Kinetics of Averaging

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Two parts

1. Averaging velocities and angles
2. Averaging opinions
Themes and concepts

1. Self-similarity, scaling
2. Multi-scaling
3. Cascades
4. Phase transitions
5. Synchronization
6. Bifurcations
7. Pattern Formation
8. Coarsening

• Naturally emerge in various kinetic theories
• Useful in complex and nonequilibrium particle systems
Part I: Averaging velocities and angles
Plan

I. Averaging velocities
   A. Kinetics of pure averaging
   B. Averaging with forcing: steady-states

II. Averaging angles
   A. Averaging with forcing: steady states
The basic averaging process

- $N$ identical particles (grains, billiard balls)
- Each particle carries a number (velocity) $v_i$
- Particles interact in pairs (collision)
- Both particles acquire the average (inelastic)

$$(v_1, v_2) \rightarrow \left( \frac{v_1 + v_2}{2}, \frac{v_1 + v_2}{2} \right)$$
Conservation laws & dissipation

- **Total number of particles is conserved**
- **Total momentum is conserved**

\[
\sum_{i=1}^{N} v_i = \text{constant}
\]

- **Energy is dissipated in each collision**

\[
\Delta E = \frac{1}{4} (v_1 - v_2)^2
\]

\[
E_i = \frac{1}{2} v_i^2
\]

We expect the velocities to shrink
Some details

- **Dynamic treatment**
  Each particle collides once per unit time

- **Random interactions**
  The two colliding particles are chosen randomly

- **Infinite particle limit is implicitly assumed**

\[ N \to \infty \]

- **Process is galilean invariant**
  \[ v \to v + v_0 \]

Set average velocity to zero \[ \langle x \rangle = 0 \]
The temperature

• Definition

\[ T = \langle v^2 \rangle \]

• Time evolution = exponential decay

\[ \frac{dT}{dt} = -\lambda T \quad \implies \quad T = T_0 e^{-\lambda t} \]

\[ \lambda = \frac{1}{2} \]

• All energy is eventually dissipated

• Trivial steady-state

\[ P(v) \rightarrow \delta(v) \]
The moments

- **Kinetic theory**

\[
\frac{\partial P(v, t)}{\partial t} = \int \int dv_1 dv_2 P(v_1, t) P(v_2, t) \left[ \delta \left( v - \frac{v_1 + v_2}{2} \right) - \delta(v - v_1) \right]
\]

- **Moments of the distribution**

\[
M_n = \int dv \, v^n P(v, t)
\]

- **Closed nonlinear recursion equations**

\[
\frac{dM_n}{dt} + \lambda_n M_n = 2^{-n} \sum_{m=2}^{n-2} \binom{n}{m} M_m M_{n-m}
\]

- **Asymptotic decay**

\[
M_n \sim e^{-\lambda_n t} \quad \text{with} \quad \lambda_n = 1 - 2^{-(n-1)}
\]

\[M_0 = 1, \quad M_{2n+1} = 0\]
Multiscaling

- Nonlinear spectrum of decay constants
  \[ \lambda_n = 1 - 2^{-(n-1)} \]
- Spectrum is concave, saturates
  \[ \lambda_n < \lambda_m + \lambda_{n-m} \]
- Each moment has a distinct behavior
  \[ \frac{M_n}{M_m M_{n-m}} \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty \]

Multiscaling Asymptotic Behavior
The Fourier transform

- The Fourier transform
  \[ F(k) = \int dv \, e^{ikv} P(v, t) \]

- Obey closed, nonlinear, nonlocal equation
  \[ \frac{\partial F(k)}{\partial t} + F(k) = F^2(k/2) \]

- Scaling behavior, scale set by second moment
  \[ F(k, t) \rightarrow f(k e^{-\lambda t}) \]
  \[ \lambda = \frac{\lambda_2}{2} = \frac{1}{4} \]

- Nonlinear differential equation
  \[ -\lambda z f'(z) + f(z) = f^2(z/2) \]
  \[ f(0) = 1 \]
  \[ f'(0) = 0 \]

- Exact solution
  \[ f(z) = (1 + |z|) e^{-|z|} \]
Closure: derivation

- The Fourier transform
  \[ F(k) = \int dv \, e^{ikv} P(v, t) \]

- The kinetic theory
  \[ \frac{\partial P(v, t)}{\partial t} + P(v, t) = \iint dv_1 dv_2 P(v_1, t) P(v_2, t) \delta \left( v - \frac{v_1 + v_2}{2} \right) \]

