Nonequilibrium Statistical Physics of Driven Granular Gases

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talk, papers available: http://cnls.lanl.gov/~ebn

August 9, 2005Pattern Formation and Transport Phenomena, João Pessoa, Paraiba, Brazil
Plan

1. Introduction
2. kinetic theory of granular gases
3. Free cooling states
4. Driven Steady states I (forcing at large scales)
5. Driven steady states II (forcing at all scales)
Experiments

- **Friction**
  D Blair, A Kudrolli 01

- **Rotation**
  K Feitosa, N Menon 04

- **Driving strength**
  W Losert, J Gollub 98

- **Dimensionality**
  J Urbach & Olafsen 98

- **Boundary**
  J van Zon, H Swinney 04

- **Fluid drag**
  K Kohlstedt, I Aronson, EB 05

- **Long range interactions**
  D Blair, A Kudrolli 01; W Losert 02
  K Kohlstedt, J Olafsen, EB 05

- **Substrate**
  G Baxter, J Olafsen 04

Deviations from equilibrium distribution
Energy dissipation in granular matter

- Responsible for collective phenomena
  - Clustering  I. Goldhirsch, G. Zanetti 93
  - Hydrodynamic instabilities  E. Khain, B. Meerson 04
  - Pattern formation  P. Umbanhower, H. Swinney 96

- Anomalous statistical mechanics
  - No energy equipartition  R. Wildman, D. Parker 02
  - Nonequilibrium energy distributions

\[ P(E) \neq \exp \left( -\frac{E}{kT} \right) \]
Driven Granular gas

- Vigorous driving
- Spatially uniform system
- Particles undergo binary collisions
- Velocities change due to
  1. **Collisions:** lose energy
  2. **Forcing:** gain energy
- What is the typical velocity (granular “temperature”)?
  \[ T = \langle v^2 \rangle \]
- What is the velocity distribution?
  \[ f(v) \]
Nonequilibrium velocity distributions

A  Mechanically vibrated beads
   F Rouyer & N Menon 00

B  Electrostatically driven powders
   I Aronson, J Olafsen, EB
   - Gaussian core
   - Overpopulated tail
     \[ f(v) \sim \exp\left(-|v|^{\delta}\right) \]
     \[ 1 \leq \delta \leq 3/2 \]
   - Kurtosis
     \[ \kappa = \begin{cases} 
       3.55 & \text{theory} \\
       3.6 & \text{experiment} 
     \end{cases} \]

Excellent agreement between theory and experiment

balance between collisional dissipation, energy injection from walls
Inelastic Collisions (1D)

- **Relative velocity reduced by** $0 < r < 1$
  
  $$v_1 - v_2 = -r(u_1 - u_2)$$

- **Momentum is conserved**
  
  $$v_1 + v_2 = u_1 + u_2$$

- **Energy is dissipated**
  
  $$\Delta E = \frac{1 - r^2}{4}(u_1 - u_2)^2$$

- **Limiting cases**
  
  $$r = \begin{cases} 
  0 & \text{completely inelastic (}\Delta E = \text{max)} \\
  1 & \text{elastic (}\Delta E = 0) 
  \end{cases}$$
Inelastic Collisions (any D)

- Normal relative velocity reduced by $0 < r < 1$
  \[(v_1 - v_2) \cdot n = -r(u_1 - u_2) \cdot n\]

- Momentum conservation
  \[v_1 + v_2 = u_1 + u_2\]

- Energy loss
  \[
  \Delta E = \frac{1 - r^2}{4} [(u_1 - u_2) \cdot n]^2
  \]

- Limiting cases
  \[
  r = \begin{cases} 
  0 & \text{completely inelastic (}\Delta E = \text{max}) \\
  1 & \text{elastic (}\Delta E = 0) 
  \end{cases}
  \]
Non-Maxwellian velocity distributions

1. Velocity distribution is isotropic

\[ f(v_x, v_y, v_z) = f(|v|) \]

2. No correlations between velocity components

\[ f(v_x, v_y, v_z) \neq f(v_x)f(v_y)f(v_z) \]

Only possibility is Maxwellian

\[ f(v_x, v_y, v_z) \neq C \exp \left( -\frac{v_x^2 + v_y^2 + v_z^2}{2T} \right) \]

Granular gases: collisions create correlations
The collision rate

- **Collision rate**

\[ K(u_1, u_2) = |(u_1 - u_2) \cdot n|^\lambda \]

