winning team will be a 15th seed in the Regional.

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Binary Tree Structure
The competition model

- Two teams play, loser is eliminated

\[ N \rightarrow N/2 \rightarrow N/4 \rightarrow \cdots \rightarrow 1 \]

- Teams have inherent strength (or fitness) \( x \)

- Outcome of game depends on team strength

\[
(x_1, x_2) \rightarrow \begin{cases} 
  x_1 & \text{probability } 1 - q \\
  x_2 & \text{probability } q 
\end{cases}
\quad \text{if } x_1 < x_2 \]
Recursive approach

- **Number of teams**
  \[ N = 2^k = 1, 2, 4, 8, \ldots \]

- \( G_N(x) \) = Cumulative probability distribution function for teams with fitness less than \( x \) to win an \( N \)-team tournament

- **Closed equations for the cumulative distribution**
  \[
  G_{2N}(x) = 2p \ G_N(x) + (1 - 2p) \ [G_N(x)]^2
  \]

  **Nonlinear Recursion Equation**
Scaling properties

1. Scale of Winner
\[ x_\ast \sim N^{\frac{\ln 2p}{\ln 2}} \]

2. Scaling Function
\[ G_N(x) \rightarrow \Psi \left(\frac{x}{x_\ast}\right) \]

3. Algebraic Tail
\[ 1 - \Psi(z) \sim z^{\frac{\ln 2p}{\ln 2q}} \]

1. Large tournaments produce strong winners
3. High probability for an upset
The scaling function

Universal shape

\[ \Psi(2pz) = 2p\Psi(z) + (1 - 2p)\Psi^2(z) \]

Broad tail

\[ \Psi'(z) \sim z^{\ln 2p / \ln 2q - 1} \]
College Basketball

- Teams ranked 1-16
- Well defined favorite
- Well defined underdog
- 4 winners each year
- Theory: $q=0.18$
- Simulation: $q=0.22$
- Data: $q=0.27$
- Data: 1978-2006
- 1600 games

2008: all four top seed advance; 1 in 150 chance!
Evolution, Men vs Women
2. Conclusions

- Strong teams fare better in large tournaments
- Tournaments can produce major upsets
- Distribution of winner relates parity with predictability
- Tournaments are efficient but not fair
3. Leagues
(regular season)
League champions

- $N$ teams with fixed ranking
- In each game, favorite and underdog are well defined
- Favorite wins with probability $p > 1/2$
  Underdog wins with probability $q < 1/2$
- Each team plays $t$ games against random opponents
  - Regular random graph
- Team with most wins is the champion

$p + q = 1$

How many games are needed for best team to win?
Random walk approach

• Probability team ranked n wins a game
  \[ P_n = p \frac{n - 1}{N - 1} + q \frac{N - n}{N - 1} \]

• Number of wins performs a biased random walk
  \[ w_n = P_n t \pm \sqrt{D_n t} \]

• Team n can finish first at early times as long as
  \[ (2p - 1) \frac{n}{N} t \sim \sqrt{t} \]

• Rank of champion as function of N and t
  \[ n_* \sim \frac{N}{\sqrt{t}} \]
Length of season

- For best team to finish first
  \[ 1 \sim \frac{N}{\sqrt{t}} \]
- Each team must play
  \[ t \sim N^2 \]
- Total number of games
  \[ T \sim N^3 \]

1. Normal leagues are too short
2. Normal leagues: rank of winner \( \sim \sqrt{N} \)
3. League champions are a transient!
Distribution of outcomes

- Scaling distribution for the rank of champion
  \[ Q_n(t) \sim \frac{1}{n_*} \psi \left( \frac{n}{n_*} \right) \quad n_* \sim \frac{N}{\sqrt{t}} \]

- Probability worse team wins decays exponentially
  \[ Q_N(t) \sim \exp(-\text{const} \times t) \]

- Gaussian tail because
  \[ \psi \left( t^{1/2} \right) \sim \exp(-t) \]
  \[ \psi(z) \sim \exp(-\text{const} \times z^2) \]

- Normal league: Prob. (weakest team wins) \( \sim \exp(-N) \)

Leagues are fair: upset champions extremely unlikely
Leagues versus Tournaments

16 teams, $q=0.4$

$n_\ast \sim \sqrt{N}$

<table>
<thead>
<tr>
<th>$n$</th>
<th>league</th>
<th>tournament</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>24.5</td>
<td>12.9</td>
</tr>
<tr>
<td>2</td>
<td>18.2</td>
<td>11.4</td>
</tr>
<tr>
<td>3</td>
<td>13.6</td>
<td>10.1</td>
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<tr>
<td>4</td>
<td>10.3</td>
<td>8.9</td>
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<td>5</td>
<td>7.9</td>
<td>7.9</td>
</tr>
<tr>
<td>6</td>
<td>6.1</td>
<td>7.1</td>
</tr>
<tr>
<td>7</td>
<td>4.7</td>
<td>6.3</td>
</tr>
<tr>
<td>8</td>
<td>3.7</td>
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<td>9</td>
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<td>5.1</td>
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<td>3.4</td>
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<tr>
<td>14</td>
<td>0.81</td>
<td>3.1</td>
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<tr>
<td>15</td>
<td>0.63</td>
<td>2.8</td>
</tr>
<tr>
<td>16</td>
<td>0.49</td>
<td>2.6</td>
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</table>
What is the likelihood the best team has best record?