- Fourier transform of the gain term
  \[
  \begin{align*}
  \int dv e^{ikv} \iint dv_1 dv_2 P(v_1, t) P(v_2, t) \delta \left( v - \frac{v_1 + v_2}{2} \right) &= \iint dv_1 dv_2 P(v_1, t) P(v_2, t) \int dv e^{ikv} \delta \left( v - \frac{v_1 + v_2}{2} \right) \\
  &= \iint dv_1 dv_2 P(v_1, t) P(v_2, t) e^{ik \frac{v_1 + v_2}{2}} \\
  &= \int dv_1 P(v_1, t) e^{ik \frac{v_1}{2}} \int dv_2 P(v_2, t) e^{ik \frac{v_2}{2}} \\
  &= F(k/2) F(k/2)
  \end{align*}
  \]

- Closed equation for Fourier Transform
  \[ \frac{\partial F(k)}{\partial t} + F(k) = F^2(k/2) \]
**Fourier transform generates the moments**

- **The Fourier transform**
  \[ F(k) = \int dv \, e^{ikv} P(v, t) \]

- **Is the generating function of the moments**
  \[ M_n = \int dv \, v^n P(v) \]

\[
\begin{align*}
F(k) &= \int dv \, e^{ikv} P(v) \\
&= \int dv \left[ 1 + ikv + \frac{1}{2!} (ikv)^2 + \frac{(ikv)^3}{3!} + \cdots \right] P(v) \\
&= \int dv P(v) + ik \int dv \, v P(v) + \frac{(ik)^2}{2!} \int dv \, v^2 P(v) + \frac{(ik)^3}{3!} \int dv \, v^3 P(v) + \cdots \\
&= M_0 + ik M_1 + \frac{(ik)^2}{2!} M_2 + \frac{(ik)^3}{3!} M_3 + \cdots \\
&= M_0 - \frac{k^2}{2!} M_2 + \frac{k^4}{4!} M_4 + \cdots
\end{align*}
\]

- **Closed equation for Fourier transform**
  \[
  \frac{\partial F(k)}{\partial t} + F(k) = F^2(k/2)
  \]

- **Generates closed equations for the moments**
  \[
  \frac{dM_2}{dt} = -\frac{M_2}{2}
  \]
The velocity distribution

- Self-similar form
  
  \[ P(v, t) \rightarrow e^{\lambda t} p(ve^{\lambda t}) \]

- Obtained by inverse Fourier transform
  
  \[ p(w) = \frac{2}{\pi} \frac{1}{(1 + w^2)^2} \]

- Power-law tail
  
  \[ p(w) \sim w^{-4} \]

1. Temperature is the characteristic velocity scale

2. Multiscaling is consequence of diverging moments of the power-law similarity function
Stationary Solutions

- Stationary solutions do exist!

\[ F(k) = F^2(k/2) \]

- Family of exponential solutions

\[ F(k) = \exp(-kv_0) \]

- Lorentz/Cauchy distribution

\[ P(v) = \frac{1}{\pi v_0} \frac{1}{1 + (v/v_0)^2} \]

How is a stationary solution consistent with dissipation?
Extreme Statistics

- Large velocities, cascade process
  \[ v \rightarrow \left( \frac{v}{2}, \frac{v}{2} \right) \]

- Linear evolution equation
  \[ \frac{\partial P(v)}{\partial t} = 4P\left( \frac{v}{2} \right) - P(v) \]

- Steady-state: power-law distribution
  \[ P(v) \sim v^{-2} \]

- Divergent energy, divergent dissipation rate

Power-law energy distribution

\[ P(E) \sim E^{-3/2} \]
Energy cascade

$v$ → $v/2$ → $v/4$

$v$ → $v/2$ → $v/4$

$v$ → $v/2$ → $v/4$
Injection, Cascade, Dissipation

\[ \ln P(|v|) \]

\[ \ln |v| \]

\[ v_0 \]

\[ V \]
Pure averaging: conclusions

- Moments exhibit multiscaling
- Distribution function is self-similar
- Power-law tail
- Stationary solution with infinite energy exists
- Driven steady-state
- Energy cascade
Averaging with diffusive forcing

Two independent competing processes

1. Averaging (nonlinear)

\[(v_1, v_2) \rightarrow \left( \frac{v_1 + v_2}{2}, \frac{v_1 + v_2}{2} \right)\]

