Strong Transport in Weakly Disordered Systems

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May 15, 2009

Talk, paper available from: http://cnls.lanl.gov/~ebn

Asymptotic Methods for Dissipative Particle Systems, IPAM, May 20, 2009
Plan

1. Model: diffusion of interacting particles in disordered one-dimensional system
2. Motion of non-interacting particles in disorder
3. Motion on interacting particles in disorder
Disorder

- Disorder underlies many interesting phenomena
  - Localization (Anderson 58)
  - Glassiness & slow relaxation (Sherrington 75, Parisi 79)
  - Frustration (Ramirez 94)

- Influence of disorder:
  - Well understood for non-interacting particles
  - Open question for interacting particles (Lee & Ramakrishnan 85)

- De-localization of two interacting particles (Shepelyansky 93)

Interplay between disorder and particle interaction
Model System

- Infinite one-dimensional lattice
- Identical particles with concentration $c$
- Dynamics: particles move left and right with two rules:
  
  (i) **Disorder**: random, uncorrelated bias at each site

  \[
  p_+ = \begin{cases} 
  \frac{1}{2} + \epsilon & \text{with probability } = \frac{1}{2} \\
  \frac{1}{2} - \epsilon & \text{with probability } = \frac{1}{2}
  \end{cases}
  \]

  (ii) **Interaction**: via exclusion, one particle per site

  Minimal model with disorder and interaction
Particle Dynamics

- Pick a particle out of $N$ randomly
- Say particle is located at site $i$.

(i) Disorder: site dependent, governs motion
- With probability $p_+(i)$ move to the right by one site
- With probability $p_-(i) = 1 - p_+(i)$ move to the left one site

(ii) Interaction: via exclusion
- Accept the move if new site is vacant
- Reject the move if new site is occupied

- Augment time by $1/N$
Parameters

- Two parameters: concentration $c$, disorder strength $\epsilon$
- Generalizes two “seminal” diffusion processes:
  1. Sinai Diffusion: no interaction, $c \to 0$  
     - Sinai 82
  2. Single-File Diffusion: no disorder, $\epsilon \to 0$  
     - Levitt 73

(i) Disorder is small

$$\epsilon \ll 1$$

(ii) Concentration is finite

$$c = \frac{1}{2}$$
One Question

• Displacement of a particle \( x \)

• No overall bias, average displacement vanishes

\[ \langle x \rangle = 0 \]

• How does the variance grow with time?

\[ \sigma^2 = \langle x^2 \rangle = ? \]
2. Non-interacting Particles
Non-interacting particles

- Particle is trapped in a stochastic potential well
  \[ U(x) = \sum_{i=1}^{x} [p_+(i) - p_-(i)] \]

- Potential well is a random walk
  \[ U \sim \epsilon \sqrt{x} \]

- Escape time is exponential with depth of well
  \[ t \sim e^{U} \sim e^{\epsilon \sqrt{x}} \]

- Logarithmically slow displacement
  \[ x \sim \epsilon^{-2} (\ln t)^2 \]

Sinai 83
Distribution of Displacements

- Scaled displacement
  \[ \xi = \frac{x}{(\ln t)^2} \]

- Distribution is exactly known
  \[ F(\xi) = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \left( n + \frac{1}{2} \right)^{-1} \exp \left[ -\pi^2 |\xi| \left( n + \frac{1}{2} \right)^2 \right] \]

- Non-gaussian statistics
  \[ F(\xi) \sim \exp \left[ - \text{const.} \times |\xi| \right] \]

Golosov 84
Kesten 86
Early time: random walk

- Ignore biases
  \[ \epsilon = 0 \]

- In each step
  \[ \langle x \rangle = 0 \]
  \[ \langle x^2 \rangle = 1 \]

- In \( t \) steps: average and variance are additive
  \[ \langle x \rangle = 0 \]
  \[ \langle x^2 \rangle = t \]

- Purely diffusive motion
  \[ \sigma = t^{1/2} \]
Two time regimes

- When disorder is small, there are two time regimes
- Early times: disorder is irrelevant, simple diffusion
- Late times: disorder is relevant, particle trapped
- Crossover obtained by matching two behaviors

\[ \sigma \sim \begin{cases} 
  t^{1/2} & t \ll \epsilon^{-4}, \\
  \epsilon^{-2} (\ln t)^2 & t \gg \epsilon^{-4}.
\end{cases} \]

Without particle interactions: disorder slows particles down
Monotonic dependence on disorder strength: stronger disorder implies smaller displacement.
3. Interacting Particles
Early times

- Disorder is irrelevant, problem reduces to single file diffusion or simple exclusion process
- Particles motion is sub-diffusive
  \[ \sigma \sim t^{1/4} \]
- Observed in colloidal rings and biological channels

Exclusion hinders motion of particles

Harris 63
Levitt 73
Alexander 78
van beijeren 83
Percus 85

Bechinger 00
Lin 02
Heuristic Derivation

- **Dense limit**
  
  \[ c \to 1 \]