<table>
<thead>
<tr>
<th>league</th>
<th>season</th>
<th>games</th>
<th>likelihood</th>
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</thead>
<tbody>
<tr>
<td>NFL</td>
<td>short</td>
<td>predictable</td>
<td>30%</td>
</tr>
<tr>
<td>MLB*</td>
<td>long</td>
<td>random</td>
<td>31%</td>
</tr>
<tr>
<td>NHL</td>
<td>moderate</td>
<td>moderate</td>
<td>32%</td>
</tr>
<tr>
<td>NBA</td>
<td>moderate</td>
<td>predictable</td>
<td>45%</td>
</tr>
</tbody>
</table>

*N90% likelihood requires 15000 games/team!!!

Interplay between length of season and predictability of games
3. Conclusions

- Leagues are fair but inefficient
- Leagues do not produce major upsets
4. Ranking Algorithm
One preliminary round

- Preliminary round
  - Teams play a small number of games \( T \sim N t \)
  - Top \( M \) teams advance to championship round \( M \sim N^\alpha \)
  - Bottom \( N-M \) teams eliminated
  - Best team must finish no worse than \( M \) place \( t \sim \frac{N^2}{M^2} \)

- Championship round: plenty of games \( T \sim M^3 \)

- Total number of games
  \[ T \sim N^{3-2\alpha} + N^{3\alpha} \]

- Minimal when
  \[ M \sim N^{3/5} \quad T \sim N^{9/5} \]
Two preliminary rounds

- Two stage elimination

\[ N \rightarrow N^{\alpha_2} \rightarrow N^{\alpha_2 \alpha_1} \rightarrow 1 \]

- Second round

\[ T_2 \sim N^{3-2\alpha_2} + N^{\alpha_2(3-2\alpha_1)} + N^{3\alpha_1 \alpha_2} \]

- Minimize number of games

\[ 3 - 2\alpha_2 = \alpha_2(3 - 2\alpha_1) \quad \longrightarrow \quad \alpha_2 = \frac{15}{19} \]

- Further improvement in efficiency

\[ T \sim N^{27/19} \]
Multiple preliminary rounds

- Each additional round further reduces $T$
  \[ T_k \sim N^{\gamma_k} \]
  \[ \gamma_k = \frac{1}{1 - (2/3)^{k+1}} \]

- Gradual elimination
  \[ \gamma_k = 3, \frac{9}{5}, \frac{27}{19}, \frac{81}{65}, \ldots \]
  \[ N \rightarrow N^{\frac{57}{65}} \rightarrow N^{\frac{57}{65} \frac{15}{19}} \rightarrow N^{\frac{57}{65} \frac{15}{19} \frac{3}{5}} \rightarrow 1 \]

- Teams play a small number of games initially

Optimal linear scaling achieved using many rounds

- $T_\infty \sim N$
- $M_\infty \sim N^{1/3}$

Preliminary elimination is very efficient!
4. Conclusions

- Gradual elimination is fair and efficient
- Preliminary rounds reduce the number of games
- In preliminary round, teams play a small number of games and almost all teams advance to next round
5. Social Dynamics
Competition and social dynamics

- Teams are agents
- Number of wins represents fitness or wealth
- Agents advance by competing against each other
- Competition is a mechanism for social differentiation
The social diversity model

- Agents advance by competition
  
  $$(i, j) \rightarrow \begin{cases} 
  (i + 1, j) & \text{probability } p \\
  (i, j + 1) & \text{probability } 1 - p 
  \end{cases} \quad i > j$$

- Agent decline due to inactivity

  $$k \rightarrow k - 1 \quad \text{with rate } r$$

- Rate equations

  $$\frac{dG_k}{dt} = r(G_{k+1} - G_k) + pG_{k-1}(G_{k-1} - G_k) + (1 - p)(1 - G_k)(G_{k-1} - G_k) - \frac{1}{2}(G_k - G_{k-1})^2$$

- Scaling equations

  $$[(p + r - 1 + x) - (2p - 1)F(x)] \frac{dF}{dx} = 0$$
1. Middle class
   Agents advance at different rates

2. Middle+lower class
   Some agents advance at different rates
   Some agents do not advance

3. Lower class
   Agents do not advance

4. Egalitarian class
   All agents advance at equal rates
Concluding remarks

- Mathematical modeling of competitions sensible
- Minimalist models are a starting point
- Randomness a crucial ingredient
- Validation against data is necessary for predictive modeling
Publications

- Randomness in Competitions
  E. Ben-Naim, N.W. Hengartner

- Efficiency of Competitions
  E. Ben-Naim, N.W. Hengartner

- Scaling in Tournaments
  E. Ben-Naim, S. Redner, F. Vazquez
  Europhysics Letters 77, 30005 (2007)

- What is the Most Competitive Sport?
  E. Ben-Naim, F. Vazquez, S. Redner

- Dynamics of Multi-Player Games
  E. Ben-Naim, B. Kahng, and J.S. Kim

- On the Structure of Competitive Societies
  E. Ben-Naim, F. Vazquez, S. Redner

- Dynamics of Social Diversity
  E. Ben-Naim and S. Redner