Kinetic Theory of Granular Gases

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talk, papers available: http://cnls.lanl.gov/~ebn
Plan

1. Basics: collision rules, collision rates
2. The Boltzmann equation
3. Extreme statistics and the linearized Boltzmann equation
4. Forced steady states
5. Freely cooling states
6. Stationary states and energy cascades
7. Hybrid solutions
Experiments

- **Friction**
  - D Blair, A Kudrolli 01

- **Rotation**
  - K Feitosa, N Menon 04

- **Driving strength**
  - W Losert, J Gollub 98

- **Dimensionality**
  - J Urbach & Olafsen 98

- **Boundary**
  - J van Zon, H Swinney 04

- **Fluid drag**
  - K Kohlstedt, I Aronson, EB 05

- **Long range interactions**
  - D Blair, A Kudrolli 01; W Losert 02
  - K Kohlstedt, J Olafsen, EB 05

- **Substrate**
  - G Baxter, J Olafsen 04

**Deviations from equilibrium distribution**
Driven Granular gas

- Vigorous driving
- Spatially uniform system
- Particles undergo binary collisions
- Velocities change due to
  1. **Collisions**: lose energy
  2. **Forcing**: gain energy
- What is the typical velocity (granular “temperature”)?
  \[ T = \langle v^2 \rangle \]
- What is the velocity distribution?
  \[ f(v) \]
Inelastic Collisions (1D)

- Relative velocity reduced by $0 < r < 1$
  \[ v_1 - v_2 = -r(u_1 - u_2) \]

- Momentum is conserved
  \[ v_1 + v_2 = u_1 + u_2 \]

- Energy is dissipated
  \[ \Delta E = \frac{1 - r^2}{4}(u_1 - u_2)^2 \]

- Limiting cases
  \[ r = \begin{cases} 
    0 & \text{completely inelastic ($\Delta E = \text{max}$)} \\
    1 & \text{elastic ($\Delta E = 0$)}
  \end{cases} \]
Inelastic Collisions (any D)

- **Normal relative velocity reduced by**  $0 < r < 1$
  \[(v_1 - v_2) \cdot n = -r(u_1 - u_2) \cdot n\]

- **Momentum conservation**
  \[v_1 + v_2 = u_1 + u_2\]

- **Energy loss**
  \[\Delta E = \frac{1 - r^2}{4} [(u_1 - u_2) \cdot n]^2\]

- **Limiting cases**
  \[r = \begin{cases} 
  0 & \text{completely inelastic (} \Delta E = \text{max) } \\
  1 & \text{elastic (} \Delta E = 0) 
\end{cases}\]
Non-Maxwellian velocity distributions

1. Velocity distribution is isotropic

\[ f(v_x, v_y, v_z) = f(|v|) \]

2. No correlations between velocity components

\[ f(v_x, v_y, v_z) \neq f(v_x)f(v_y)f(v_z) \]

Only possibility is Maxwellian

\[ f(v_x, v_y, v_z) \neq C \exp \left( -\frac{v_x^2 + v_y^2 + v_z^2}{2T} \right) \]

Granular gases: collisions create correlations

JC Maxwell, Phil Trans Roy. Soc. 157 49 (1867)
The Boltzmann equation (1D)

- Collision rule (linear) \( r = 1 - 2p, \quad p + q = 1 \)
  \((u_1, u_2) \rightarrow (pu_1 + qu_2, pu_2 + qu_1)\)

- Boltzmann equation (nonlinear and nonlocal)

\[
\frac{\partial f(v)}{\partial t} = \int \int du_1 du_2 f(u_1)f(u_2)|u_1 - u_2|^\lambda \left[ \delta(v - pu_1 - qu_2) - \delta(v - u_2) \right]
\]

- Collision rate related to interaction potential

\[
U(r) \sim r^{-\gamma} \quad \lambda = 1 - 2 \frac{d - 1}{\gamma} = \begin{cases} 0 & \text{Maxwell molecules} \\ 1 & \text{Hard spheres} \end{cases}
\]

