Bifurcations and Patterns in Opinion Dynamics

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Thanks
Paul Krapivsky and Sidney Redner (Boston)

http://cnls.lanl.gov/~ebn
1. Motivation: modeling social systems
2. Continuous opinions: simulations, scaling
3. Discrete opinions: general features, theory
4. Pattern selection: linear stability analysis
5. Extensions: initial conditions, 2D, noise
Modeling social dynamics

- Goal: predictive models of human opinions
- Relevance: politics, economics, consumer, sports

Questions

- Are "physics concepts useful?"
- Are human interactions predictable?

This should help

- Large data sets available
- Large number of humans $N \sim 10^9$
- Human opinions are quantitative
Quantifying opinions
Humans interact, opinions evolve

tendency to reach consensus?

<table>
<thead>
<tr>
<th>Rank</th>
<th>AP Top 25</th>
<th>USA Today/ESPN</th>
</tr>
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<td>1.</td>
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The Compromise Process

- Opinion measured by continuum variable
  \[- \Delta < x < \Delta\]
- Compromise: reached via pairwise interactions
  \[(x_1, x_2) \rightarrow \left( \frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2} \right)\]
- Conviction: restricted interaction range
  \[|x_1 - x_2| < 1\]
- Minimal, one parameter model
- Mimics competition between compromise and conviction

Weisbuch 2001
Problem

- **Given initial distribution**

\[
P_0(x) = \begin{cases} 
1 & |x| < \Delta \\
0 & |x| > \Delta 
\end{cases}
\]

- **Find final distribution**

\[
P_\infty(x) = ?
\]

- **Multitude of final states**

\[
P_\infty(x) = \sum_{i=1}^{N} m_i \delta(x - x_i) \quad |x_i - x_j| > 1
\]

- **Dynamics selects one (deterministically)**

Multiple localized clusters (parties)
Numerical integration of probability distribution

Kinetic theory: nonlinear rate equations

\[ \frac{\partial}{\partial t} P(x, t) = \int \int d\xi_1 d\xi_2 P(x_1, t) P(x_2, t) \left[ 2\delta(x - (x_1 + x_2)/2) - \delta(x - x_1) - \delta(x - x_2) \right] \]

Direct simulation of stochastic process
Rise and fall of central party

\[ 0 < \Delta < 1.871 \quad \text{and} \quad 1.871 < \Delta < 2.724 \]

Central party may or may not exist!
Reemergence of central party

\[ 2.724 < \Delta < 4.079 \]

\[ 4.079 < \Delta < 4.956 \]
Hidden clusters

Tiny parties (mass $<10^{-3}$), extremists
Bifurcations and Patterns

![Graph showing bifurcations and patterns](image)

- Lines represent different categories:
  - Blue: Major
  - Central
  - Red: Minor

Axes:
- **X-axis**: \( \Delta \)
- **Y-axis**: \( X \)

Numbers 1, 2, and 3 indicate specific points or regions on the graph.
Self-similar structure, universality

- **Periodic sequence of bifurcations**
  1. Nucleation of minor cluster branch
  2. Nucleation of major cluster branch
  3. Nucleation of central cluster

- **Alternating major-minor pattern**

- **Clusters are equally spaced**

- **Period gives major cluster mass, separation**

\[ x(\Delta) = x(\Delta + L) \quad L = 2.155 \]
How many political parties?

- Data: CIA world factbook 2002
- 120 countries with multi-party parliaments
- Average = 5.8, standard deviation = 2.9
Cluster mass

- Masses are periodic
  \[ m(\Delta) = m(\Delta + L) \]
- Major mass
  \[ M \rightarrow L = 2.155 \]
- Minor mass
  \[ m \rightarrow 3 \times 10^{-4} \]
Scaling near bifurcation points

- Minor mass vanishes

\[ m \sim (\Delta - \Delta_c)^\alpha \]

- Universal exponents

\[ \alpha = \begin{cases} 
3 & \text{type 1} \\
4 & \text{type 3} 
\end{cases} \]

L-2 is the small parameter explains small saturation mass
Heuristic derivation of exponents

