Extinction and Survival in Two-Species Annihilation

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Diffusion-controlled two-species annihilation with finite number of particles

- Initial condition: uniform density in compact domain
- Number of majority & minority particles is $N_+$ & $N_-$
- Total number of particles

$$N = N_+ + N_-$$

- Number difference is a conserved quantity

$$\Delta = N_+ - N_-$$
Main result (three dimensions): two scenarios for fate of system

- Average number of surviving majority & minority particles is $M_+ & M_-$.
- Some majority particles must survive $M_+ \geq \Delta$.
- Number difference controls the behavior.
- There is a critical number difference $\Delta_c \sim N^{1/3}$.
- Subcritical difference: minority species survives $M_+ \sim M_- \sim N^{1/3}$ when $\Delta \ll \Delta_c$.
- Supercritical difference: minority species goes extinct $M_+ \sim N^{1/2}$ and $M_- \sim N^{1/6}$ when $\Delta \gg \Delta_c$. 
Sufficiently small dimensions: extinction

- Probability a random walk returns to origin
  \[ P = 1 \quad \text{when} \quad d \leq 2 \]
- The separation between two random walks itself performs a random walk
- Two diffusing particles are guaranteed to meet
  All minority particles eventually disappear

Above critical dimension: survival feasible

- Probability a random walk at distance \( r \) returns to origin
  \[ P \sim r^{-(d-2)} \quad \text{when} \quad d > 2 \]
- Two diffusing particles may or may not meet
Uniform-density approximation

- Concentrations obey reaction-diffusion equation
  \[
  \frac{\partial c_- (r, t)}{\partial t} = D \nabla^2 c_- (r, t) - K c_- (r, t) c_+ (r, t)
  \]

- Dimensionless form \( D = K = a = 1 \)

- Total number of particles obeys rate equation
  \[
  n_- (t) = \int dr \ c_- (r, t) \quad \Rightarrow \quad \frac{dn_-}{dt} = - \int dr \ c_- (r, t) \ c_+ (r, t)
  \]

- Two major simplifying assumptions
  1. Particles confined to volume \( V \)
  2. Spatial distribution remains uniform

- Closed equation for number of remaining particles
  \[
  \frac{dn_-}{dt} = - \frac{n_- n_+}{V}
  \]
Equal populations \((\Delta = 0)\)

- Particles still inside initial-occupied domain
  \[ n_+ = n_- = n/2 \quad \& \quad V \sim N \quad \implies \quad \frac{dn}{dt} = -\frac{n^2}{N} \]

- Mean-field like decay
  \[ n(t) \sim N t^{-1} \]

- Valid until particles exit initially-occupied domain
  \[ \ell^{3/2} \sim t^{3/2} \sim N \quad \implies \quad T \sim N^{2/3} \]

- Diffusion time scale gives number of particles
  \[ n(T) \sim N^{1/3} \]

EB, Krapivsky 2016
Numerical simulations I

equal populations ( $\Delta = 0$)
Fixed number difference ( $\Delta \neq 0$)

- Rate equation
  \[
  \frac{dn_-}{dt} = -n_- (n_- + \Delta) \frac{N}{N}
  \]

- Average number of minority particles
  \[
  n_-(t) = N_- \Delta \frac{N_- (e^{t\Delta/N} - 1) + \Delta}{N_- (e^{t\Delta/N} - 1) + \Delta}
  \]

- Average number of surviving minority particles
  \[
  M_- \sim n_-(T) \quad \Rightarrow \quad M_- \sim N_- \Delta \frac{N_- (e^{\Delta/N^{1/3}} - 1) + \Delta}{N_- (e^{\Delta/N^{1/3}} - 1) + \Delta}
  \]

- Emergence of critical difference
  \[
  M_- \sim \begin{cases} 
  N^{1/3} & \Delta \ll N^{1/3} \\
  \Delta \exp(-c \Delta/N^{1/3}) & \Delta \gg N^{1/3}
  \end{cases}
  \]

Transition from extinction to survival
Finite-size scaling

- Number of surviving particles
  \[ M_- \sim N_- \frac{\Delta}{N_-(e^{\Delta/N^{1/3}} - 1) + \Delta} \]

- Scaling laws for surviving number and critical number difference
  \[ M_- \sim N^{1/3} \quad \text{and} \quad \Delta_c \sim N^{1/3} \]

- Universal scaling form for number of surviving particles
  \[ \frac{M_-}{N^{1/3}} = G \left( \frac{\Delta}{N^{1/3}} \right) \]

- Scaling function
  \[ G(x) = \frac{x}{e^c x - 1} \]

- Two regimes of behavior
  \[ G(x) \sim \begin{cases} 
    1 & x \ll 1 \\
    x e^{-c x} & x \gg 1 
  \end{cases} \]

Critical difference fully characterizes the behavior
Numerical simulations II
finite-size scaling

\[ 1 + \frac{x}{G} \]

- \( N = 82,519 \)
- \( N = 816,577 \)
- \( N = 7,058,099 \)
Equal concentrations

- Massive imbalance, surviving majority population is large
  $$\Delta \sim N^{1/2} \implies M_+ \sim N^{1/2}$$

- Number difference is normally distributed
  $$P(\Delta) = (2\pi N)^{-1/2} \exp[-\Delta^2/(2N)] \rightarrow \begin{cases} N^{-1/2} & \Delta < N^{1/2} \\ 0 & \Delta > N^{1/2} \end{cases}$$

- Minority survives with tiny probability
  $$\Delta < \Delta_c$$ with probability $$N^{-1/2} \times N^{1/3} \sim N^{-1/6}$$

- Minority goes extinct otherwise
  $$\Delta > \Delta_c$$ with probability $$1 - N^{-1/6}$$

- Average number of surviving minority particles
  $$M_- \sim N^{-1/6} \times N^{1/3} \sim N^{1/6}$$

Lack of self-averaging, huge fluctuations
Two distinct scaling laws for majority and minority
Numerical simulations III

equal concentrations \( (M_+ \sim \Delta \sim N^{1/2}) \)
General spatial dimensions

- Critical difference, one-tier scaling law
  \[ \Delta_c \sim N^{\delta} \quad \delta = \frac{d - 2}{d} \]

- Majority population, two-tier scaling law
  \[ M_+ \sim N^{\beta_+} \quad \beta_+ = \begin{cases} \frac{1}{2} & d \leq 4 \\ \frac{d-2}{d} & 4 \leq d \end{cases} \]

- Minority population, three-tier scaling law
  \[ M_- \sim N^{\beta_-} \quad \beta_- = \begin{cases} 0 & d \leq \frac{8}{3} \\ \frac{3d-8}{2d} & \frac{8}{3} \leq d \leq 4 \\ \frac{d-2}{d} & 4 \leq d \end{cases} \]

Surviving minority population does not grow with \( N \) when \( d < \frac{8}{3} \)
Conclusions

- Diffusion-controlled two-species annihilation
- Starting with finite number of particles
- Finite number of particles escape annihilation
- Number difference controls the behavior
- Subcritical phase: minority species survives
- Supercritical phase: minority species goes extinct
- Equal concentrations: two distinct scaling laws for minority and majority populations
- Opposite to infinite systems: survival probability is enhanced as the dimension increases
- Exact analytical methods to treat finite number of particles