Nontrivial Exponents in Records Statistics

Eli Ben-Naim
Los Alamos National Laboratory

with: Pearson Miller (Yale)


Talk, paper available from: http://cnls.lanl.gov/~ebn

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## Marathon World Record

<table>
<thead>
<tr>
<th>Year</th>
<th>Athlete</th>
<th>Country</th>
<th>Record</th>
<th>Improvement</th>
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</thead>
<tbody>
<tr>
<td>2002</td>
<td>Khalid Khannuchi</td>
<td>USA</td>
<td>2:05:38</td>
<td></td>
</tr>
<tr>
<td>2003</td>
<td>Paul Tergat</td>
<td>Kenya</td>
<td>2:04:55</td>
<td>0:43</td>
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<td>2007</td>
<td>Haile Gebrsellassie</td>
<td>Ethiopia</td>
<td>2:04:26</td>
<td>0:29</td>
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<tr>
<td>2008</td>
<td>Haile Gebrsellassie</td>
<td>Ethiopia</td>
<td>2:03:59</td>
<td>0:27</td>
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<tr>
<td>2011</td>
<td>Patrick Mackau</td>
<td>Kenya</td>
<td>2:03:38</td>
<td>0:21</td>
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<tr>
<td>2013</td>
<td>Wilson Kipsang</td>
<td>Kenya</td>
<td>2:03:23</td>
<td>0:15</td>
</tr>
</tbody>
</table>

**Incremental sequence of records**

*every record improves upon previous record by yet smaller amount*

**Are incremental sequences of records common?**

source: wikipedia
Incremental Records

Incremental sequence of records

every record improves upon previous record by yet smaller amount

random variable = \{0.4, 0.4, 0.6, 0.7, 0.5, 0.1\}

latest record = \{0.4, 0.4, 0.6, 0.7, 0.7, 0.7\}  \uparrow

latest increment = \{0.4, 0.4, 0.2, 0.1, 0.1, 0.1\}  \downarrow

What is the probability all records are incremental?
Probability all records are incremental

$$S_N \sim N^{-\nu} \quad \nu = 0.31762101$$

Power law decay with nontrivial exponent

Question is free of parameters!
Uniform distribution

- The variable $x$ is randomly distributed in $[0:1]$
  \[ \rho(x) = 1 \quad \text{for} \quad 0 \leq x \leq 1 \]
- Probability record is smaller than $x$
  \[ R_N(x) = x^N \]
- Average record
  \[ A_N = \frac{N}{N + 1} \quad \Rightarrow \quad 1 - A_N \approx N^{-1} \]
- Number of records
  \[ M_N = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \]
Distribution of records

- Probability a sequence is incremental and record \( < x \)

\[
G_N(x) \implies S_N = G_N(1)
\]

- One variable

\[
G_1(x) = x \implies S_1 = 1
\]

- Two variables

\[
x_2 - x_1 > x_1, \quad G_2(x) = \frac{3}{4}x^2 \implies S_2 = \frac{3}{4}
\]

- In general, conditions are scale invariant

\[
x \rightarrow a \ x
\]

- Distribution of records for incremental sequences

\[
G_N(x) = S_N x^N
\]

- Distribution of records for all sequences equals

\[
x^N
\]

Statistics of records are standard
Scaling behavior

- Distribution of records for incremental sequences

\[
G_N(x)/S_N = x^N = [1 - (1 - x)]^N \to e^{-s}
\]

- Scaling variable

\[
s = (1 - x)N
\]

Exponential scaling function
Distribution of increment+records

- **Probability density** $S_N(x,y)dx\,dy$ that:
  1. Sequence is incremental
  2. Current record is in range $(x,x+dx)$
  3. Latest increment is in range $(y,y+dy)$ with $0<y<x$

- Gives the probability a sequence is incremental

$$S_N = \int_0^1 dx \int_0^x dy \, S_N(x,y)$$

- **Recursion equation incorporates memory**

$$S_{N+1}(x,y) = x \, S_N(x,y) + \int_y^{x-y} dy' \, S_N(x-y,y')$$

  old record holds \hspace{1cm} a new record is set

- **Evolution equation includes integral, has memory**

$$\frac{\partial S_N(x,y)}{\partial N} = -(1-x)S_N(x,y) + \int_y^{x-y} dy' \, S_N(x-y,y')$$
Scaling transformation

• Assume record and increment scale similarly
  \[ y \sim 1 - x \sim N^{-1} \]

• Introduce a scaling variable for the increment
  \[ s = (1 - x)N \quad \text{and} \quad z = yN \]

• Seek a scaling solution
  \[ S_N(x, y) = N^2 S_N \Psi(s, z) \]

• Eliminate time out of the master equation
  \[
  \left(2 - \nu + s + s \frac{\partial}{\partial s} + z \frac{\partial}{\partial z}\right) \Psi(s, z) = \int_z^\infty dz' \Psi(s + z, z')
  \]

Reduce problem from three variables to two
Factorizing solution

- Assume record and increment decouple
  \[ \Psi(s, z) = e^{-s} f(z) \]

- Substitute into equation for similarity solution
  \[ \left(2 - \nu + s + s \frac{\partial}{\partial s} + z \frac{\partial}{\partial z}\right) \Psi(s, z) = \int_{z}^{\infty} dz' \Psi(s + z, z') \]

- First order integro-differential equation
  \[ zf'(z) + (2 - \nu) f(z) = e^{-z} \int_{z}^{\infty} f(z') dz' \]

- Cumulative distribution of scaled increment
  \[ g(z) = \int_{z}^{\infty} f(z') dz' \]

- Convert into a second order differential equation
  \[ zg''(z) + (2 - \nu) g'(z) + e^{-z} g(z) = 0 \]

  \[ g(0) = 1 \]
  \[ g'(0) = -1/(2 - \nu) \]

Reduce problem from two variable to one
Distribution of increment

- Assume record and increment decouple
  \[ zg''(z) + (2 - \nu)g'(z) + e^{-z}g(z) = 0 \]
  \[ g(0) = 1 \]
  \[ g'(0) = -1/(2 - \nu) \]

- Two independent solutions
  \[ g(z) = z^{\nu-1} \quad \text{and} \quad g(z) = \text{const.} \quad \text{as} \quad z \to \infty \]

- The exponent is determined by the tail behavior
  \[ \beta = 0.317621014462... \]

- The distribution of increment has a broad tail
  \[ P_N(y) \sim N^{-1} y^{\nu-2} \]

Increments can be relatively large
problem reduced to second order ODE
Numerical confirmation

Monte Carlo simulation versus integration of ODE

\[ g(0) = 1 \]
\[ g'(0) = -\frac{1}{2 - \nu} \]

\[ \frac{\langle s \hat{z} \rangle}{\langle s \rangle \langle \hat{z} \rangle} \rightarrow 1 \]

Increment and record become uncorrelated
Conclusions

- Studied persistent configuration of record sequences
- Linear evolution equations (but with memory)
- Dynamic formulation: treat sequence length as time
- Similarity solutions for distribution of records
- Probability a sequence of records is incremental decays as power-law with sequence length
- Power-law exponent is nontrivial, obtained analytically
- Distribution of record increments is broad

First-passage properties of extreme values are interesting