From grains to rods

Eli Ben-Naim
Los Alamos National Laboratory

with: Paul Krapivsky (Boston University)

thanks: Igor Aronson (Argonne), Lev Tsimring (UC, San Diego)

Talk, papers available from: http://cnls.lanl.gov/~ebn
Plan

I. Driven Grains: nonequilibrium steady states
II. Driven Rods: nonequilibrium phase transitions
I. Driven grains
“A shaken box of marbles”
Driven Granular Gas

- Vigorous driving
- Spatially uniform system
- Velocities change due to:
  - ★ Collisions: lose energy
  - ★ Forcing: gain energy
- Time irreversibility

Nonequilibrium steady state
Theoretical Model

Two independent competing processes

1. Inelastic collisions (nonlinear)

\[
(v_1, v_2) \rightarrow \left( \frac{v_1 + v_2}{2}, \frac{v_1 + v_2}{2} \right)
\]

2. Random uncorrelated white noise (linear)

\[
\frac{dv_j}{dt} = \eta_j(t)

\langle \eta_j(t)\eta_j(t') \rangle = 2D\delta(t - t')
\]

System reaches a nontrivial steady-state

Energy injection balances dissipation
Kinetic theory

- Boltzmann equation
  \[
  \frac{\partial P(v)}{\partial t} = D \frac{\partial^2 P(v)}{\partial v^2} + \int \int dv_1 dv_2 P(v_1)P(v_2)\delta \left( v - \frac{v_1 + v_2}{2} \right) - P(v)
  \]

- Fourier transform
  \[
  F(k) = \int dv \, e^{ikv} P(v)
  \]

- Closed nonlinear and nonlocal equation
  \[
  (1 + D k^2) F(k) = F^2(k/2)
  \]

- Invariance
  \[
  v \rightarrow v/\sqrt{D}
  \]

Shape of distribution is independent of forcing strength
Infinite product solution

- **Solution by iteration**
  \[ F(k) = \frac{1}{1 + Dk^2} F^2(k/2) = \frac{1}{1 + Dk^2} \left( \frac{1}{1 + D(k/2)^2} \right)^2 F^4(k/4) = \cdots \]

- **Infinite product solution**
  \[ F(k) = \prod_{i=0}^{\infty} \left[ 1 + D(k/2^i)^2 \right]^{-2^i} \]

- **Exponential tail** \( v \rightarrow \infty \)
  \[ P(v) \propto \exp \left( -|v|/\sqrt{D} \right) \]

- **Also follows from**
  \[ D \frac{\partial^2 P(v)}{\partial v^2} = -P(v) \]

Non-Maxwellian distribution/Overpopulated tails

Ernst 97
Cumulant solution

- Steady-state equation

\[ F(k)(1 + Dk^2) = F^2(k/2) \]

- Take the logarithm \( \psi(k) = \ln F(k) \)

\[ \psi(k) + \ln(1 + Dk^2) = 2\psi(k/2) \]

- Cumulant solution

\[ F(k) = \exp \left[ \sum_{n=1}^{\infty} \psi_n (-Dk^2)^n / n \right] \]

- Generalized fluctuation-relaxation relations

\[ \psi_n = \lambda_n^{-1} = \left[ 1 - 2^{1-n} \right]^{-1} \]

\[ \psi_n - \psi_n(\infty) \sim e^{-\lambda_n t} \]
Experiment

- Experiment A
- Experiment B
- Theory
- Maxwellian
Stationary Solutions

• Stationary solutions do exist!
  \[ F(k) = F^2(k/2) \]

• Family of exponential solutions
  \[ F(k) = \exp(-k v_0) \]

• Lorentz/Cauchy distribution
  \[ P(v) = \frac{1}{\pi v_0} \frac{1}{1 + (v/v_0)^2} \]

How is a stationary solution consistent with energy dissipation?
Extreme Statistics

- Large velocities, cascade process
  \[ v \rightarrow \left( \frac{v_1}{2}, \frac{v_2}{2} \right) \]

- Linear evolution equation
  \[ \frac{\partial P(v)}{\partial t} = 4P\left(\frac{v}{2}\right) - P(v) \]