- **Collision rate related to interaction potential (elastic)**

\[ U(r) \sim r^{-\gamma} \quad \lambda = 1 - 2 \frac{d-1}{\gamma} = \begin{cases} 
0 & \text{Maxwell molecules} \\
1 & \text{Hard spheres} 
\end{cases} \]

- **Balance kinetic and potential energy**

\[ v^2 \sim r^{-\gamma} \quad \implies \quad r \sim v^{-2/\gamma} \]

- **Collisional cross-section**

\[ \sigma \sim vr^{d-1} \quad \implies \quad \sigma \sim v^{1-\frac{2}{\gamma}(d-1)} \]
The Inelastic Boltzmann equation (1D)

- **Collision rule (linear)** \( r = 1 - 2p, \quad p + q = 1 \)
  \((u_1, u_2) \rightarrow (pu_1 + qu_2, pu_2 + qu_1)\)

- **General collision rate**
  \[ K(v_1, v_2) = |v_1 - v_2|^\lambda = \begin{cases} 
 0 & \text{Maxwell molecules} \\
 1 & \text{Hard spheres} 
\end{cases} \]

- **Boltzmann equation (nonlinear and nonlocal)**
  \[
  \frac{\partial f(v)}{\partial t} = \int \int du_1 du_2 f(u_1)f(u_2)|u_1-u_2|^\lambda \left[ \delta(v-\rho u_1 - \rho u_2) - \delta(v-u_2) \right]
  \]
  collision rate, gain, loss

**Theory:** non-linear, non-local, dissipative
The Inelastic Boltzmann equation

Spatially homogeneous systems

\[
\frac{\partial f(v)}{\partial t} = \iint d\mathbf{u}_1 d\mathbf{u}_2 f(\mathbf{u}_1)f(\mathbf{u}_2) |\mathbf{u}_1 - \mathbf{u}_2|^\lambda [\delta(v - p\mathbf{u}_1 - q\mathbf{u}_2) - \delta(v - \mathbf{u}_2)]
\]

What are the stationary solutions of this equation?
What is the nature of the velocity distribution?
Homogeneous cooling state: temperature decay

- Energy loss: \( \Delta T \sim (\Delta v)^2 \)
- Collision rate: \( \Delta t \sim 1/(\Delta v)^\lambda \)
- Energy balance equation:
  \[
  \frac{\Delta T}{\Delta t} \sim -(\Delta v)^{2+\lambda} \implies \frac{dT}{dt} \sim -T^{1+\lambda/2}
  \]
- Temperature decays, system comes to rest:
  \[T \sim t^{-2/\lambda} \implies f(v) \rightarrow \delta(v)\]

Trivial stationary solution

Haff, JFM 1982
Homogeneous cooling states: similarity solutions

- **Similarity solution**

  \[ f(v, t) = t^{1/\lambda} \Phi(vt^{1/\lambda}) \]

- **Stretched exponentials (overpopulation)**

  \[ \Phi(z) \sim \exp\left(-|z|^\lambda\right) \]
Are there nontrivial stationary solutions?

- **Stationary Boltzmann equation**

\[ 0 = \int \int du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)] \]

Collision rate  Gain  Loss

*Naive answer: NO!*

- According to the energy balance equation

\[ \frac{dT}{dt} = -\Gamma \]

- Dissipation rate is positive

\[ \Gamma > 0 \]
An exact solution (1D, $\lambda=0$)

- One-dimensional Maxwell molecules
- Fourier transform obeys a closed equation
  \[ F(k) = \int dv e^{ikv} f(v) \]
  \[ F(k) = F(pk)F(qk) \]
- Exponential solution
  \[ F(k) = \exp(-v_0|k|) \]
- Lorentzian velocity distribution
  \[ f(v) = \frac{1}{\pi} \frac{1}{1 + v^2} \]

A nontrivial stationary solution does exist!
Properties of stationary solution

- Perfect balance between collisional loss and gain
- Purely collisional dynamics (no source term)
- Family of solutions: scale invariance $v \rightarrow v/v_0$

$$f(v) = \frac{1}{\pi v_0} \frac{1}{1 + (v/v_0)^2}$$

- Power-law high-energy tail

$$f(v) \sim v^{-2}$$

- Infinite energy, infinite dissipation rate!

