Energy Cascades in Granular Gases

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talk, papers available: http://cnls.lanl.gov/~ebn
1. Granular gases in nature
2. Nonequilibrium distributions
3. Driven steady-states
4. Cascade dynamics and stationary states
5. Associated time dependent states
6. Conclusions & outlook
Frozen granular gases

Saturn’s rings  snow avalanche

Christoph Hormann  Swiss institute for snow and avalanche research
Filaments in granular gases

X Nie, S Chen, EB 02
Energy dissipation in granular matter

- Responsible for collective phenomena
  - Clustering  I Goldhirsch, G Zanetti 93
  - Hydrodynamic instabilities  E Khain, B Meerson 04
  - Pattern formation  P Umbanhower, H Swinney 96

- Anomalous statistical mechanics
  - No energy equipartition  R Wildman, D Parker 02
  - Nonequilibrium energy distributions

\[ P(E) \neq \exp \left( -\frac{E}{kT} \right) \]
Experiments

- **Friction**
  - D Blair, A Kudrolli 01

- **Rotation**
  - K Feitosa, N Menon 04

- **Driving strength**
  - W Losert, J Gollub 98

- **Dimensionality**
  - J Urbach & Olafsen 98

- **Boundary**
  - J van Zon, H Swinney 04

- **Fluid drag**
  - K Kohlstedt, I Aronson, EB 05

- **Long range interactions**
  - D Blair, A Kudrolli 01; W Losert 02
  - K Kohlstedt, J Olafsen, EB 05

- **Substrate**
  - G Baxter, J Olafsen 04

Deviations from equilibrium distribution
Driven Granular gas

- Vigorous driving
- Spatially uniform system
- Particles undergo binary collisions
- Velocities change due to
  1. **Collisions**: lose energy
  2. **Forcing**: gain energy

- What is the typical velocity (granular “temperature”)?
  \[ T = \langle v^2 \rangle \]
- What is the velocity distribution?
  \[ f(v) \]
Non-Maxwellian velocity distributions

1. Velocity distribution is isotropic

\[ f(v_x, v_y, v_z) = f(|v|) \]

2. No correlations between velocity components

\[ f(v_x, v_y, v_z) \neq f(v_x)f(v_y)f(v_z) \]

Only possibility is Maxwellian

\[ f(v_x, v_y, v_z) \neq C \exp \left( -\frac{v_x^2 + v_y^2 + v_z^2}{2T} \right) \]

Granular gases: collisions create correlations
Kurtosis $\kappa$

$$\kappa = \frac{\langle v^4 \rangle}{\langle v^2 \rangle^2}$$

$$\kappa = 3 + \frac{18(1 - r)^2(1 + r)}{33 - 25r + 3r^2 - 3r^3}$$

Restitution coefficient $r$

$$\Delta E \propto \left(1 - r^2\right) (\Delta v)^2$$

1. Velocity distribution independent of driving strength
2. Stronger dissipation yields stronger deviation

Exact solution of Maxwell’s kinetic theory: thermal forcing balances dissipation

EB, Krapivsky 02
Nonequilibrium velocity distributions

A  Mechanically vibrated beads
   F Rouyer & N Menon 00

B  Electrostatically driven powders
   I Aronson & J Olafsen 05

- **Gaussian core**
- **Overpopulated tail**
  \[ f(v) \sim \exp(-|v|^\delta) \]
  \[ 1 \leq \delta \leq 3/2 \]
- **Kurtosis**
  \[ \kappa = \begin{cases} 3.55 & \text{theory} \\ 3.6 & \text{experiment} \end{cases} \]

Excellent agreement between theory and experiment

balance between collisional dissipation, energy injection from walls
Inelastic Collisions

