Extinction and Survival in Two-Species Annihilation

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Talk, publications available from: http://cnls.lanl.gov/~ebn

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Plan

Diffusion-controlled two-species annihilation
with finite number of particles

1. Equal populations
2. Fixed number difference
3. Equal concentrations
Diffusion-controlled two-species annihilation with finite number of particles

- Initial condition: uniform density in compact domain
- Number of majority & minority particles is $N_+ \& N_-$
- Total number of particles
  \[ N = N_+ + N_- \]
- Number difference is a conserved quantity
  \[ \Delta = N_+ - N_- \]
Main result (three dimensions)

- Average number of surviving majority particles is $M_+$
- Average number of surviving minority particles is $M_-$
- Conservation law implies majority never goes extinct
  $$M_+ - M_- = \Delta$$
- Equal populations
  $$M_+ \sim M_- \sim N^{1/3}$$
- Equal concentrations
  $$M_+ \sim N^{1/2} \quad \text{and} \quad M_- \sim N^{1/6}$$
Sufficiently small dimensions: extinction

- **Probability a random walk returns to origin**
  \[ P = 1 \text{ when } d \leq 2 \]

- **Separation between two random walks itself performs a random walk**

- **Two diffusing particles are guaranteed to meet**
  All minority particles eventually disappear

Above critical dimension: survival feasible

- **Probability a random walk at distance \( r \) returns to origin**
  \[ P \sim r^{-(d-2)} \text{ when } d > 2 \]

- **Two diffusing particles may or may not meet**
Uniform-density approximation

- Concentrations obey reaction-diffusion equation
  \[
  \frac{\partial c_-(\mathbf{r}, t)}{\partial t} = D \nabla^2 c_-(\mathbf{r}, t) - K c_-(\mathbf{r}, t) c_+(\mathbf{r}, t)
  \]

- Dimensionless form \( D = K = a = 1 \)

- Total number of particles obeys rate equation
  \[
  n_-(t) = \int d\mathbf{r} c_-(\mathbf{r}, t) \implies \frac{dn_-}{dt} = - \int d\mathbf{r} c_-(\mathbf{r}, t) c_+(\mathbf{r}, t)
  \]

- Two major simplifying assumptions
  1. Particles confined to volume \( V \)
  2. Spatial distribution remains uniform

- Closed equation for number of remaining particles
  \[
  \frac{dn_-}{dt} = - \frac{n_-n_+}{V}
  \]
Equal populations ($\Delta = 0$)

- Particles still inside initial-occupied domain

$$n_+ = n_- = n/2 \quad & \quad V \sim N \quad \Rightarrow \quad \frac{dn}{dt} = -\frac{n^2}{N}$$

- Mean-field like decay

$$n(t) \sim N t^{-1}$$

- Valid until particles exit initially-occupied domain

$$\ell^{3/2} \sim t^{3/2} \sim N \quad \Rightarrow \quad T \sim N^{2/3}$$

- Diffusion time scale gives number of particles

$$n(T) \sim N^{1/3}$$
Numerical simulations I

equal populations ( $\Delta = 0$)
Fixed number difference ($\Delta \neq 0$)

- Rate equation
  \[
  \frac{dn_-}{dt} = - \frac{n_-(n_- + \Delta)}{N} 
  \]

- Average number of minority particles
  \[
  n_-(t) = N_- \frac{\Delta}{N_- (e^{t\Delta/N} - 1) + \Delta} 
  \]

- Average number of surviving minority particles
  \[
  M_- \sim n_-(T) \implies M_- \sim N_- \frac{\Delta}{N_- (e^{\Delta/N^{1/3}} - 1) + \Delta} 
  \]

- Emergence of critical difference
  \[
  M_- \sim \begin{cases} 
  N^{1/3} & \Delta \ll N^{1/3} \\
  \Delta \exp(-c \Delta/N^{1/3}) & \Delta \gg N^{1/3} 
  \end{cases} 
  \]

Transition from extinction to survival
Finite-size scaling

- Number of surviving particles

\[ M_- \sim N_- \frac{\Delta}{N_-(e^{\Delta/N^{1/3}} - 1) + \Delta} \]

- Scaling laws for surviving number and critical number difference

\[ M_- \sim N^{1/3} \quad \text{and} \quad \Delta_c \sim N^{1/3} \]

- Universal scaling form for number of surviving particles

\[ M_-/N^{1/3} = G\left(\Delta/N^{1/3}\right) \]

- Scaling function

\[ G(x) = \frac{x}{e^{c}x - 1} \]

