First Passage in High Dimensions

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Talk, publications available from: http://cnls.lanl.gov/~ebn

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Plan

I. First Passage 101
II. Ordering of Diffusing Particles
III. Mixing of Diffusing Particles
Part I: First Passage 101
First-Passage Processes

- Process by which a fluctuating quantity reaches a threshold for the first time.

- **First-passage probability:** for the random variable to reach the threshold as a function of time.

- **Total probability:** that threshold is ever reached. May or may not equal 1.

- **First-passage time:** the mean duration of the first-passage process. Can be finite or infinite.
Relevance

- Economics: specify stock orders, define bear/bull markets
- Politics: redistricting
- Geophysics: earthquakes, avalanches
- Biological Physics: transport in channels, translocation
- Polymer Physics: dynamics of knots
- Population dynamics: epidemic outbreaks

Connections

- Electrostatics
- Heat conduction
- Probability theory
- Quantum Mechanics
- Diffusion-limited aggregation
You versus casino. Fair coin. Your wealth = \( n \), Casino = \( N-n \)

Game ends with ruin. What is your winning probability \( E_n \)?

Winning probability satisfies discrete Laplace equation

\[
E_n = \frac{E_{n-1} + E_{n+1}}{2} \quad \nabla^2 E = 0
\]

Boundary conditions are crucial

\[
E_0 = 0 \quad \text{and} \quad E_N = 1
\]

Winning probability is proportional to your wealth

\[
E_n = \frac{n}{N}
\]

First-passage probability satisfies a simple equation

Feller 1968
**First-Passage Time**

- Average duration of game is $T_n$
- Duration satisfies discrete Poisson equation

\[
T_n = \frac{T_{n-1}}{2} + \frac{T_{n+1}}{2} + 1
\]

- Boundary conditions: $T_0 = T_N = 0$
- Duration is quadratic

\[
T_n = n(N - n)
\]

- Small wealth = short game, big wealth = long game

\[
T_n \sim \begin{cases} 
N & n = \mathcal{O}(1) \\
N^2 & n = \mathcal{O}(N) 
\end{cases}
\]

First-passage time satisfies a simple equation
Brute Force Approach

• Start with time-dependent diffusion equation

\[
\frac{\partial P(x, t)}{\partial t} = D \nabla^2 P(x, t)
\]

• Impose absorbing boundary conditions & initial conditions

\[P(x, t)\bigg|_{x=0} = P(x, t)\bigg|_{x=N} = 0 \quad \text{and} \quad P(x, t = 0) = \delta(x - n)\]

• Obtain full time-dependent solution

\[P(x, t) = \frac{2}{N} \sum_{l \geq 1} \sin \frac{l \pi x}{N} \sin \frac{l \pi n}{N} e^{-(l \pi)^2 D t / N^2}\]

• Integrate flux to calculate winning probability and duration

\[E_n = - \int_0^\infty dt \, D \frac{\partial P(x, t)}{\partial x} \bigg|_{x=N} \implies E_n = \frac{n}{N}\]

Lesson: focus on quantity of interest
Knots in Vibrated Granular Polymers

- Represent knot by three random walks (with exclusion)
- Solve gambler ruin problem in three dimensions

\[
\sigma_{\text{exp}} = 0.62 \pm 0.01
\]

\[
\sigma_{\text{theory}} = 0.63047
\]

EB, Daya, Vorobieff, Ecke, PRL (2001); Maryland 2001
Part II: Ordering of Diffusing Particles
The capture problem

- System: $N$ independent diffusing particles in one dimension
- What is the probability that original leader maintains the lead?
- $N$ Diffusing particles
  \[
  \frac{\partial \varphi_i(x, t)}{\partial t} = D \nabla^2 \varphi_i(x, t)
  \]
- Initial conditions
  \[
  x_N(0) < x_{N-1}(0) < \cdots < x_2(0) < x_1(0)
  \]
- Survival probability $S(t) = \text{probability "lamb" survives "lions" until } t$
- Independent of initial conditions, power-law asymptotic behavior
  \[
  S(t) \sim t^{-\beta} \quad \text{as} \quad t \to \infty
  \]
- Monte Carlo: nontrivial exponents that depend on $N$