Are these stationary solutions physical?
Extreme Statistics (1D)

- Collision rule: arbitrary velocities
  \[(u_1, u_2) \rightarrow (pu_1 + qu_2, pu_2 + qu_1)\]

- Large velocities: linear but nonlocal process
  \[v \xrightarrow{\nu^\lambda} (pv, qv)\]

- High-energies: linear equation
  \[
  \frac{\partial f(v)}{\partial t} = \nu^\lambda \left[ \frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right) - f(v) \right]
  \]

Linear, nonlocal evolution equation
Stationary solution (1D)

- **High-energies: linear equation**

\[
f(v) = \frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right)
\]

  loss  gain  gain

- **Power-law tail**

\[
f(v) \sim v^{-2-\lambda}
\]
Energy Cascades (1D)

Energetic particles “see” a static medium

\[ \nu \rightarrow (p\nu, q\nu) \]
Collision process: large velocities

\[ v |v \cos \theta|^\lambda (\alpha v, \beta v) \]

Stretching parameters related to impact angle

\[ \alpha = (1 - p) \cos \theta \quad \beta = \sqrt{1 - (1 - p^2) \cos^2 \theta} \]

Energy decreases, velocity magnitude increases

\[ \alpha^2 + \beta^2 \leq 1 \quad \alpha + \beta \geq 1 \]

Linear equation

\[ \frac{\partial f(v)}{\partial t} = \left\langle (v \cos \theta)^\lambda \left[ \frac{1}{\alpha^{d+\lambda}} f \left( \frac{v}{\alpha} \right) + \frac{1}{\beta^{d+\lambda}} f \left( \frac{v}{\beta} \right) - f(v) \right] \right\rangle \]
Power-laws are generic

- Velocity distribution always has power-law tail
  \[ f(v) \sim v^{-\sigma} \]

- Characteristic exponent varies with parameters
  \[
  1 - 2F_1\left(\frac{d+\lambda-\sigma}{2}, \frac{\lambda+1}{2}, \frac{d+\lambda}{2}, 1-p^2\right) = \frac{\Gamma\left(\frac{\sigma-d+1}{2}\right)\Gamma\left(\frac{d+\lambda}{2}\right)}{\Gamma\left(\frac{\sigma}{2}\right)\Gamma\left(\frac{\lambda+1}{2}\right)}
  \]

- Tight bounds \(1 \leq \sigma - d - \lambda \leq 2\)

- Elastic limit is singular \(\sigma \to d + 2 + \lambda\)

Dissipation rate always divergent
Energy finite or infinite
The characteristic exponent $\sigma (d=2,3)$

$\sigma$ varies with spatial dimension, collision rules
Monte Carlo Simulations: Driven Steady States

- **Compact** initial distribution
- **Inject energy at very large velocity scales only**
- **Maintain constant total energy**
- **“Lottery” implementation:**
  - Keep track of total energy dissipated, $E_T$
  - With small rate, boost a particle by $E_T$

Excellent agreement between theory and simulation
Further confirmation: extremal statistics

Maxwell molecules (1D, 2D)  Hard spheres (1D, 2D)

\begin{align*}
N &= 10^7 \\
N &= 10^5
\end{align*}

\begin{array}{|c|c|c|}
\hline
d & \text{theory} & \text{simulation} \\
\hline
1 & 2 & 1.995 \\
2 & 3.19520 & 3.19 \\
\hline
\end{array}

\begin{array}{|c|c|c|}
\hline
d & \text{theory} & \text{simulation} \\
\hline
1 & 3 & 2.994 \\
2 & 4.14922 & 4.15 \\
\hline
\end{array}
Injection, cascade, dissipation

- Energy is injected ONLY AT LARGE VELOCITY SCALES!
- Energy cascades from large velocities to small velocities
- Energy dissipated at small velocity scales

Experimental realization?
Energetic particle “shot” into static medium

Energy balance
\[ \Gamma \sim \gamma V^2 \]
Extreme statistics

- Scaling function

\[ \Phi(x) = \sum_{n=1}^{\infty} A_n \exp \left[ -(2^n x)^{\lambda} \right] \]

\[ A_n = \prod_{\substack{k=1 \atop k \neq n}}^{\infty} \frac{1}{1 - 2^{\lambda(n-k)}} \]