- Two regimes of behavior

\[ G(x) \sim \begin{cases} 
1 & x \ll 1 \quad \text{survival} \\
z e^{-c z} & z \gg 1 \quad \text{extinction}
\end{cases} \]

Critical difference fully characterizes the behavior
Numerical simulations II

finite-size scaling

\[ 1 + \frac{x}{G} \]

- \( N = 82,519 \)
- \( N = 816,577 \)
- \( N = 7,058,099 \)
The critical difference

• There is a critical number difference

\[ \Delta_c \sim N^{1/3} \]

• Subcritical difference: minority species survives

\[ M_- \sim N^{1/3} \quad \text{when} \quad \Delta \ll \Delta_c \]

• Supercritical difference: minority species becomes extinct

\[ M_- \sim \Delta \exp\left(-\frac{c\Delta}{N^{1/3}}\right) \quad \text{when} \quad \Delta \gg \Delta_c \]

• In particular, for typical difference (equal concentrations)

\[ M_- \sim N^{1/2} \exp\left(-\frac{cN^{1/6}}{}\right) \quad \text{when} \quad \Delta = bN^{1/2} \]

Number difference controls the behavior
Numerical simulations III

Typical difference ($\Delta = N^{1/2}$)

$$M_{-}/N^{1/2} \sim \exp(-cN^{1/6})$$

Theory “rescues” simulations!
Equal concentrations

- Massive imbalance, surviving majority population is large
  \[ \Delta \sim N^{1/2} \implies M_+ \sim N^{1/2} \]

- Number difference is normally distributed
  \[ P(\Delta) = (2\pi N)^{-1/2} \exp[-\Delta^2/(2N)] \]
  \[ \rightarrow \begin{cases} 
  N^{-1/2} & \Delta < N^{1/2} \\
  0 & \Delta > N^{1/2} 
\end{cases} \]

- Minority survives with tiny probability
  \[ \Delta < \Delta_c \text{ with probability } N^{-1/2} \times N^{1/3} \sim N^{-1/6} \]

- Minority goes extinct otherwise
  \[ \Delta > \Delta_c \text{ with probability } 1 - N^{-1/6} \]

- Average number of surviving minority particles
  \[ M_- \sim N^{-1/6} \times N^{1/3} \sim N^{1/6} \]

Lack of self-averaging, huge fluctuations
Two distinct scaling laws for majority and minority
Equal concentrations

- Number difference is normally distributed

\[ P(\Delta) = \left( \frac{1}{2\pi N} \right)^{1/2} \exp \left( -\frac{\Delta^2}{2N} \right) \]

- Separate supercritical and subcritical contributions

\[
M_- \sim \int_0^{N^{1/3}} d\Delta \left( \frac{1}{2\pi N} \right)^{1/2} \exp \left( -\frac{\Delta^2}{2N} \right) \times N^{1/3} \\
+ \int_{N^{1/3}}^{\infty} d\Delta \left( \frac{1}{2\pi N} \right)^{1/2} \exp \left( -\frac{\Delta^2}{2N} \right) \times \Delta \exp \left( -\frac{\Delta}{N^{1/3}} \right)
\]

- System is almost always supercritical (extinction)
- Rare subcritical cases dominate the behavior (survival)
Numerical simulations IV

equal concentrations

\[ \beta \sim N^{1/6} \]

\[ \text{Inset graph showing data points for } \beta \text{ vs. } N \]

\[ \text{Main graph showing data points for } \beta \text{ vs. } N \]
General spatial dimensions

- Critical difference, one-tier scaling law
  \[ \Delta_c \sim N^\delta \quad \delta = \frac{d - 2}{d} \]

- Majority population, two-tier scaling law
  \[ M_+ \sim N^{\beta_+} \quad \beta_+ = \begin{cases} \frac{1}{2} & d \leq 4 \\ \frac{d-2}{d} & 4 \leq d \end{cases} \]

- Minority population, three-tier scaling law
  \[ M_- \sim N^{\beta_-} \quad \beta_- = \begin{cases} 0 & d \leq \frac{8}{3} \\ \frac{3d-8}{2d} & \frac{8}{3} \leq d \leq 4 \\ \frac{d-2}{d} & 4 \leq d \end{cases} \]

Surviving minority population does not grow with \( N \) when \( d < \frac{8}{3} \)
Conclusions

- Diffusion-controlled two-species annihilation, starting with finite number of particles
- Finite number of particles escape annihilation
- Number difference controls the behavior
- Subcritical phase: minority species survives
- Supercritical phase: minority species goes extinct
- Equal concentrations: two distinct scaling laws for minority and majority populations
- Opposite to infinite systems: survival probability is enhanced as the dimension increases
- Exact analytical methods to treat finite number of particles