Theory: non-linear, non-local, dissipative
The collision rate

- **Collision rate**
  \[
  K(u_1, u_2) = |(u_1 - u_2) \cdot n|^\lambda
  \]

- **Collision rate related to interaction potential**
  \[
  U(r) \sim r^{-\gamma}
  \]
  \[
  \lambda = 1 - 2 \frac{d - 1}{\gamma} = \begin{cases}
  0 & \text{Maxwell molecules} \\
  1 & \text{Hard spheres}
  \end{cases}
  \]

- **Balance kinetic and potential energy**
  \[
  v^2 \sim r^{-\gamma} \quad \Rightarrow \quad r \sim v^{-2/\gamma}
  \]

- **Collisional cross-section**
  \[
  \sigma \sim v r^{d - 1} \quad \Rightarrow \quad \sigma \sim v^{1 - \frac{2}{\gamma}(d - 1)}
  \]
Extreme Statistics (1D)

- **Collision rule: arbitrary velocities**
  \[ (u_1, u_2) \rightarrow (pu_1 + qu_2, pu_2 + qu_1) \]

- **Large velocities: linear but nonlocal process**
  \[ v \xrightarrow{v^\lambda} (pv, qv) \]

- **High-energies: linear equation**
  \[
  \frac{\partial f(v)}{\partial t} = v^\lambda \left[ \frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right) - f(v) \right]
  \]

**Linear, nonlocal evolution equation**
Collision process: large velocities

\[ v \left( v \cos \theta \right)^{\lambda} \rightarrow (\alpha v, \beta v) \]

Stretching parameters related to impact angle

\[ \alpha = (1 - p) \cos \theta \quad \beta = \sqrt{1 - (1 - p^2) \cos^2 \theta} \]

Energy decreases, velocity magnitude increases

\[ \alpha^2 + \beta^2 \leq 1 \quad \alpha + \beta \geq 1 \]

Linear equation

\[
\frac{\partial f(v)}{\partial t} = \left\langle \left( v \cos \theta \right)^{\lambda} \left[ \frac{1}{\alpha^{d+\lambda}} f \left( \frac{v}{\alpha} \right) + \frac{1}{\beta^{d+\lambda}} f \left( \frac{v}{\beta} \right) - f(v) \right] \right\rangle
\]
Forced steady states: overpopulated tails

- **Energy injection**: thermal forcing *(at all scales)*

\[ \frac{dv}{dt} = \eta \]

- **Energy dissipation**: inelastic collision

\[ v \rightarrow (pv, qv) \]

- **Steady state equation**

\[
0 = D \frac{d^2 f(v)}{d^2 v} + \nu^\lambda \left[ \frac{1}{p^{1+\lambda}} f\left( \frac{v}{p} \right) + \frac{1}{q^{1+\lambda}} f\left( \frac{v}{q} \right) - f(v) \right]
\]

- **Stretched exponentials**

\[ f(v) \sim \exp \left( -v^{1+\lambda/2} \right) \]

T van Noije, M Ernst 97
Nonequilibrium velocity distributions

A  Mechanically vibrated beads
   F Rouyer & N Menon 00

B  Electrostatically driven powders
   I Aronson & J Olafsen 05

   - Gaussian core
   - Overpopulated tail

\[ f(v) \sim \exp\left(-|v|^{\delta}\right) \]
\[ 1 \leq \delta \leq 3/2 \]

- Kurtosis

\[ \kappa = \begin{cases} 
  3.55 & \text{theory} \\
  3.6 & \text{experiment} 
\end{cases} \]

Excellent agreement between theory and experiment

balance between collisional dissipation, energy injection from walls
Comparing kinetic theories

\( f(v) \)

- Black: Maxwell molecules, \( \xi = 1 \)
- Red: Hard spheres, \( \xi = 3/2 \)
- Blue: Maxwellian, \( \xi = 2 \)
- Purple: Experiment - Rouyer/Menon
Freely cooling states: temperature decay

