Opinion Dynamics: Rise and Fall of Political Parties

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Kinetic description of Social Dynamics, College Park, Maryland, November 6, 2012
DC, Election Day
Plan

I. Pure compromise dynamics
   A. Continuous opinions
   B. Discrete opinions

II. Noisy compromise dynamics
   A. Single-party dynamics
   B. Two-party dynamics
   C. Multi-party dynamics
Themes

1. Bifurcations
2. Pattern Formation
3. Scaling
4. Coarsening
1. Pure compromise dynamics
The compromise process

- Opinion measured by a continuum variable
  
  $-\Delta < x < \Delta$

1. **Compromise**: reached by pairwise interactions

   \[(x_1, x_2) \rightarrow \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right)\]

2. **Conviction**: restricted interaction range

   \[|x_1 - x_2| < 1\]

- Minimal, one parameter model
- Mimics competition between compromise and conviction

  "bounded confidence"

Problem set-up

- Given uniform initial (un-normalized) distribution

$$P_0(x) = \begin{cases} 
1 & |x| < \Delta \\
0 & |x| > \Delta
\end{cases}$$

- Find final distribution

$$P_\infty(x) = ?$$

- Multitude of final steady-states

$$P_0(x) = \sum_{i=1}^{N} m_i \delta(x - x_i) \quad |x_i - x_j| > 1$$

- Dynamics selects one (deterministically!)

Multiple localized clusters
Further details

• Dynamic treatment
  Each individual interacts once per unit time

• Random interactions
  Two interacting individuals are chosen randomly

• Infinite particle limit is implicitly assumed
  \[ N \to \infty \]

• Process is galilean invariant
  \[ x \to x + x_0 \]

Set average opinion to zero
  \[ \langle x \rangle = 0 \]
Numerical methods, kinetic theory

- **Same master equation, restricted integration**

\[
\frac{\partial P(x, t)}{\partial t} = \int \int dx_1dx_2 P(x_1, t)P(x_2, t) \left[ \delta \left( x - \frac{x_1 + x_2}{2} \right) - \delta(x - x_1) \right]
\]

\(|x_1 - x_2| < 1\)

- Direct Monte Carlo simulation of stochastic process

- Numerical integration of rate equations
Two Conservation Laws

- **Total population is conserved**

\[
\int_{-\Delta}^{\Delta} dx \, P(x) = 2\Delta
\]

- **Average opinion is conserved**

\[
\int_{-\Delta}^{\Delta} dx \, x \, P(x) = 0
\]
Rise and fall of central party

$0 < \Delta < 1.871$

$1.871 < \Delta < 2.724$

Central party may or may not exist!
Resurrection of central party

2.724 < Δ < 4.079

4.079 < Δ < 4.956

Parties may or may not be equal in size
Emergence of extremists

Tiny fringe parties ($m \sim 10^{-3}$)
Bifurcations and Patterns

![Graph showing bifurcations and patterns with lines labeled major, central, and minor. The x-axis is labeled \( \Delta \) and the y-axis is labeled X. There are arrows indicating changes at certain points.]
Self-similar structure, universality

- **Periodic sequence of bifurcations**
  1. Nucleation of minor cluster branch
  2. Nucleation of major cluster branch
  3. Nucleation of central cluster

- **Alternating major-minor pattern**

- **Clusters are equally spaced**

- **Period $L$ gives major cluster mass, separation**

\[ x(\Delta) = x(\Delta) + L \quad \text{with} \quad L = 2.155 \]
How many political parties?

- Data: CIA world factbook
- 120 countries with multi-party parliaments
- Average=5.8; Standard deviation=2.9
Cluster mass

- **Masses are periodic**
  \[ m(\Delta) = m(\Delta + L) \]

- **Major mass**
  \[ M \rightarrow L = 2.155 \]

- **Minor mass**
  \[ m \rightarrow 3 \times 10^{-4} \]

Why are the minor clusters so small?
Scaling near bifurcation points

- Minor mass vanishes
  \[ m \sim (\Delta - \Delta_c)^\alpha \]
- Universal exponent
  \[ \alpha = \begin{cases} 
  3 & \text{type 1} \\
  4 & \text{type 3} 
\end{cases} \]

L-2 is the small parameter explains small saturation mass
Consensus dynamics

- Integrable for $\Delta < 1/2$
  \[
  \langle x^2(t) \rangle = \langle x^2(0) \rangle e^{-\Delta t}
  \]
- Final state: localized
  \[
  P_\infty(x) = 2\Delta \delta(x)
  \]
- Rate equations in Fourier space
  \[
  P_t(k) + P(k) = P^2(k/2)
  \]
- Self-similar collapse dynamics
  \[
  \Phi(z) \propto (1 + z^2)^{-2} \quad \text{for} \quad z = x/\sqrt{\langle x^2 \rangle}
  \]