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
N & 2 & 3 & 4 & 5 & 6 & 10 \\
\hline
\beta(N) & 1/2 & 3/4 & 0.913 & 1.032 & 1.11 & 1.37 \\
\hline
\end{array}
\]

No theoretical computation of exponents

Lebowitz 82
Fisher 84
Bramson 91
Redner 96
benAvraham 02
Grassberger 03
Two Particles

- We need the probability that two particles do not meet
- Map two one-dimensional walks onto one two-dimensional walk
- Space is divided into allowed and forbidden regions
- Boundary separating the two regions is absorbing
- Coordinate $x_1 - x_2$ performs one-dimensional random walk
- Survival probability decays as power-law
  \[ S_1(t) \sim t^{-1/2} \]
- In general, map $N$ one-dimensional walk onto one walk in $N$ dimension with complex boundary conditions
Order Statistics

- Generalize the capture problem: $S_m(t)$ is the probability that the leader does not fall below rank $m$ until time $t$.
- $S_1(t)$ is the probability that leader maintains the lead.
- $S_{N-1}(t)$ is the probability that leader never becomes laggard.
- Power-law asymptotic behavior is generic:
  \[ S_m(t) \sim t^{-\beta_m(N)} \]
- Spectrum of first-passage exponents:
  \[ \beta_1(N) > \beta_2(N) > \cdots > \beta_{N-1}(N) \]

Can’t solve the problem? Make it bigger!
Three Particles

- Diffusion in three dimensions; now, allowed regions are wedges

\[
\begin{array}{c|c|c}
 m = 2 & m = 1 & x_2 = x_3 \\
 312 & 132 & \\
 m = 3 & 321 & 123 \\
 231 & 213 & m = 1 \\
 m = 3 & 321 & m = 2 \\
 x_1 = x_3 & x_1 = x_2 &
\end{array}
\]

- Survival probability in wedge with opening angle $0 < \alpha < \pi$

\[S(t) \sim t^{-\pi/(4\alpha)}\]

- Survival probabilities decay as power-law with time

\[S_1 \sim t^{-3/4} \quad \text{and} \quad S_2 \sim t^{-3/8}\]

- Indeed, a family of nontrivial first-passage exponents

\[S_m \sim t^{-\beta_m} \quad \text{with} \quad \beta_1 > \beta_2 > \cdots > \beta_{N-1}\]

Large spectrum of first-passage exponents

Spitzer 58
Fisher 84
First Passage in a Wedge

• Survival probability obeys the diffusion equation
  \[
  \frac{\partial S(r, \theta, t)}{\partial t} = D \nabla^2 S(r, \theta, t)
  \]

• Focus on long-time limit
  \[
  S(r, \theta, t) \simeq \Phi(r, \theta) t^{-\beta}
  \]

• Amplitude obeys Laplace’s equation
  \[
  \nabla^2 \Phi(r, \theta) = 0
  \]

• Use dimensional analysis
  \[
  \Phi(r, \theta) \sim \left(\frac{r^2}{D}\right)^{\beta} \psi(\theta)
  \implies \quad \psi_{\theta\theta} + (2\beta)^2 \psi = 0
  \]

• Enforce boundary condition
  \[
  S|_{\theta=\alpha} = \Phi|_{\theta=\alpha} = \psi|_{\theta=\alpha}
  \]

• Lowest eigenvalue is the relevant one
  \[
  \psi_2(\theta) = \cos(2\beta \theta) \implies \beta = \frac{\pi}{4\alpha}
  \]
Monte Carlo Simulations

3 particles

\[ \beta_1 = 0.913 \]

\[ \beta_2 = 0.556 \]

\[ \beta_3 = 0.306 \]

as expected, there are 3 nontrivial exponents

confirm wedge theory results
Kinetics of First Passage in a Cone

- Repeat wedge calculation step by step
  \[ S(r, \theta, t) \sim \psi(\theta)(Dt/r^2)^{-\beta} \]

- Angular function obeys Poisson-like equation
  \[
  \frac{1}{(\sin \theta)^{d-2}} \frac{d}{d\theta} \left[ (\sin \theta)^{d-2} \frac{d\psi}{d\theta} \right] + 2\beta(2\beta + d - 2)\psi = 0
  \]