- Energy loss \( \Delta T \sim (\Delta v)^2 \)
- Collision rate \( \Delta t \sim 1/(\Delta v)^\lambda \)
- Energy balance equation
  \[
  \frac{\Delta T}{\Delta t} \sim -(\Delta v)^{2+\lambda} \quad \Rightarrow \quad \frac{dT}{dt} \sim -T^{1+\lambda/2}
  \]
- Temperature decays, system comes to rest
  \[
  T \sim t^{-2/\lambda} \quad \Rightarrow \quad f(v) \rightarrow \delta(v)
  \]
Temperature decay: dimensional analysis

- Collision rate

\[ K \sim (\Delta v)^\lambda \sim v^\lambda \]

- Collision rate inversely proportional to time

\[ K \sim t^{-1} \quad \Rightarrow \quad v \sim t^{-1/\lambda} \]
Freely cooling states: similarity solutions

- **Linearized equation**
  \[
  \frac{\partial f(v)}{\partial t} = v^\lambda \left[ \frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right) - f(v) \right]
  \]

- **Similarity solution**
  \[
  f(v) \rightarrow t^{1/\lambda} \Phi(v t^{1/\lambda})
  \]

- **Steady state equation**
  \[
  \frac{1}{\lambda} [\Phi(z) + z \frac{d}{dz} \Phi(z)] = z^\lambda \left[ \frac{1}{p^{1+\lambda}} \Phi\left(\frac{z}{p}\right) + \frac{1}{q^{1+\lambda}} \Phi\left(\frac{z}{q}\right) - \Phi(z) \right]
  \]

- **Stretched exponentials (overpopulation)**
  \[
  \Phi(z) \sim \exp\left(-z^{\lambda}\right)
  \]

Esipov, Poeschel 97
An exact solution

- One-dimensional Maxwell molecules
- Fourier transform obeys a closed equation
  \[ F(k) = \int dv e^{ikv} f(v) \]
  \[ F(k) = F(pk)F(qk) \]
- Exponential solution
  \[ F(k) = \exp(-v_0|k|) \]
- Lorentzian velocity distribution
  \[ f(v) = \frac{1}{\pi v_0} \frac{1}{1 + (v/v_0)^2} \]

Nontrivial stationary states do exist!
Are there nontrivial steady states?

- **Stationary Boltzmann equation**

\[
0 = \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda \left[ \delta(v - pu_1 - qu_2) - \delta(v - u_2) \right]
\]

collision rate  gain  loss

**Naive answer: NO!**

- According to the energy balance equation

\[
\frac{dT}{dt} = -\Gamma
\]

- Dissipation rate is positive

\[
\Gamma > 0
\]
Cascade Dynamics (1D)

- **Collision rule:** arbitrary velocities
  \[(u_1, u_2) \rightarrow (pu_1 + qu_2, pu_2 + qu_1)\]

- **Large velocities:** linear but nonlocal process
  \[v \rightarrow (pv, qv)\]

- **High-energies:** linear equation
  \[f(v) = \frac{1}{p^{1+\lambda}}f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}}f\left(\frac{v}{q}\right)\]

- **Power-law tail**
  \[f(v) \sim v^{-2-\lambda}\]
Cascade Dynamics (any D)

- Collision process: large velocities
  \[ v \rightarrow (\alpha v, \beta v) \]

- Stretching parameters related to impact angle
  \[ \alpha = (1 - p) \cos \theta \quad \beta = \sqrt{1 - (1 - p^2) \cos^2 \theta} \]

- Energy decreases, velocity magnitude increases
  \[ \alpha^2 + \beta^2 \leq 1 \quad \alpha + \beta \geq 1 \]

- Steady state equation
  \[ f(v) = \left\langle \frac{1}{\alpha^{d+\lambda}} f \left( \frac{v}{\alpha} \right) + \frac{1}{\beta^{d+\lambda}} f \left( \frac{v}{\beta} \right) \right\rangle \]
Power-laws are generic