Heuristic derivation of exponent

- Perturbation theory \( \Delta = 1 + \epsilon \)
- Major cluster \( x(\infty) = 0 \)
- Minor cluster \( x(\infty) = \pm(1 + \epsilon/2) \)
- Rate of transfer from minor cluster to major cluster
  \[
  \frac{dm}{dt} = -m M \quad \rightarrow \quad m \sim \epsilon e^{-t}
  \]
- Process stops when
  \[
  x \sim e^{-t_f/2} \sim \epsilon \quad \langle x^2 \rangle \sim e^{-t}
  \]
- Final mass of minor cluster
  \[
  m(\infty) \sim m(t_f) \sim \epsilon^3 \quad \alpha = 3
  \]
Pattern selection

- **Linear stability analysis**
  
  \[ P - 1 \propto e^{i(kx + \omega t)} \implies w(k) = \frac{8}{k} \sin \frac{k}{2} - \frac{2}{k} \sin k - 2 \]

- **Fastest growing mode**

  \[ \frac{dw}{dk} \implies L = \frac{2\pi}{k} = 2.2515 \]

- **Traveling wave (FKPP saddle point analysis)**

  \[ v = \frac{dw}{dk} = \frac{\text{Im}[w]}{\text{Im}[k]} \implies k_{\text{select}} = k_* - \frac{w_*}{v} \]

  Patterns induced by wave propagation from boundary

  However, emerging period is different

  \[ L_{\text{select}} = \frac{2\pi}{k_{\text{select}}} = 2.148644 \]

  \[ L_* = 2.037514 \]

  \[ L_{\text{num}} = 2.155 \]

  Wavelength obtained analytically!
Discrete opinions

• Compromise process
  \[(n - 1, n + 1) \rightarrow (n, n)\]

• Master equation
  \[
  \frac{dP_n}{dt} = 2P_{n-1}P_{n+1} - P_n(P_{n-2} + P_{n+2})
  \]

• Simplest example: 6 states

• Symmetry + normalization:

• Two-dimensional problem

  Initial condition determines final state

  Isolated fixed points, lines of fixed points
Discrete opinions

- Dissipative system, volume contracts
- Energy (Lyapunov) function exists
- No cycles or strange attractors
- Uniform state is unstable (Cahn-Hilliard)

$$P_i = 1 + \phi_i$$

$$\phi_t + (\phi + a \phi_{xx} + b \phi^2)_{xx}$$

Discrete case yields useful insights
Pattern selection

- Linear stability analysis
  
  \[ P - 1 \propto e^{i(kx + \omega t)} \rightarrow w(k) = 4 \cos k - 4 \cos 2k - 2 \]

- Fastest growing mode
  
  \[
  \frac{dw}{dk} \quad \Rightarrow \quad L = \frac{2\pi}{k} = 6
  \]

- Traveling wave (FKPP saddle point analysis)
  
  \[
  v = \frac{dw}{dk} = \frac{\text{Im}[w]}{\text{Im}[k]} \quad \Rightarrow \quad k_{\text{select}} = k_* - \frac{w_*}{v}
  \]

  \[ L_* = 5.671820 \]

Again, wavelength can be obtained analytically.
I. Conclusions

- Clusters form via bifurcations
- Periodic structure
- Alternating major-minor pattern
- Central party does not always exist
- Power-law behavior near transitions
- Pattern selection understood
I. Outlook

- Gaps
- Role of initial conditions, classification
- Role of spatial dimension, correlations
- Disorder, inhomogeneities
- Tiling/Packing in 2D
- Discord dynamics (seceder model, Halpin-Heally 03)

Many open questions
II. Noisy compromise dynamics
Diffusion (noise)

- **Diffusion**: Individuals change opinion spontaneously

\[ n \xrightarrow{D} n \pm 1 \]

- Adds noise ("temperature")
- Linear process: no interaction
- Mimics unstable, varying opinion
- Influence of environment, news, editorials, events
Rate equations

- Compromise: reached through pairwise interactions
  \[(n - 1, n + 1) \rightarrow (n, n)\]

- Conserved quantities: total population, average opinion

- Probability distribution \( P_n(t) \)

- Kinetic theory: nonlinear rate equations

\[
\frac{dP_n}{dt} = 2P_{n-1}P_{n+1} - P_n(P_{n-2} + P_{n+2}) + D(P_{n-1} + P_{n+1} - 2P_n)
\]

Direct Monte Carlo simulations of stochastic process

Numerical integration of rate equations
Single-party dynamics

- Initial condition: large isolated party
  \[ P_n(0) = m(\delta_{n,0} + \delta_{n,-1}) \]