2. Random uncorrelated white noise (linear)

\[\frac{dv_j}{dt} = \eta_j(t) \quad \langle \eta_j(t)\eta_j(t') \rangle = 2D\delta(t - t')\]

- Add diffusion term to equation (Fourier space)

\[(1 + Dk^2)F(k) = F^2(k/2)\]

System reaches a nontrivial steady-state

Energy injection balances dissipation
Infinite product solution

• **Solution by iteration**

\[
F(k) = \frac{1}{1 + Dk^2} F^2(k/2) = \frac{1}{1 + Dk^2} \frac{1}{1 + D(k/2)^2} F^4(k/4) = \cdots
\]

• **Infinite product solution**

\[
F(k) = \prod_{i=0}^{\infty} \left[ 1 + D(k/2^i)^2 \right]^{-2^i}
\]

• **Exponential tail**  \( v \to \infty \)

\[
P(v) \propto \exp \left( -|v|/\sqrt{D} \right)
\]

• **Also follows from**

\[
D \frac{\partial^2 P(v)}{\partial v^2} = -P(v)
\]

Non-Maxwellian distribution/Overpopulated tails
Cumulant solution

- **Steady-state equation**
  \[ F(k)(1 + Dk^2) = F^2(k/2) \]

- **Take the logarithm** \( \psi(k) = \ln F(k) \)
  \[ \psi(k) + \ln(1 + Dk^2) = 2\psi(k/2) \]

- **Cumulant solution**
  \[ F(k) = \exp \left[ \sum_{n=1}^{\infty} \psi_n (-Dk^2)^n / n \right] \]

- **Generalized fluctuation-dissipation relations**
  \[ \psi_n = \lambda_n^{-1} = [1 - 2^{1-n}]^{-1} \]
Experiments

“A shaken box of marbles”  
Menon 01  
Aronson 05
Averaging with forcing: conclusions

- Nonequilibrium steady-states
- Energy pumped and dissipated by different mechanisms
- Overpopulation of high-energy tail with respect to equilibrium distribution
Averaging angles

• Each rod has an orientation

\[ 0 \leq \theta \leq \pi \]

• Alignment by pairwise interactions (nonlinear)

\[ \langle \eta_j(t) \eta_j(t') \rangle = 2D \delta(t - t') \]

• Diffusive wiggling (linear)

\[ \frac{d\theta_j}{dt} = \eta_j(t) \]
Relevance

- Biology: molecular motors
- Ecology: flocking
- Granular matter: granular chains and solid rods
- Phase synchronization
Kinetic Theory

- **Nonlinear integro-differential equation**
  \[
  \frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial \theta^2} + \int_{-\pi}^{\pi} d\phi \, P \left( \theta - \frac{\phi}{2} \right) P \left( \theta + \frac{\phi}{2} \right) - P.
  \]

- **Fourier transform**
  \[
  P_k = \langle e^{-ik\theta} \rangle = \int_{-\pi}^{\pi} d\theta e^{-ik\theta} P(\theta)
  \]
  \[
  P(\theta) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} P_k e^{ik\theta}
  \]

- **Closed nonlinear equation**
  \[
  (1 + Dk^2)P_k = \sum_{i+j=k} A_{i-j} P_i P_j
  \]

- **Coupling constants**
  \[
  A_q = \sin \frac{\pi q}{2} = \begin{cases} 
  1 & q = 0 \\
  0 & q = 2, 4, \ldots \\
  (-1)^{\frac{q-1}{2}} \frac{2}{\pi|q|} & \text{otherwise}
  \end{cases}
  \]
The order parameter

• **Lowest order Fourier mode**

\[ R = |\langle e^{i\theta} \rangle| = |P_{-1}| \]

• **Probes state of system**

\[
R = \begin{cases} 
0 & \text{disordered state} \\
0.4 & \text{partially ordered} \\
1 & \text{perfectly ordered state}
\end{cases}
\]
The Fourier equation

- **Compact Form**

\[ P_k = \sum_{i+j=k} G_{i,j} P_i P_j \]

- **Transformed coupling constants**

\[ G_{i,j} = \frac{A_{i-j}}{1+D(i+j)^2-2A_{i+j}} \]

- **Properties**

\[
\begin{align*}
G_{i,j} & = G_{j,i} \\
G_{i,j} & = G_{-i,-j} \\
G_{i,j} & = 0, \quad \text{for} \quad |i - j| = 2, 4, \ldots
\end{align*}
\]
Solution

