From grains to rods

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Talk, papers available from: http://cnls.lanl.gov/~ebn
Plan

I. Driven Grains: nonequilibrium steady states
II. Driven Rods: nonequilibrium phase transitions
I. Driven grains
“A shaken box of marbles”
Driven Granular Gas

- Vigorous driving
- Spatially uniform system
- Velocities change due to:
  - ★ Collisions: lose energy
  - ★ Forcing: gain energy
- Time irreversibility

Nonequilibrium steady state
Theoretical Model

Two independent competing processes

1. Inelastic collisions (nonlinear)

\[(v_1, v_2) \rightarrow \left( \frac{v_1 + v_2}{2}, \frac{v_1 + v_2}{2} \right)\]

2. Random uncorrelated white noise (linear)

\[
\frac{dv_j}{dt} = \eta_j(t) \quad \langle \eta_j(t)\eta_j(t') \rangle = 2D\delta(t - t')
\]

System reaches a nontrivial steady-state
Energy injection balances dissipation
Kinetic theory

- **Boltzmann equation**

\[
\frac{\partial P(v)}{\partial t} = D \frac{\partial^2 P(v)}{\partial v^2} + \int \int dv_1 dv_2 P(v_1)P(v_2)\delta \left(v - \frac{v_1 + v_2}{2}\right) - P(v)
\]

- **Fourier transform**

\[
F(k) = \int dv \, e^{ikv} P(v)
\]

- **Closed nonlinear and nonlocal equation**

\[
(1 + Dk^2)F(k) = F^2(k/2)
\]

- **Invariance**

\[
v \rightarrow v/\sqrt{D}
\]

Shape of distribution is independent of forcing strength
Infinite product solution

- Solution by iteration

\[
F(k) = \frac{1}{1 + Dk^2} F^2(k/2) = \frac{1}{1 + Dk^2} \frac{1}{1 + D(k/2)^2} F^4(k/4) = \ldots
\]

- Infinite product solution

\[
F(k) = \prod_{i=0}^{\infty} \left[ 1 + D(k/2^i)^2 \right]^{-2^i}
\]

- Exponential tail \( v \to \infty \)

\[
P(v) \propto \exp \left( -\frac{|v|}{\sqrt{D}} \right)
\]

- Also follows from

\[
D \frac{\partial^2 P(v)}{\partial v^2} = -P(v)
\]

Non-Maxwellian distribution/Overpopulated tails

Ernst 97
Cumulant solution

- **Steady-state equation**

\[ F(k)(1 + Dk^2) = F^2(k/2) \]

- **Take the logarithm**

\[ \psi(k) = \ln F(k) \]

\[ \psi(k) + \ln(1 + Dk^2) = 2\psi(k/2) \]

- **Cumulant solution**

\[ F(k) = \exp \left[ \sum_{n=1}^{\infty} \frac{\psi_n(-Dk^2)^n}{n} \right] \]

- **Generalized fluctuation-relaxation relations**

\[ \psi_n = \lambda_n^{-1} = \left[ 1 - 2^{1-n} \right]^{-1} \]

\[ \psi_n - \psi_n(\infty) \sim e^{-\lambda_n t} \]
Stationary Solutions

• Stationary solutions do exist!

\[ F(k) = F^2(k/2) \]

• Family of exponential solutions

\[ F(k) = \exp(-kv_0) \]

• Lorentz/Cauchy distribution

\[ P(v) = \frac{1}{\pi v_0} \frac{1}{1 + (v/v_0)^2} \]

How is a stationary solution consistent with energy dissipation?
Extreme Statistics

- Large velocities, cascade process
  \[ v \rightarrow \left( \frac{v_1}{2}, \frac{v_2}{2} \right) \]
  \[ (v_1, v_2) \rightarrow \left( \frac{v_1 + v_2}{2}, \frac{v_1 + v_2}{2} \right) \]

- Linear evolution equation
  \[ \frac{\partial P(v)}{\partial t} = 4P \left( \frac{v}{2} \right) - P(v) \]

- Steady-state: power-law distribution
  \[ P(v) \sim v^{-2} \]

- Divergent energy, divergent dissipation rate
Injection, Cascade, Dissipation

Experiment: rare, powerful energy injections

Lottery MC: award one particle all dissipated energy

Injection selects the typical scale!
I. Conclusions

• Nonequilibrium steady-states

• Energy pumped and dissipated by different mechanisms

• Overpopulation of high-energy tail with respect to equilibrium distribution
II. Driven rods
“A shaken dish of toothpicks”
Motivation

- Biology: molecular motors
- Ecology: flocking
- Granular matter: granular rods and chains
- Phase synchronization
The rod alignment model

- Each rod has an orientation $-\pi \leq \theta \leq \pi$

I. Alignment by pairwise interactions (nonlinear)

$$\left( \frac{\theta_1 + \theta_2}{2}, \frac{\theta_1 + \theta_2}{2} \right) \quad |\theta_1 - \theta_2| < \pi$$

$$\left( \frac{\theta_1 + \theta_2 + 2\pi}{2}, \frac{\theta_1 + \theta_2 + 2\pi}{2} \right) \quad |\theta_1 - \theta_2| > \pi$$