- Steady-state: power-law distribution
  \[ P(v) \sim v^{-2} \]

- Divergent energy, divergent dissipation rate
Injection, Cascade, Dissipation

Experiment: rare, powerful energy injections

Lottery MC: award one particle all dissipated energy

Injection selects the typical scale!
I. Conclusions

- Nonequilibrium steady-states
- Energy pumped and dissipated by different mechanisms
- Overpopulation of high-energy tail with respect to equilibrium distribution
II. Driven rods
“A shaken dish of toothpicks”
Motivation

- Biology: molecular motors
- Ecology: flocking
- Granular matter: granular rods and chains
- Phase synchronization
The rod alignment model

- Each rod has an orientation

\[-\pi \leq \theta \leq \pi\]

I. Alignment by pairwise interactions (nonlinear)

\[(\theta_1, \theta_2) \rightarrow \left\{ \begin{align*}
\left( \frac{\theta_1 + \theta_2}{2}, \frac{\theta_1 + \theta_2}{2} \right) & \quad |\theta_1 - \theta_2| < \pi \\
\left( \frac{\theta_1 + \theta_2 + 2\pi}{2}, \frac{\theta_1 + \theta_2 + 2\pi}{2} \right) & \quad |\theta_1 - \theta_2| > \pi
\end{align*} \right\} \]

II. Diffusive wiggling (linear)

\[\frac{d\theta_j}{dt} = \eta_j(t)\]

\[\langle \eta_j(t)\eta_j(t') \rangle = 2D\delta(t - t')\]
Kinetic theory

• Nonlinear integro-differential equation

\[
\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial \theta^2} + \int_{-\pi}^{\pi} d\phi \, P \left( \theta - \frac{\phi}{2} \right) P \left( \theta + \frac{\phi}{2} \right) - P.
\]

• Fourier transform

\[
P_k = \langle e^{-ik\theta} \rangle = \int_{-\pi}^{\pi} d\theta e^{-ik\theta} P(\theta)
\]

\[
P(\theta) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} P_k e^{ik\theta}
\]

• Closed nonlinear equation

\[
(1 + Dk^2)P_k = \sum_{i+j=k} A_{i-j} P_i P_j
\]

• Coupling constants

\[
A_q = \frac{\sin \frac{\pi q}{2}}{\frac{\pi q}{2}} = \begin{cases} 
1 & q = 0 \\
0 & q = 2, 4, \ldots \\
(-1)^{\frac{q-1}{2}} \frac{2}{\pi|q|} & q = 1, 3, \ldots 
\end{cases}
\]
Linear Stability Analysis

- Small perturbation to uniform state
  \[ P(\theta, t) = \frac{1}{2\pi} + p(\theta, t) \]

- Linear evolution equation for small perturbation
  \[ \frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial \theta^2} + \int_{-\pi}^{\pi} d\phi \frac{p(\theta - \phi/2) + p(\theta + \phi/2)}{2\pi} - p \]

- Growth rate of perturbation
  \[ p(\theta, t) \propto e^{ik\theta + \lambda t} \quad \Rightarrow \quad \lambda_k = 2A_k - 1 - Dk^2 \]

- Uniform state stable only when diffusion large
  \[ D > D_c = 2A_1 - 1 = \frac{4}{\pi} - 1 \]

Aronson, Tsimring
The order parameter

- **Lowest order Fourier mode**

\[ R = |\langle e^{i\theta} \rangle| = |P_{-1}| \]

- **Probes the state of the system**

\[ R = \begin{cases} 
0 & \text{disordered} \\
0.4 & \text{partially ordered} \\
1 & \text{perfectly ordered} 
\end{cases} \]
The Fourier Equation

- **Compact Form**

\[ P_k = \sum_{i+j=k, i\neq 0, j\neq 0} G_{i,j} P_i P_j \]

- **Transformed coupling constants**

\[ G_{i,j} = \frac{A_{i-j}}{1+D(i+j)^2-2A_{i+j}} \]