- Repeated iterations (product of three modes)

\[ P_k = \sum_{i+j=k} \sum_{l+m=j} G_{i,j} G_{l,m} P_i P_l P_m. \]

\[ P_2 = G_{1,1} P_1^2 \]
\[ P_4 = G_{2,2} P_2^2 = G_{2,2} G_{1,1} P_1^4. \]

- When \( k=2,4,8,... \)

- Generally

\[ P_3 = 2G_{1,2} P_1 P_2 + 2G_{-1,4} P_{-1} P_4 + \cdots \]
\[ = 2G_{1,2} G_{1,1} P_1^3 + 2G_{-1,4} G_{2,2} G_{1,1} P_1^4 P_{-1} + \cdots \]
Partition of Integers

- **Diagramatic solution**

\[ P_k = R^k \sum_{n=0}^{\infty} p_{k,n} R^{2n} \]

- **Partition**

\[ k = 1 + 1 + \cdots + 1 + 1 - 1 - \cdots - 1. \]

- **Partition rules**

\[
\begin{align*}
  k &= i + j \\
  i &\neq 0 \\
  j &\neq 0 \\
  G_{i,j} &\neq 0
\end{align*}
\]

All modes expressed in terms of order parameter
The order parameter

- **Diagramatic solution**
  \[ R = R^k \sum_{n=0}^{\infty} p_{1,n} R^{2n} \]

- **Landau theory**
  \[ R = \frac{C}{D_c - D} R^3 + \cdots \]

- **Critical diffusion constant**
  \[ D_c = \frac{4}{\pi} - 1 \]

Closed equation for order parameter
Nonequilibrium phase transition

- Critical diffusion constant \( D_c = \frac{4}{\pi} - 1 \)
- Weak diffusion: ordered phase \( R > 0 \)
- Strong diffusion: disordered phase \( R = 0 \)
- Critical behavior \( R \sim (D_c - D)^{1/2} \)
Distribution of orientation

- Fourier modes decay exponentially with $R$

  $$P_k \sim R^k$$

- Small number of modes sufficient

\[
P(\theta) = \frac{1}{2\pi} + \frac{1}{\pi} R \cos \theta + \frac{1}{\pi} G_{1,1} R^2 \cos (2\theta) + \frac{2}{\pi} G_{1,2} G_{1,1} R^3 \cos (3\theta) + \cdots
\]
Arbitrary alignment rates

- **Kinetic theory:** arbitrary alignment rates

\[ 0 = D \frac{d^2 P}{d\theta^2} + \int_{-\pi}^{\pi} d\phi K(\phi) P \left( \theta - \frac{\phi}{2} \right) P \left( \theta + \frac{\phi}{2} \right) - P(\theta) \int_{-\pi}^{\pi} d\phi K(\phi) P(\theta + \phi) \]

- **Fourier transform of alignment rate**

\[ A_q = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{iq\phi/2} K(\phi) \]

- **Recover same Fourier equation using**

\[ G_{i,j} = \frac{1}{2} \frac{A_{i-j} + A_{j-i} - A_{2i} - A_{2j}}{1 + D(i+j)^2 - 2A_{i+j}} \]

**When Fourier spectrum is discrete:**

exact solution is possible for arbitrary alignment rates
Experiments

“A shaken dish of toothpicks”
Averaging angles: conclusions

- Nonequilibrium phase transition
- Weak noise: ordered phase (nematic)
- Strong noise: disordered phase (isotropic)
- Solution relates to iterated partition of integers
- Kinetic theory of synchronization
- Only when Fourier spectrum is discrete: exact solution possible for arbitrary averaging rates
1. E. Ben-Naim and P.L. Krapivsky, 

2. E. Ben-Naim and P.L. Krapivsky, 

3. E. Ben-Naim and P.L. Krapivsky, 

4. E. Ben-Naim and J. Machta, 


6. E. Ben-Naim and P.L. Krapivsky, 
Part 2: Averaging Opinions
Plan

I. Restricted averaging as a compromise process
   A. Continuous opinions
   B. Discrete opinions

II. Restricted averaging with noise
   A. Single-party dynamics
   B. Two-party dynamics
   C. Multi-party dynamics
I. Restricted averaging
The compromise process

- Opinion measured by a continuum variable
  \(-\Delta < x < \Delta\)

1. **Compromise**: reached by pairwise interactions

   \[(x_1, x_2) \rightarrow \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right)\]

2. **Conviction**: restricted interaction range

   \[|x_1 - x_2| < 1\]

- **Restricted averaging process**
- **One parameter model**
- **Mimics competition between compromise and conviction**

Deffuant & Weisbuch (2000)
Problem set-up

• Given uniform initial (un-normalized) distribution

\[ P_0(x) = \begin{cases} 
1 & |x| < \Delta \\
0 & |x| > \Delta 
\end{cases} \]

• Find final distribution

\[ P_\infty(x) = ? \]

• Multitude of final steady-states

\[ P_0(x) = \sum_{i=1}^{N} m_i \delta(x - x_i) \quad |x_i - x_j| > 1 \]

• Dynamics selects one (deterministically!)