- **Particles move by exchanging position with vacancies**
  
  \[ x = N_+ - N_- \]

- **Excess vacancies**
  
  \[ |N_+ - N_-| \sim N^{1/2} \]

- **Total number of vacancies over the diffusive length** \[ t^{1/2} \]
  
  \[ N \sim (1 - c)t^{1/2} \]

- **Displacement**
  
  \[ x \sim t^{1/4} \]
Random Velocity Field

Magnitude of velocity diminishes with length
Local biases lead to directed motion

bouchaud 90
Intermediate times

- Local biases exist, cause directed motion
- Particle visits $\sigma = n_+ + n_- \text{ distinct sites in time } t$
- Disorder is random, so there is a diffusive excess
  $$\Delta = |n_+ - n_-| \sim \sigma^{1/2}$$
- Local drift velocity is proportional to excess
  $$\nu \sim \epsilon \Delta/\sigma \quad \Rightarrow \quad \nu \sim \epsilon \sigma^{-1/2}$$
- The displacement is super-diffusive
  $$\sigma \sim \nu t \sim \epsilon t \sigma^{-1/2} \quad \Rightarrow \quad \sigma \sim (\epsilon t)^{2/3}$$

With particle interactions: disorder speeds particles up!
Late times

- Interaction is irrelevant, problem reduces to Sinai diffusion
- The exponential escape time is dominant
- Imagine particles lines up to exit the cage
  \[ t \sim e^U \text{ replaced by } t \sim xe^U \]
- Particles motion remains logarithmically slow
  \[ \sigma \sim \epsilon^{-2}(\ln t)^2 \]

Ultimate asymptotic behavior: particles motion is logarithmic slow
Three time regimes

- **Early times:** interaction is relevant, sub-diffusion
- **Intermediate times:** disorder & interaction both relevant, super-diffusion
- **Late times:** disorder relevant, caging

\[ \sigma \sim \begin{cases} 
  t^{1/4} & t \ll \epsilon^{-8/5}, \\
  (\epsilon t)^{2/3} & \epsilon^{-8/5} \ll t \ll \epsilon^{-4}, \\
  \epsilon^{-2} (\ln t)^2 & t \gg \epsilon^{-4}.
\end{cases} \]

**Small disorder:** mobility is strong over a long period
Can we ignore the cage at intermediate times?

- Of course, the hopping time is of order one
- The escape time is appreciable when
  \[ t \sim \exp(U) \Rightarrow U \gg 1 \Rightarrow \epsilon \sqrt{x} \gg 1 \]
- The cage is relevant only at late times
  \[ x \gg \epsilon^{-2} \]

Yes, we can (ignore the cage)!
Heuristic argument does not utilize particle interactions!

- Therefore, super-diffusive transport must be relevant for non-interacting particles.
- However, the diffusive transport overwhelms the super-diffusive transport

\[
t^{1/2} \gg (\epsilon t)^{2/3} \quad \text{for} \quad t \ll \epsilon^{-4}
\]
Early and intermediate time behavior for a weak disorder

![Graph showing early and intermediate time behavior](image)

- slope = \( \frac{2}{3} \)
- weak disorder
- no disorder

Monte Carlo number of particles: \( N = 4 \times 10^5 \)

Qualitative and quantitative agreement with scaling theory
Early and intermediate time behavior for two different weak disorders

Universal scaling function for the displacement
Late time behavior for a moderate disorder

Suggests that interaction becomes irrelevant

$\epsilon = 0.1$
1. Transport is enhanced at all disorder strengths
2. Displacement is not monotonic with disorder
3. Eventually, no-disorder catches up
4. But why is the crossover time so large?
Giant crossover time

• Compare ultimate asymptotic behaviors with interaction

• Without disorder

\[ \sigma \sim t^{1/4} \]

• With small disorder

\[ \sigma \sim \epsilon^{-2} (\ln t)^2 \]

• The crossover time is astronomical

\[ t \sim \epsilon^{-8} \]

In practice, small disorder generates stronger transport in an interacting particle system
Generalizations

✓ Different concentrations
✓ Disorder with variable strengths
• Synchronous dynamics = parallel updates

Qualitative behavior appears to be robust
Summary

• Without interactions: disorder slows particles down

• With interactions: disorder speeds particles up, at least for a very long time
  - Early times: sub-diffusive displacements
  - Intermediate times: super-diffusive displacement
  - Late times: logarithmically slow displacement

• Intricate interplay between interaction and disorder
Outlook

• Beyond scaling theory: a mathematical theory
• Distribution of displacements
• Different types of disorder
• Periodic modulations
• Self-averaging?
• Experiments: colloids, biology, microfluids
• Disorder as a mechanism to control transport in matter
Experiments

\[ \sigma \sim t^\alpha \quad \alpha > 0.6 \]

Modulated (irregular) quasi-1D colloidal channel

Coupier, Saint Jean, Guthman 07