- **Relative velocity reduced by** \( 0 < r < 1 \)
  \[ v_1 - v_2 = -r(u_1 - u_2) \]
- **Momentum is conserved**
  \[ v_1 + v_2 = u_1 + u_2 \]
- **Energy is dissipated**
  \[ \Delta E \propto (1 - r)(\Delta v)^2 \]
- **Limiting cases**
  \[ r = \begin{cases} 
 0 & \text{completely inelastic } (\Delta E = \text{max}) \\
 1 & \text{elastic } (\Delta E = 0) 
\end{cases} \]
Time dependent states

- Energy loss: $\Delta T \sim (\Delta v)^2$
- Collision rate: $\Delta t \sim 1/(\Delta v)^\lambda$
- Energy balance equation:
  \[
  \frac{\Delta T}{\Delta t} \sim - (\Delta v)^{2+\lambda} \quad \Rightarrow \quad \frac{dT}{dt} \sim -T^{1+\lambda/2}
  \]
- Temperature decays, system comes to rest:
  \[
  T \sim t^{-2/\lambda} \quad \Rightarrow \quad f(v) \to \delta(v)
  \]

Trivial steady-state

Haff, JFM 1982
- **Collision rule (linear)** \( r = 1 - 2p, \quad p + q = 1 \)

\[ (u_1, u_2) \rightarrow (pu_1 + qu_2, pu_2 + qu_1) \]

- **Boltzmann equation (nonlinear and nonlocal)**

\[
\frac{\partial P(v)}{\partial t} = \iint du_1 du_2 f(u_1)f(u_2)|u_1-u_2|^{\lambda} \left[ \delta(v-pu_1-qu_2) - \delta(v-u_2) \right]
\]

Collision rate \quad Gain \quad Loss

- **Collision rate related to interaction potential**

\[
U(r) \sim r^{-\gamma} \quad \lambda = 1 - 2 \frac{d-1}{\gamma} = \begin{cases} 0 & \text{Maxwell molecules} \\ 1 & \text{Hard spheres} \end{cases}
\]

Theory: non-linear, non-local, dissipative
Are there nontrivial steady states?

- Stationary Boltzmann equation

\[ 0 = \int \int du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)] \]

Naive answer: NO!

- According to the energy balance equation

\[ \frac{dT}{dt} = -\Gamma \]

- Dissipation rate is positive

\[ \Gamma > 0 \]
An exact solution

- One-dimensional Maxwell molecules
- Fourier transform obeys a closed equation
  \[ F(k) = \int dv \, e^{ikv} f(v) \]
  \[ F(k) = F(pk) F(qk) \]
- Exponential solution
  \[ F(k) = \exp(-v_0 |k|) \]
- Lorentzian velocity distribution
  \[ f(v) = \frac{1}{\pi v_0} \frac{1}{1 + (v/v_0)^2} \]

Nontrivial stationary states do exist!
Cascade Dynamics (1D)

- **Collision rule:** arbitrary velocities
  \[(u_1, u_2) \rightarrow (pu_1 + qu_2, pu_2 + qu_1)\]

- **Large velocities:** linear but nonlocal process
  \[v \rightarrow (pv, qv)\]

- **High-energies:** linear equation
  \[f(v) = \frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right)\]

- **Power-law tail**
  \[f(v) \sim v^{-2-\lambda}\]
Collision process: large velocities

\[ \nu \rightarrow (\alpha \nu, \beta \nu) \]

Stretching parameters related to impact angle

\[ \alpha = (1 - p) \cos \theta \quad \beta = \sqrt{1 - (1 - p^2) \cos^2 \theta} \]

Energy decreases, velocity magnitude increases

\[ \alpha^2 + \beta^2 \geq 1 \quad \alpha + \beta \leq 1 \]

Steady state equation

\[ f(\nu) = \left\langle \frac{1}{\alpha^{d+\lambda}} f \left( \frac{\nu}{\alpha} \right) + \frac{1}{\beta^{d+\lambda}} f \left( \frac{\nu}{\beta} \right) \right\rangle \]
Power-laws are generic

- Velocity distributions always have power-law tail
  \[ f(v) \sim v^{-\sigma} \]