- Velocity distributions always have power-law tail
  \[ f(v) \sim v^{-\sigma} \]

- Exponent varies with parameters
  \[
  1 - 2F_1\left( \frac{d+\lambda-\sigma}{2}, \frac{\lambda+1}{2}, \frac{d+\lambda}{2}, 1-p^2 \right) \frac{\Gamma\left(\frac{\sigma-d+1}{2}\right)\Gamma\left(\frac{d+\lambda}{2}\right)}{\Gamma\left(\frac{\sigma}{2}\right)\Gamma\left(\frac{\lambda+1}{2}\right)} = (1 - p)^{\sigma-d-\lambda}
  \]

- Tight bounds \(1 \leq \sigma - d - \lambda \leq 2\)

- Elastic limit is singular \(\sigma \to d + 2 + \lambda\)

Dissipation rate always divergent
Energy finite or infinite
The characteristic exponent $\sigma$ varies with spatial dimension, collision rules.
Monte Carlo Simulations

- **Compact** initial distribution
- **Inject energy at very large velocity scales only**
- **Maintain constant total energy**
- **“Lottery”** implementation:
  - Keep track of total energy dissipated, $E_T$
  - With small rate, boost a particle by $E_T$

Excellent agreement between theory and simulation
Further confirmation

Maxwell molecules (1D, 2D)

Hard spheres (1D, 2D)

\[ N = 10^7 \]

\[ N = 10^5 \]

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Injection, cascade, dissipation

- Energy is injected at large velocity scales
- Energy cascades from large velocities to small velocities
- Energy dissipated at small velocity scales

Experimental realization?
Energetic particle “shot” into static medium

Energy balance
\[ \Gamma \sim \gamma V^2 \]
Energy balance

- Energy injection rate $\gamma$
- Energy injection scale $V$
- Typical velocity scale $v_0$
- Balance between energy injection and dissipation

$$\gamma \sim V^\lambda (V/v_0)^{d-\sigma}$$

- For “lottery” injection: injection scale diverges with injection rate

$$V \sim \begin{cases} \gamma^{1/(2-\lambda)} & \text{if } \sigma < d + 2 \\ \gamma^{-1/(\sigma-d-\lambda)} & \text{if } \sigma > d + 2 \end{cases}$$
Self-similar collapse

- Self-similar distribution
  \[ f(\nu, t) \sim \nu^{-\sigma} \Phi \left( \frac{\nu}{V(t)} \right) \]

- Cutoff velocity decays
  \[ V(t) \sim t^{-1/\lambda} \]

- Scaling function
  \[ \Phi(x) = \sum_{n=1}^{\infty} A_n \exp \left[ -(2^n x)^\lambda \right] \]

  \[ A_n = \prod_{k=1 \atop k \neq n}^{\infty} \frac{1}{1 - 2^\lambda(n-k)} \]

Hybrid between steady-state and time dependent state
A third family of solutions exists.
Conclusions

- New class of nonequilibrium stationary states
- Energy cascades from large to small velocities
- Power-law high-energy tail
- Energy input at large scales balances dissipation
- Associated similarity solutions exist as well
- Temperature insufficient to characterize velocities
- Experimental realization: requires a different driving mechanism
Outlook

- Spatially extended systems
- Spatial structures
- Polydisperse granular media
- Experimental realization

E. Ben-Naim, B. Machta, and J. Machta, cond-mat/0504187
Deviation from Maxwell-Boltzmann

- **Kurtosis** $\kappa$
  \[
  \kappa = \frac{\langle v^4 \rangle}{\langle v^2 \rangle^2} = 3 + \frac{18(1-r)^2(1+r)}{33 - 25r + 3r^2 - 3r^3}
  
- **Restitution coefficient** $r$
  \[
  \Delta E \propto (1 - r^2)(\Delta v)^2
  
1. Velocity distribution independent of driving strength
2. Stronger dissipation yields stronger deviation

Exact solution of Maxwell’s kinetic theory: thermal forcing balances dissipation

EB, Krapivsky 02