- Large velocities: as in free cooling

\[ \Phi(x) \sim \exp(-x^{\lambda}) \quad x \to \infty \]

- Small velocities: non-analytic behavior

\[ 1 - \Phi(x) \sim \exp \left[ -(\ln x)^2 \right] \quad x \to 0 \]

Hybrid between steady-state and time dependent state

Maxwell Model (\(\lambda=0\)) only unsolved case!
Time dependent solutions (1D, $\lambda > 0$)

- Self-similar distribution
  \[ f(\nu, t) \simeq \nu^{-\sigma} \Phi \left( \frac{\nu}{V(t)} \right) \]

- Cutoff velocity decays
  \[ V(t) \sim t^{-1/\lambda} \]

- Scaling function
  \[ \Phi(x) = \sum_{n=1}^{\infty} A_n \exp \left[ -\left( 2^n x \right)^\lambda \right] \]
  \[ A_n = \prod_{\substack{k=1 \atop k \neq n}}^{\infty} \frac{1}{1 - 2^\lambda(n-k)} \]

Hybrid between steady-state and time dependent state

with Ben Machta (Brown)
A third family of solutions exists
Energy balance

- Energy injection rate $\gamma$
- Energy injection scale $V$
- Typical velocity scale $v_0$
- Balance between energy injection and dissipation

$$\gamma \sim V^\lambda (V/v_0)^{d-\sigma}$$

- For “lottery” injection: injection scale diverges with injection rate

$$V \sim \begin{cases} 
\gamma^{-1/(2-\lambda)} & \sigma < d + 2 \\
\gamma^{-1/(\sigma-d-\lambda)} & \sigma > d + 2
\end{cases}$$

Energy injection selects stationary solution
Summary: solutions of kinetic theory

- **Time dependent solution**
  \[ f(v, t) = t^{1/\lambda} \Psi(vt^{1/\lambda}) \]

- **Time independent solution**
  \[ f_s(v) \sim v^{-\sigma} \]

- **Hybrid solution**
  \[ f(v, t) = f_s(v) \Phi(vt^{1/\lambda}) \]

Are there other types of solutions?
Conclusions I

- New class of nonequilibrium steady states
- Energy cascades from large to small velocities
- Power-law high-energy tail
- Energy input at large scales balances dissipation
- Associated similarity solutions exist as well
- Temperature insufficient to characterize velocities
- Experimental realization: requires a different driving mechanism
The Thermally Forced Inelastic Boltzmann equation

- **Energy injection: thermal forcing (at all scales)**
  \[ \frac{dv}{dt} = \eta \]

- **Energy dissipation: inelastic collision**
  \[ \frac{\partial f(v)}{\partial t} = D \nabla^2 f(v) + \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)] \]

- **Steady state equation**
  \[ 0 = D \nabla^2 f(v) + \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)] \]
Driven Steady States: extremal statistics

- **Energy injection: thermal forcing (at all scales)**
  \[ \frac{dv}{dt} = \eta \]

- **Energy dissipation: inelastic collision**
  \[ v \to (p v, q v) \]

- **Steady state equation**
  \[ 0 = D \frac{d^2 f(v)}{d^2 v} + v^\lambda \left[ \frac{1}{p^{1+\lambda}} f \left( \frac{v}{p} \right) + \frac{1}{q^{1+\lambda}} f \left( \frac{v}{q} \right) - f(v) \right] \]

- **Stretched exponentials**
  \[ f(v) \sim \exp \left( -v^{1+\lambda/2} \right) \]
Nonequilibrium velocity distributions

A  Mechanically vibrated beads
F Rouyer & N Menon 00

B  Electrostatically driven powders
I Aronson, J Olafsen, EB

- **Gaussian core**
- **Overpopulated tail**
  \[ f(v) \sim \exp\left(-|v|^\delta\right) \]
  \[ 1 \leq \delta \leq 3/2 \]
- **Kurtosis**
  \[ \kappa = \begin{cases} 
  3.55 & \text{theory} \\
  3.6 & \text{experiment} 
\end{cases} \]

Excellent agreement between theory and experiment

balance between collisional dissipation, energy injection from walls
Conclusions II

- Conventional nonequilibrium steady states
- Energy cascades from large to small velocities
- Energy input at ALL scales balances dissipation
- Stretched exponential tails
- Low order moments (temperature, kurtosis) useful
- Excellent agreement between experiments and kinetic theory