- Solution in terms of associated Legendre functions
  \[
  \psi_d(\theta) = \begin{cases} 
  (\sin \theta)^{-\delta} P_{2\beta+\delta}(\cos \theta) & d \text{ odd,} \\
  (\sin \theta)^{-\delta} Q_{2\beta+\delta}(\cos \theta) & d \text{ even}
  \end{cases}
  \]
  \[ \delta = \frac{d - 3}{2} \]

- Enforce boundary condition, choose \textit{lowest} eigenvalue
  \[
  P_{2\beta+\delta}(\cos \alpha) = 0 \quad d \text{ odd,}
  \]
  \[
  Q_{2\beta+\delta}(\cos \alpha) = 0 \quad d \text{ even.}
  \]

Exponent is nontrivial root of Legendre function
Additional Results

- Explicit results in 2d and 4d
  \[ \beta_2(\alpha) = \frac{\pi}{4\alpha} \quad \text{and} \quad \beta_4(\alpha) = \frac{\pi - \alpha}{2\alpha} \]

- Root of ordinary Legendre function in 3d
  \[ P_{2\beta}(\cos \alpha) = 0 \]

- Flat cone is equivalent to one-dimension
  \[ \beta_d(\alpha = \pi/2) = 1/2 \]

- First-passage time obeys Poisson’s equation
  \[ D \nabla^2 T(r, \theta) = -1 \]

- First-passage time (when finite)
  \[ T(r, \theta) = \frac{r^2}{2D} \left( \frac{\cos^2 \theta - \cos^2 \alpha}{d \cos^2 \alpha - 1} \right) \]
  \( \alpha < \cos^{-1}(1/\sqrt{d}) \)
High Dimensions

- Exponent varies sharply for opening angles near $\pi/2$
- Universal behavior in high dimensions
  $$\beta_d(\alpha) \to \beta(\sqrt{N} \cos \alpha)$$
- Scaling function is smallest root of parabolic cylinder function
  $$D_{2\beta}(y) = 0$$

Exponent is function of one scaling variable, not two
Asymptotic Analysis

- Limiting behavior of scaling function
  \[ \beta(y) \simeq \begin{cases} \sqrt{y^2/8\pi} \exp(-y^2/2) & y \to -\infty, \\ y^2/8 & y \to \infty. \end{cases} \]

- Thin cones: exponent diverges
  \[ \beta_d(\alpha) \simeq B_d \alpha^{-1} \quad \text{with} \quad J_\delta(2B_d) = 0 \]

- Wide cones: exponent vanishes when \( d \geq 3 \)
  \[ \beta_d(\alpha) \simeq A_d (\pi - \alpha)^{d-3} \quad \text{with} \quad A_d = \frac{1}{2} B \left( \frac{1}{2}, \frac{d-3}{2} \right) \]

- A needle is reached with certainty only when \( d < 3 \)

- Large dimensions
  \[ \beta_d(\alpha) \simeq \begin{cases} \frac{d}{4} \left( \frac{1}{\sin \alpha} - 1 \right) & \alpha < \pi/2, \\ C(\sin \alpha)^d & \alpha > \pi/2. \end{cases} \]
**Diffusion in High Dimensions**

- In general, map $N$ one-dimensional walk onto one walk in $N$ dimension with complex boundary conditions.
- There are $\binom{N}{2} = \frac{N(N-1)}{2}$ planes of the type $x_i = x_j$.
- These planes divide space into $N!$ “chambers”.
- Particle order is unique to each chamber.
- The absorbing boundary encloses multiple chambers.
- We do not know the shape of the allowed region.
- However, we do know the volume of the allowed region.
- Equilibrium distribution of particle order:

$$V_m = \frac{m}{N}$$
Equilibrium versus Nonequilibrium

- Diffusion is an ergodic process
- Wait long enough and initial order is completely forgotten
- Equilibrium distribution: each chamber has weight $P = 1/N$

First passage as a nonequilibrium process
Cone Approximation

- Fractional volume of allowed region given by equilibrium distribution of particle order
  \[ V_m(N) = \frac{m}{N} \]