- Steady-state: compromise and diffusion balance
  \[ DP_n = P_{n-1}P_{n+1} \]

- Core of party: localized to a few opinion states
  \[ P_0 = m \quad P_1 = D \quad P_2 = D^2m^{-1} \]

- Compromise negligible for \( n > 2 \)

Party has a well defined core
The tail

• Diffusion dominates outside the core
  \[ \frac{dP_n}{dt} = D(P_{n-1} + P_{n+1} - 2P_n) \quad P \ll D \]

• Standard problem of diffusion with source
  \[ P_n \sim m^{-1} \Psi(n t^{-1/2}) \]

• Tail mass
  \[ M_{\text{tail}} \sim m^{-1} t^{1/2} \]

• Party dissolves when
  \[ M_{\text{tail}} \sim m \quad \implies \quad \tau \sim m^4 \]

Party lifetime grows dramatically with its size
Core versus tail

\[ m = 10^3 \]

Party height = \( m \)

Party depth \( \sim m^{-1} \)

Self-similar shape

Gaussian tail
Qualitative features

• Exists in a quasi-steady state
• Tight core localized to a few sites
• Random opinion changes of members do not affect party position
• Party lifetime grows very fast with size
• Ultimate fate of a party: demise
• Its remnant: a diffusive cloud
• Depth inversely proportional to size, the larger the party the more stable
Two party dynamics

- Initial condition: two large isolated parties
  \[ P_n(0) = m_>(\delta_{n,0} + \delta_{n,-1}) + m_<(\delta_{n,l} + \delta_{n,l+1}) \]

- Interaction between parties mediated by diffusion
  \[ 0 = P_{n-1} + P_{n+1} - 2P_n \]

- Boundary conditions set by parties depths
  \[ P_0 = \frac{1}{m_>}, \quad P_l = \frac{1}{m_<} \]

- Steady state: linear profile
  \[ P_n = \frac{1}{m_<} + \left( \frac{1}{m_<} - \frac{1}{m_>} \right) \frac{n}{l} \]
Merger

- Steady flux from small party to larger one
  \[ J \sim \frac{1}{l} \left( \frac{1}{m_\text{<}} - \frac{1}{m_\text{>}} \right) \sim \frac{1}{lm_\text{<}} \]
- Merger time
  \[ T \sim \frac{m_\text{<}}{J} \sim lm_\text{<}^2 \]
- Lifetime grows with separation (“niche”)
- Outcome of interaction is deterministic
- Larger party position remains fixed throughout merger process

Small party absorbed by larger one
Merger: numerical results
Multiple party dynamics

- **Initial condition: large isolated party**
  \[ P_n(0) = \text{randomly chosen number in } [1 - \epsilon : 1 + \epsilon] \]

- **Linear stability analysis**
  \[ P_n - 1 \sim e^{i k n + \lambda t} \]

- **Growth rate of perturbations**
  \[ \lambda(k) = (4 \cos k - 4 \cos 2k - 2) - 2D(1 - \cos 2k) \]

- **Long wavelength perturbations unstable**
  \[ k < k_0 \quad \cos k_0 = D/2 \]

\[ P=1 \text{ stable only for strong diffusion } D > D_c = 2 \]
Strong noise ($D > D_c$)

• Regardless of initial conditions

\[ P_n \to \langle P_n(0) \rangle \]

• Relaxation time

\[ \lambda \approx (D_c - D) k^2 \quad \Rightarrow \quad \tau \sim (D - D_c)^{-2} \]

No parties, disorganized political system
Weak noise ($D < D_c$): Coarsening

- Smaller parties merge into large parties
- Party size grows indefinitely
- Assume a self-similar process, size scale $m$
- Conservation of populations implies separation
  \[ l \sim m \]
- Use merger time to estimate size scale
  \[ t \sim lm^2 \sim m^3 \implies m \sim t^{1/3} \]
- Self-similar size distribution
  \[ P_m \sim t^{-1/3} F(m t^{-1/3}) \]
Coarsening: numerical results

- Parties are static throughout process.
- A small party with a large niche may still outlast a larger neighbor!
Three scenarios

$D = 0$

$D < D_c$

$D > D_c$

early intermediate late
II. Conclusions

• Isolated parties
  - Tight, immobile core and diffusive tail
  - Lifetime grows fast with size

• Interaction between two parties
  - Large party grows at expense of small one
  - Deterministic outcome, steady flux

• Multiple parties
  - Strong noise: disorganized political system, no parties
  - Weak noise: parties form, coarsening mosaic
  - No noise: pattern formation
Publications