II. Diffusive wiggling (linear)

$$\frac{d\theta_j}{dt} = \eta_j(t)$$

$$\langle \eta_j(t)\eta_j(t') \rangle = 2D\delta(t - t')$$
Kinetic theory

- Nonlinear integro-differential equation
  \[
  \frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial \theta^2} + \int_{-\pi}^{\pi} d\phi \, P \left( \theta - \frac{\phi}{2} \right) P \left( \theta + \frac{\phi}{2} \right) - P.
  \]

- Fourier transform
  \[
  P_k = \langle e^{-ik\theta} \rangle = \int_{-\pi}^{\pi} d\theta e^{-ik\theta} P(\theta)
  \]
  \[
  P(\theta) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} P_k e^{ik\theta}
  \]

- Closed nonlinear equation
  \[
  (1 + Dk^2) P_k = \sum_{i+j=k} A_{i-j} P_i P_j
  \]

- Coupling constants
  \[
  A_q = \sin \frac{\pi q}{2} = \begin{cases} 
  1 & q = 0 \\
  0 & q = 2, 4, \cdots \\
  (-1)^{\frac{q-1}{2}} \frac{2}{\pi |q|} & q = 1, 3, \cdots
  \end{cases}
  \]
The order parameter

- **Lowest order Fourier mode**
  \[ R = |\langle e^{i\theta} \rangle| = |P_{-1}| \]

- **Probes the state of the system**

  \[
  R = \begin{cases} 
    0 & \text{disordered} \\
    0.4 & \text{partially ordered} \\
    1 & \text{perfectly ordered}
  \end{cases}
  \]

  - Disordered: \[ \begin{array}{c} \nwarrow \uparrow \downarrow \leftarrow \uparrow \end{array} \]
  - Partially ordered: \[ \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \end{array} \]
  - Perfectly ordered: \[ \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \end{array} \]
The Fourier Equation

- **Compact Form**

\[ P_k = \sum_{i+j=k} G_{i,j} P_i P_j \]
\[ \text{where } i \neq 0, j \neq 0 \]

- **Transformed coupling constants**

\[ G_{i,j} = \frac{A_{i-j}}{1+D(i+j)^2-2A_{i+j}} \]

- **Properties**

\[ G_{i,j} = G_{j,i} \]
\[ G_{i,j} = G_{-i,-j} \]
\[ G_{i,j} = 0, \quad \text{for } |i - j| = 2, 4, \ldots \]
Solution

- Repeated iterations (product of three modes)

\[ P_k = \sum_{i+j=k} \sum_{l+m=j} G_{i,j} G_{l,m} P_i P_l P_m. \]

- When \( k=2,4,8,\ldots \)

\[ P_2 = G_{1,1} P_1^2 \]
\[ P_4 = G_{2,2} P_2^2 = G_{2,2} G_{1,1} P_1^4 \]

- Generally

\[ P_3 = 2G_{1,2} P_1 P_2 + 2G_{-1,4} P_{-1} P_4 + \cdots \]
\[ = 2G_{1,2} G_{1,1} P_1^3 + 2G_{-1,4} G_{2,2} G_{1,1} P_1^4 P_{-1} + \cdots \]
Partition of Integers

- **Diagramatic solution**
  \[
P_k = R^k \sum_{n=0}^{\infty} p_{k,n} R^{2n}
\]

- **Partition**
  \[
k = 1 + 1 + \cdots + 1 + 1 - 1 - \cdots - 1.
\]

- **Partitions rules**
  \[
k = i + j
  
i \neq 0
  
j \neq 0
  
G_{i,j} \neq 0
\]

All modes expressed in terms of order parameter
The order parameter

• Infinite series solution

\[ R = R^k \sum_{n=0}^{\infty} p_{1,n} R^{2n} \]

• Landau theory

\[ R = \frac{C}{D_c - D} R^3 + \cdots \]

• Critical diffusion constant

\[ D_c = \frac{4}{\pi} - 1 \]

Close equation for order parameter
Nonequilibrium phase transition

- Critical diffusion constant \( D_c = \frac{4}{\pi} - 1 \)
- Subcritical: ordered phase \( R > 0 \)
- Supercritical: disordered phase \( R = 0 \)
- Critical behavior \( R \sim (D_c - D)^{1/2} \)
**Distribution of orientation**

- Fourier modes decay exponentially with $R$

$$P_k \sim R^k$$

- Small number of modes sufficient in practice

$$P(\theta) = \frac{1}{2\pi} \left[ 1 + 2R \cos \theta + 2G_{1,1}R^2 \cos(2\theta) + 4G_{1,2}G_{1,1}R^3 \cos(3\theta) + \cdots \right]$$
General alignment rates

- **Alignment rate**
  \[ K(|\theta_1 - \theta_2|) \]

- **Diagramatic solution holds**

- **Hard-rods**
  \[ K(\phi) \propto |\sin \phi| \quad D_c = \frac{1}{3} \]

- **Hard-spheres: system always disordered**
  \[ K(\phi) \propto |\phi| \]

Boltzmann equation can be solved!
Phase transition may or may not exist
Experiments
II. Conclusions

- Nonequilibrium phase transition
- Weak noise: ordered phase (nematic)
- Strong noise: disordered phase
- Solution relates to iterated partition of integers
- Only when Fourier spectrum is discrete: exact solution possible for arbitrary averaging rates