- Replace allowed region with cone of same fractional volume
  \[ V(\alpha) = \frac{\int_0^\alpha d\theta (\sin \theta)^{N-3}}{\int_0^\pi d\theta (\sin \theta)^{N-3}} \]
  \[ d\Omega \propto \sin^{d-2} \theta \, d\theta \quad d = N - 1 \]

- Use analytically known exponent for first passage in cone
  \[ Q^{\gamma}_{2\beta+\gamma}(\cos \alpha) = 0 \quad N \text{ odd}, \quad \gamma = \frac{N - 4}{2} \]
  \[ P^{\gamma}_{2\beta+\gamma}(\cos \alpha) = 0 \quad N \text{ even.} \]

- Good approximation for four particles

<table>
<thead>
<tr>
<th>( m )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_m )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{3}{4} )</td>
</tr>
<tr>
<td>( \beta_{cone}^m )</td>
<td>0.888644</td>
<td>( \frac{1}{2} )</td>
<td>0.300754</td>
</tr>
<tr>
<td>( \beta_m )</td>
<td>0.913</td>
<td>0.556</td>
<td>0.306</td>
</tr>
</tbody>
</table>
Small Number of Particles

- By construction, cone approximation is exact for $N=3$
- Cone approximation gives a formal lower bound

Excellent, consistent approximation!
Very Large Number of Particles \( (N \to \infty) \)

- Equilibrium distribution is simple

\[ V_m = \frac{m}{N} \]

- Volume of cone is also given by error function

\[ V(\alpha, N) \to \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{-y}{\sqrt{2}} \right) \quad \text{with} \quad y = (\cos \alpha)\sqrt{N} \]

- First-passage exponent has the scaling form

\[ \beta_m(N) \to \beta(x) \quad \text{with} \quad x = m/N \]

- Scaling function is root of equation involving parabolic cylinder function

\[ D_{2\beta} \left( \sqrt{2} \text{erfc}^{-1}(2x) \right) = 0 \]

**Scaling law for scaling exponents!**
Simulation Results

Numerical simulation of diffusion in 10,000 dimensions!
Only 10 measurements confirm scaling function!
Cone approximation is asymptotically exact!
Scaling function converges quickly

Is spherical one as a limiting shape?
Small Number of Particles

<table>
<thead>
<tr>
<th>N</th>
<th>$\beta_1^{\text{cone}}$</th>
<th>$\beta_1$</th>
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<tbody>
<tr>
<td>3</td>
<td>3/4</td>
<td>3/4</td>
</tr>
<tr>
<td>4</td>
<td>0.888644</td>
<td>0.91</td>
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<tr>
<td>5</td>
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<td>7</td>
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<tr>
<td>10</td>
<td>1.258510</td>
<td>1.37</td>
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<table>
<thead>
<tr>
<th>N</th>
<th>$\beta_{N-1}^{\text{cone}}$</th>
<th>$\beta_{N-1}$</th>
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<tr>
<td>2</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>3</td>
<td>3/8</td>
<td>3/8</td>
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<tr>
<td>4</td>
<td>0.300754</td>
<td>0.306</td>
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<tr>
<td>5</td>
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<td>6</td>
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<tr>
<td>10</td>
<td>0.150221</td>
<td>0.165</td>
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</table>

Decent approximation for the exponents even for small number of particles
Extreme Exponents

- Extremal behavior of first-passage exponents

\[
\beta(x) \simeq \begin{cases} 
\frac{1}{4} \ln \frac{1}{2x} & x \to 0 \\
(1 - x) \ln \frac{1}{2(1-x)} & x \to 1 
\end{cases}
\]

- Probability leader never loses the lead (capture problem)

\[
\beta_1 \simeq \frac{1}{4} \ln N
\]

- Probability leader never becomes last (laggard problem)

\[
\beta_{N-1} \simeq \frac{1}{N} \ln N
\]

- Both agree with previous heuristic arguments

Extremal exponents can not be measured directly
Indirect measurement via exact scaling function

Krapivsky 02
Summary

• First-passage kinetics are rich
• Family of first-passage exponents
• Cone approximation gives good estimates for exponents
• Exponents follow a scaling behavior in high dimensions
• Cone approximation yields the exact scaling function
• Combine equilibrium distribution and geometry to obtain exact or approximate nonequilibrium behavior, namely, first-passage kinetics
Part III:
Mixing of Diffusing Particles
Diffusion in One Dimension