- **Properties**

\[ G_{i,j} = G_{j,i} \]
\[ G_{i,j} = G_{-i,-j} \]
\[ G_{i,j} = 0, \quad \text{for} \quad |i-j| = 2, 4, \ldots. \]
Solution

• Repeated iterations (product of three modes)

\[ P_k = \sum_{i+j=k} \sum_{l+m=j} G_{i,j} G_{l,m} P_i P_l P_m. \]

• When \( k=2,4,8,... \)

\[ P_2 = G_{1,1} P_1^2 \]
\[ P_4 = G_{2,2} P_2^2 = G_{2,2} G_{1,1}^2 P_1^4 \]

• Generally

\[ P_3 = 2G_{1,2} P_1 P_2 + 2G_{-1,4} P_{-1} P_4 + \cdots \]
\[ = 2G_{1,2} G_{1,1} P_1^3 + 2G_{-1,4} G_{2,2} G_{1,1}^2 P_1^4 P_{-1} \cdots \]
Partition of Integers

- **Diagramatic solution**

\[ P_k = R^k \sum_{n=0}^{\infty} p_{k,n} R^{2n} \]

- **Partition**

\[ k = 1 + 1 + \cdots + 1 + 1 - 1 - \cdots - 1. \]

- **Partitions rules**

\[ \begin{align*}
  k &= i + j \\
  i &\neq 0 \\
  j &\neq 0 \\
  G_{i,j} &\neq 0
\end{align*} \]

\[ p_{3,1} = 2G_{-1,4}G_{2,2}G_{1,1}^2 \]

All modes expressed in terms of order parameter
The order parameter

- Infinite series solution
  \[ R = R^k \sum_{n=0}^{\infty} p_{1,n} R^{2n} \]

- Landau theory
  \[ R = \frac{C}{D_c - D} R^3 + \cdots \]

- Critical diffusion constant
  \[ D_c = \frac{4}{\pi} - 1 \]

Close equation for order parameter
Nonequilibrium phase transition

- **Critical diffusion constant** \( D_c = \frac{4}{\pi} - 1 \)
- **Subcritical: ordered phase** \( R > 0 \)
- **Supercritical: disordered phase** \( R = 0 \)
- **Critical behavior** \( R \sim (D_c - D)^{1/2} \)
Distribution of orientation

- Fourier modes decay exponentially with \( R \)

\[ P_k \sim R^k \]

- Small number of modes sufficient in practice

\[
P(\theta) = \frac{1}{2\pi} \left[ 1 + 2R \cos \theta + 2G_{1,1} R^2 \cos (2\theta) + 4G_{1,2} G_{1,1} R^3 \cos (3\theta) + \cdots \right]
\]
General alignment rates

- **Alignment rate**
  \[ K(|\theta_1 - \theta_2|) \]

- **Diagramatic solution holds**

- **Hard-rods**
  \[ K(\phi) \propto |\sin \phi| \quad D_c = \frac{1}{3} \]

- **Hard-spheres: system always disordered**
  \[ K(\phi) \propto |\phi| \]

Boltzmann equation can be solved!
Phase transition may or may not exist
Arbitrary alignment rates

- Kinetic theory: arbitrary alignment rates

\[ 0 = D \frac{d^2 P}{d\theta^2} + \int_{-\pi}^{\pi} d\phi K(\phi) P \left( \theta - \frac{\phi}{2} \right) P \left( \theta + \frac{\phi}{2} \right) - P(\theta) \int_{-\pi}^{\pi} d\phi K(\phi) P(\theta + \phi) \]

- Fourier transform of alignment rate

\[ A_q = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{iq\phi/2} K(\phi) \]

- Recover same Fourier equation using

\[ G_{i,j} = \frac{1}{2} \frac{A_{i-j} + A_{j-i} - A_{2i} - A_{2j}}{1 + D(i + j)^2 - 2A_{i+j}} \]

When Fourier spectrum is discrete: exact solution is possible for arbitrary alignment rates
Experiments
II. Conclusions

• Nonequilibrium phase transition
• Weak noise: ordered phase (nematic)
• Strong noise: disordered phase
• Solution relates to iterated partition of integers
• Only when Fourier spectrum is discrete: exact solution possible for arbitrary averaging rates