- Exponent varies with parameters
  \[ 1 - 2F_1 \left( \frac{d+\lambda-\sigma}{2}, \frac{\lambda+1}{2}, \frac{d+\lambda}{2}, 1-p^2 \right) = \frac{\Gamma\left(\frac{\sigma-d+1}{2}\right)\Gamma\left(\frac{d+\lambda}{2}\right)}{\Gamma\left(\frac{\sigma}{2}\right)\Gamma\left(\frac{\lambda+1}{2}\right)} \]

- Tight bounds \( 1 \leq \sigma - d - \lambda \leq 2 \)

- Elastic limit is singular \( \sigma \to d+2+\lambda \)

Dissipation rate always divergent
Energy finite or infinite
The characteristic exponent $\sigma$ varies with spatial dimension, collision rules.
Monte Carlo Simulations

- **Compact** initial distribution
- Inject energy at very large velocity scales only
- Maintain constant total energy
- "Lottery" implementation:
  - Keep track of total energy dissipated, $E_T$
  - With small rate, boost a particle by $E_T$

Excellent agreement between theory and simulation
Further confirmation

Maxwell molecules (1D, 2D)  

Hard spheres (1D, 2D)

\[ N = 10^7 \]

\[ N = 10^5 \]

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Injection, cascade, dissipation

- Energy is injected at large velocity scales
- Energy cascades from large velocities to small velocities
- Energy dissipated at small velocity scales

Experimental realization?
Energetic particle “shot” into static medium

Energy balance
\[ \Gamma \sim \gamma V^2 \]
Energy balance

- Energy injection rate $\gamma$
- Energy injection scale $V$
- Typical velocity scale $v_0$
- Balance between energy injection and dissipation

\[ \gamma \sim V^\lambda (V/v_0)^{d-\sigma} \]

- For “lottery” injection: injection scale diverges with injection rate

\[ V \sim \begin{cases} 
\gamma^{-1/(2-\lambda)} & \sigma < d + 2 \\
\gamma^{-1/(\sigma-d-\lambda)} & \sigma > d + 2 
\end{cases} \]
Traditional forcing: Injection, dissipation

- **Energy injection:** thermal forcing *(at all scales)*
  \[ \frac{d v}{d t} = \eta \]

- **Energy dissipation:** inelastic collision
  \[ v \rightarrow (p v, q v) \]

- **Steady state equation**
  \[ 0 = D \frac{d^2 f(v)}{d^2 v} + v^\lambda \left[ \frac{1}{p^{1+\lambda}} f \left( \frac{v}{p} \right) + \frac{1}{q^{1+\lambda}} f \left( \frac{v}{q} \right) - f(v) \right] \]

- **Stretched exponentials**
  \[ f(v) \sim \exp \left( -v^{1+\lambda/2} \right) \]

T van Noije, M Ernst 97
Self-similar collapse

- Self-similar distribution
  \[ f(\nu, t) \sim \nu^{-\sigma} \Phi \left( \frac{\nu}{V(t)} \right) \]

- Cutoff velocity decays
  \[ V(t) \sim t^{-1/\lambda} \]

- Scaling function
  \[ \Phi(x) = \sum_{n=1}^{\infty} A_n \exp \left[ -(2^nx)^\lambda \right] \]
  \[ A_n = \prod_{\substack{k=1 \atop k \neq n}}^{\infty} \frac{1}{1 - 2^\lambda(n-k)} \]

Hybrid between steady-state and time dependent state
A third family of solutions exists

Numerical confirmation

Velocity distribution

Scaling function
Conclusions

- New class of nonequilibrium stationary states
- Energy cascades from large to small velocities
- Power-law high-energy tail
- Energy input at large scales balances dissipation
- Associated similarity solutions exist as well
- Temperature insufficient to characterize velocities
- Experimental realization: requires a different driving mechanism
Outlook

- Spatially extended systems
- Spatial structures
- Polydisperses granular media
- Experimental realization

E. Ben-Naim and J. Machta, PRL 91 (2005) cond-mat/0411473