- **Mixing:** well-studied in fluids, granular media, not in diffusion

- **System:** $N$ independent random walks in one dimension

Strong Mixing

- trajectories cross many times

Poor Mixing

- trajectories rarely cross

How to quantify mixing of diffusing particles?
The Inversion Number

• Measures how “scrambled” a list of numbers is
• Used for ranking, sorting, recommending (books, songs, movies)
  – I rank: 1234, you rank 3142
  – There are three inversions: {1,3}, {2,3}, {2,4}
• Definition: The inversion number \( m \) equals the number of pairs that are inverted = out of sort
• Bounds:
  \[
  0 \leq m \leq \frac{N(N - 1)}{2}
  \]

McMahon 1913
Random Walks and Inversion Number

- **Initial conditions: particles are ordered**
  \[ x_1(0) < x_2(0) < \cdots < x_{N-1}(0) < x_N(0) \]

- **Each particle is an independent random walk**
  \[
  x \rightarrow \begin{cases}
  x - 1 & \text{with probability } 1/2 \\
  x + 1 & \text{with probability } 1/2 
  \end{cases}
  \]

- **Inversion number**
  \[
  m(t) = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \Theta(x_i(t) - x_j(t))
  \]

- **Strong mixing: large inversion number**
- **Weak mixing: small inversion number persists**

**Inversion number is a natural measure of mixing**
Equilibrium Distribution

• Diffusion is ergodic, order is completely random when \( t \to \infty \)
• Every permutation occurs with the same weight \( 1/N! \)
• Probability \( P_m(N) \) of inversion number \( m \) for \( N \) particles

\[
(P_0, P_1, \ldots, P_M) = \frac{1}{N!} \times \begin{cases} 
(1) & N = 1, \\
(1, 1) & N = 2, \\
(1, 2, 2, 1) & N = 3, \\
(1, 3, 5, 6, 5, 3, 1) & N = 4.
\end{cases}
\]

• Recursion equation

\[
P_m(N) = \frac{1}{N} \sum_{l=0}^{N-1} P_{m-l}(N-1)
\]

• Generating Function

\[
\sum_{m=0}^{M} P_m(N) s^m = \frac{1}{N!} \prod_{n=1}^{N} \left( 1 + s + s^2 + \cdots + s^{n-1} \right)
\]

Knuth 1998
Equilibrium Properties

- Average inversion number scales quadratically with $N$
  \[ \langle m \rangle = \frac{N(N - 1)}{4} \]

- Variance scales cubically with $N$
  \[ \sigma^2 = \frac{N(N - 1)(2N + 5)}{72} \]

- Asymptotic distribution is Gaussian
  \[ P_m(N) \simeq \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(m - \langle m \rangle)^2}{2\sigma^2} \right] \]

- Large fluctuations
  \[ m - N^2/4 \sim N^{3/2} \]
**Transient Behavior**

- Assume particles well mixed on a growing length scale
- Use equilibrium result for the sub-system \( \langle m \rangle / N \sim \ell \)
- Length scale must be diffusive \( \ell \sim \sqrt{t} \)

\[
\langle m(t) \rangle \sim N \sqrt{t} \quad \text{when} \quad t \ll N^2
\]
- Equilibrium behavior reached after a transient regime
- Nonequilibrium distribution is Gaussian as well
First-Passage Kinetics

- Survival probability $S_m(t)$ that inversion number $< m$ until time $t$

1. Probability there are no crossing

$$S_1(t) \sim t^{-N(N-1)/4}$$

Fisher 1984

2. Two-particles: coordinate $x_1 - x_2$ performs a random walk

$$S_1(t) \sim t^{-1/2}$$

- Map $N$ 1-dimensional walks to 1 walk in $N$ dimensions
  - Allowed region: inversion number $< m$
  - Forbidden region: inversion number $\geq m$

- Boundary is absorbing

Problem reduces to diffusion in $N$ dimensions in presence of complex absorbing boundary
Three Particles

- Diffusion in three dimensions; Allowed regions are wedges

\[
\begin{array}{c|c|c}
  m = 2 & m = 1 & \text{allowed regions} \\
  312 & 132 & x_2 = x_3 \\
  m = 3 & 123 & m = 0 \\
  321 & 213 & x_1 = x_2 \\
  m = 2 & m = 1 & V = 1/6 \\
  x_1 = x_3 & & V = 1/2 \\
  & & V = 5/6 \\
\end{array}
\]

