Popularity-Driven Networking

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E. Ben-Naim and P.L. Krapivsky, EPL 97, 48003 (2012)
Talk, paper available from: http://cnls.lanl.gov/~ebn

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Plan

• Growing random graphs: uniform linking
  - Degree distribution
  - Component size distribution

• Growing random graphs: popularity-driven linking
  - Degree distribution
  - Component size distribution
Random Graphs: Uniform Linking

• Initial state: $N$ isolated nodes

• Dynamical linking
  1. Pick two nodes at random
  2. Connect the two nodes with a link
  3. Augment time \( t \to t + \frac{1}{2N} \)

• Each node experiences one linking event per unit time

Percolation: one component contains fraction of mass
Condensation: one component contains all mass

Percolation time is finite; Condensation time is divergent
Degree Distribution & Condensation

- Distribution of nodes with degree $j$ at time $t$ is $n_j(t)$
- Linking process is simple augmentation
  
  \[ j \rightarrow j + 1 \]
- **Linear** evolution equation

\[
\frac{dn_j}{dt} = n_{j-1} - n_j
\]

- Degree distribution is Poissonian

\[
n_j(t) = \frac{t^j}{j!}e^{-t}
\]

- Isolated nodes disappear when $N n_0(t_c) = 1$

\[
t_c \sim \ln N
\]

Condensation time diverges with system size
Component Size Distribution & Percolation

- Component = a connected set of nodes
- Merger rate = product of component sizes
  \[ [i] + [j] \xrightarrow{K_{i,j}} [i, j] \]
  \[ K_{i,j} = i \cdot j \]
- Nonlinear evolution equation
  \[ \frac{dc_k}{dt} = \frac{1}{2} \sum_{i+j=k} ijc_i c_j - k c_k \]
  \[ c_k(t = 0) = \delta_{k,1} \]
- Component size distribution
  \[ c_k(t) = \frac{1}{k \cdot k!} (kt)^{k-1} e^{-kt} \]
  \[ \sum_k k c_k = \begin{cases} 
  1 & t < 1 \\
  1 - g & t > 1 
\end{cases} \]
- Percolation: finite clusters contain only fraction of mass
- Giant component with macroscopic size emerges

Percolation time is finite, independent of N
Two Phases

finite components

$g = 1 - e^{-gt}$

giant component + finite components
Random Graphs: Popularity-Driven Linking

- Initial state: \( N \) isolated nodes
- Dynamical “popularity-driven” linking,
  1. Pick 2 nodes, each with probability proportional to degree
  2. Connect the 2 nodes with a link
- Motivation: online social networks, friends seek and accept friends according to popularity (facebook)
- Rich-gets-richer mechanism as in preferential attachment
- Hybrid between random graph and preferential attachment

Barabasi-Albert 99

Nature of percolation and condensation transitions?
Degree Distribution

- Distribution of nodes with degree $j$ is $n_j$
- Linking process with linear linking rate
  \[ \frac{C_{i,j}}{(i, j)} \xrightarrow{C_{i,j}} (i + 1, j + 1) \quad C_{i,j} = (i + 1)(j + 1) \]
- Linear evolution equation
  \[ \frac{dn_j}{dt} = (1 + \langle j \rangle) \left[ j n_{j-1} - (j + 1) n_j \right] \]
- Exponential degree distribution
  \[ n_j = (1 - t) t^j \]
- Isolated nodes disappear in finite time!
- Rich gets richer may not produce broad distribution
  \[ \langle j \rangle = \frac{1}{1 - t} \]
  \[ t_c = 1 \]

Condensation in finite time!
Component Size Distribution

- Components are trees: total degree gives total links
- Merger rate = product of number of links
  \[ [l] + [m] \xrightarrow{K_{l,m}} [l + m] \quad K_{l,m} = (3l - 2)(3m - 2) \]
- Closed nonlinear evolution equation
  \[
  \frac{dc_k}{dt} = \frac{1}{2} \sum_{l+m=k} (3l - 2)(3m - 2)c_l c_m - \langle j + 1 \rangle (3k - 2)c_k, \\
  c_k(t = 0) = \delta_{k,1}
  \]
- Component size distribution
  \[
  c_k = \frac{(3k-3)!}{k!(2k-1)!} t^{k-1} (1 - t)^{2k-1}
  \]
  \[
  \sum_k k c_k = \begin{cases} 
  1 & t < 1/3 \\
  1 - g & t > 1/3 
  \end{cases}
  \]
- Percolation: finite clusters contain only fraction of mass
- Second moment diverges
  \[
  \sum_k k^2 c_k = \frac{1 - 2t}{1 - 3t} \\
  t_g = 1/3
  \]

Percolation time is smaller than condensation time!
Three Phases

Moments of size distribution diverge prior to percolation

\[ \langle k^2 \rangle = \frac{1 - 2t}{1 - 3t} \]

Average degree diverges prior to condensation

\[ \langle j \rangle = \frac{1}{1 - t} \]

size distribution

\[ c_k \sim k^{-5/2} e^{-k/k_*} \]

precisely at critical point

\[ c_k \sim k^{-5/2} \]

Two successive finite time singularities!
Generalized linking rates

• Linking rate is a general power of degree
  \[ C_{ij} = (ij)^\alpha \]

• Average degree
  \[ \langle j \rangle \sim \begin{cases} 
    t^{1/(1-2\alpha)} & \alpha < 1/2, \\
    e^{\text{const.} \times t} & \alpha = 1/2, \\
    (t_c - t)^{-1/(2\alpha-1)} & 1/2 < \alpha \leq 1.
  \end{cases} \]

• Instantaneous condensation
  \[ t_c \sim (\ln N)^{-\gamma} \text{ when } \alpha > 1 \]

Condensation time: divergent, finite, or, vanishing
Summary

• Popularity-driven evolution of a random graphs
• Linking rate is proportional to degree (rich-get-richer)
• Degree distribution is exponential
• Percolation time is finite
• Condensation time is finite

Outlook

• Cycle structure: number, size, distribution, etc.
• Analysis of condensate: cycles, shortest paths