- Perturbation theory \( \Delta = 1 + \varepsilon \)
- Central cluster \( x(\infty) = 0 \)
- Extremist minor cluster \( x(\infty) = 1 + \varepsilon / 2 \)

\[ \frac{dm}{dt} = -mM \rightarrow m(t) \sim \varepsilon e^{-t} \]

\( x \sim e^{-t_f/2} \sim \varepsilon \)

\( m(\infty) \sim m(t_f) \sim \varepsilon^3 \)
Integrable for $\Delta < 1/2$

$$\langle x^2(t) \rangle = \langle x^2(0) \rangle e^{-\Delta t}$$

Final state: localized

$$P_\infty(x) = 2\Delta \delta(x)$$

Rate equations in Fourier space

$$P_t(k) + P(k) = P^2(k/2)$$

Self-similar collapse dynamics

$$\Phi(z) \propto (1 + z^2)^{-2} \quad z = \frac{x}{\langle x^2(t) \rangle}$$

Discrete opinions

- **Compromise process**
  
  \[(i-1, i+1) \rightarrow (i, i)\]

- **Master equation**
  
  \[
dP_i / dt = 2P_{i-1}P_{i+1} - P_i(P_{i-2} + P_{i+2})\]

- **Example: 6 states**

- **Symmetry + normalization:**
  
  two-dimensional problem

Initial conditions determine final state

**Isolated fixed points, lines of fixed points**
General features

- Dissipative system, volume contracts
- Energy (Lyapunov) function exists: $<x^2>$
- No cycles or strange attractors
- Uniform state is unstable (Cahn-Hilliard)

\[ P_i = 1 + \phi_i \quad \phi_t + \left( \phi + a\phi_{xx} + b\phi^2 \right)_{xx} = 0 \]

Discrete case yields useful insights
Pattern selection

- **Linear stability analysis**
  \[ P - 1 \propto e^{i(kx + \omega t)} \Rightarrow \omega(k) = \frac{8}{k} \sin \frac{k}{2} - \frac{2}{k} \sin k - 2 \]

- **Fastest growing mode**
  \[ \frac{d\omega}{dk} = 0 \Rightarrow L = \frac{2\pi}{k} = 2.2515 \]

- **Traveling wave (FKPP extremal selection)**
  \[ \frac{d\omega}{dk} = \text{Im}(\omega) / \text{Im}(k) \Rightarrow L = \frac{2\pi}{k} = 2.0375 \]

Patterns induced by wave propagating from boundary. However, emerging period is different \( L = 2.155 \)!
Traveling waves

\[ P - 1 \propto \exp[-\lambda(x - vt) + i(kx + wt)] \]

Discrete case

\[
L_{\text{max}} = 6 \quad L = 5.67 \quad L_{\text{trav wave}} = 5.31
\]
Exponential initial conditions

- Bifurcations induced at the boundary
- Periodic structure, nontrivial period
- Two types of bifurcations
  1. Nucleation of major branch
  2. Nucleation of minor branch

Central cluster is stable
Two kinds of opinions

symmetry breaking, packing
Noisy dynamics

- Opinions change due to interaction with environment (news, events, editorials)
  \[ i \rightarrow i \pm 1 \quad \text{with rate } D \]
- Add diffusion term
  \[ \lambda \rightarrow \lambda - Dk^2 \quad \Rightarrow \quad \lambda \approx (D_c - D)k^2 + \cdots \]
- Sub-critical noise: slow coarsening (single party)
  \[ D_m = D / m, \quad m \sim (D_m t)^{1/2} \quad \Rightarrow \quad m \sim (Dt)^{1/3} \]
- Super-critical noise: flat state stable (no parties)
Conclusions

- Clusters form via bifurcations
- Periodic structure
- Alternating minor-major pattern
- Central party not always exists
- Power-law behavior near transitions

Outlook

- Pattern selection criteria
- Gaps
- Role of initial conditions, classification
- Role of spatial dimension, correlations
- Disorder, noise, inhomogeneities
- Tiling/Packing in 2D
- Discord dynamics (seceder model, Halpin-Heally 03)

Many open questions