- Survival probability in wedge with “fractional volume” 0 < V < 1

\[S(t) \sim t^{-1/(4V)}\]

- Survival probabilities decay as power-law with time

\[S_1 \sim t^{-3/2}, \quad S_2 \sim t^{-1/2}, \quad S_3 \sim t^{-3/10}\]

- In general, a series of nontrivial first-passage exponents

\[S_m \sim t^{-\beta_m} \quad \text{with} \quad \beta_1 > \beta_2 > \cdots > \beta_{N(N-1)/2}\]

Huge spectrum of first-passage exponents
Cone Approximation

- Fractional volume of allowed region given by equilibrium distribution of inversion number

\[ V_m(N) = \sum_{l=0}^{m-1} P_l(N) \]

- Replace allowed region with cone of same fractional volume

\[ V(\alpha) = \frac{\int_0^\alpha d\theta (\sin \theta)^{N-3}}{\int_0^\pi d\theta (\sin \theta)^{N-3}} \]

- Use analytically known exponent for first-passage in cone

\[ Q_{2\beta+\gamma}^\gamma(\cos \alpha) = 0 \quad N \text{ odd}, \quad \gamma = \frac{N - 4}{2} \]
\[ P_{2\beta+\gamma}^\gamma(\cos \alpha) = 0 \quad N \text{ even.} \]

- Good approximation for four particles

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<td>( \frac{3}{5} )</td>
<td>( \frac{23}{24} )</td>
</tr>
<tr>
<td>( \alpha_m )</td>
<td>0.41113</td>
<td>0.84106</td>
<td>1.31811</td>
<td>1.82347</td>
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<tr>
<td>( \beta_m^{\text{cone}} )</td>
<td>2.67100</td>
<td>1.17208</td>
<td>0.64975</td>
<td>0.39047</td>
<td>0.24517</td>
<td>0.14988</td>
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<tr>
<td>( \beta_m )</td>
<td>3.00</td>
<td>1.39</td>
<td>0.839</td>
<td>0.455</td>
<td>0.275</td>
<td>0.160</td>
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Small Number of Particles

- By construction, cone approximation is exact for \( N=3 \)
- Cone approximation produces close estimates for first-passage exponents when the number of particles is small
- Cone approximation gives a formal lower bound
Very Large Number of Particles ($N \to \infty$)

- Gaussian equilibrium distribution implies
  \[ V_m(N) \to \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{z}{\sqrt{2}} \right) \quad \text{with} \quad z = \frac{m - \langle m \rangle}{\sigma} \]

- Volume of cone is also given by error function
  \[ V(\alpha, N) \to \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{-y}{\sqrt{2}} \right) \quad \text{with} \quad y = (\cos \alpha) \sqrt{N} \]

- First-passage exponent has the scaling form
  \[ \beta_m(N) \to \beta(z) \quad \text{with} \quad z = \frac{m - \langle m \rangle}{\sigma} \]

- Scaling function is root of equation involving parabolic cylinder function
  \[ D_{2\beta}(-z) = 0 \]

Scaling exponents have scaling behavior!
Simulation Results

Cone approximation is asymptotically exact!
Summary

- Inversion number as a measure for mixing
- Distribution of inversion number is Gaussian
- First-passage kinetics are rich
- Large spectrum of first-passage exponents
- Cone approximation gives good estimates for exponents
- Exponents follow a scaling behavior
- Cone approximation yields the exact scaling function
- Use inversion number to quantify mixing in 2 & 3 dimensions
Counter example: cone is not limiting shape
Outlook

- Heterogeneous Diffusion
- Fractional Diffusion
- Accelerated Monte Carlo methods
- Scaling occurs in general
- Cone approach is not always asymptotically exact
- Geometric proof for exactness
- Limiting shapes in general
Publications

1. E. Ben-Naim,

2. E. Ben-Naim and P.L. Krapivsky,

3. E. Ben-Naim and P.L. Krapivsky,

4. T. Antal, E. Ben-Naim, and P.L. Krapivsky,