Multiple localized clusters
Further details

- **Dynamic treatment**
  Each individual interacts once per unit time

- **Random interactions**
  Two interacting individuals are chosen randomly

- **Infinite particle limit is implicitly assumed**
  \[ N \to \infty \]

- **Process is galilean invariant**
  \[ x \to x + x_0 \]

  Set average opinion to zero
  \[ \langle x \rangle = 0 \]
Numerical methods, kinetic theory

• Same master equation, restricted integration

\[
\frac{\partial P(x, t)}{\partial t} = \int \int dx_1 dx_2 P(x_1, t) P(x_2, t) \left[ \delta \left( x - \frac{x_1 + x_2}{2} \right) - \delta (x - x_1) \right]
\]  

\[|x_1 - x_2| < 1\]

☐ Direct Monte Carlo simulation of stochastic process

☑ Numerical integration of rate equations
Two Conservation Laws

• Total population is conserved

\[ \int_{-\Delta}^{\Delta} dx \, P(x) = 2\Delta \]

• Average opinion is conserved

\[ \int_{-\Delta}^{\Delta} dx \, x \, P(x) = 0 \]
Rise and fall of central party

$0 < \Delta < 1.871$

$1.871 < \Delta < 2.724$

Central party may or may not exist!
Resurrection of central party

$2.724 < \Delta < 4.079$

$4.079 < \Delta < 4.956$

Parties may or may not be equal in size
Emergence of extremists

Tiny fringe parties ($m \sim 10^{-3}$)
Bifurcations and Patterns
Self-similar structure, universality

- Periodic sequence of bifurcations
  1. Nucleation of minor cluster branch
  2. Nucleation of major cluster branch
  3. Nucleation of central cluster
- Alternating major-minor pattern
- Clusters are equally spaced
- Period $L$ gives major cluster mass, separation

\[ x(\Delta) = x(\Delta) + L \quad L = 2.155 \]
How many political parties?

- Data: CIA world factbook 2002
- 120 countries with multi-party parliaments
- Average=5.8; Standard deviation=2.9
Cluster mass

- **Masses are periodic**
  \[ m(\Delta) = m(\Delta + L) \]

- **Major mass**
  \[ M \rightarrow L = 2.155 \]

- **Minor mass**
  \[ m \rightarrow 3 \times 10^{-4} \]

Why are the minor clusters so small?
Scaling near bifurcation points

• Minor mass vanishes

\[ m \sim (\Delta - \Delta_c)^\alpha \]

• Universal exponent \( m \)

\[ \alpha = \begin{cases} 
3 & \text{type 1} \\
4 & \text{type 3} 
\end{cases} \]

L-2 is the small parameter explains small saturation mass
Consensus = pure averaging

- Integrable for $\Delta < 1/2$
  $$\langle x^2(t) \rangle = \langle x^2(0) \rangle e^{-\Delta t}$$

- Final state: localized
  $$P_\infty(x) = 2\Delta \delta(x)$$

- Rate equations in Fourier space
  $$P_t(k) + P(k) = P^2(k/2)$$

- Self-similar collapse dynamics
  $$\Phi(z) \propto (1 + z^2)^{-2} \quad z = x/\sqrt{\langle x^2 \rangle}$$

Heuristic derivation of exponent

- **Perturbation theory** \( \Delta = 1 + \epsilon \)
- **Major cluster** \( x(\infty) = 0 \)
- **Minor cluster** \( x(\infty) = \pm (1 + \epsilon/2) \)

- **Rate of transfer from minor cluster to major cluster**
  \[
  \frac{dm}{dt} = -m M \quad \rightarrow \quad m \sim \epsilon e^{-t}
  \]

- **Process stops when**
  \[
  x \sim e^{-t_f/2} \sim \epsilon \quad \langle x^2 \rangle \sim e^{-t}
  \]

- **Final mass of minor cluster**
  \[
  m(\infty) \sim m(t_f) \sim \epsilon^3 \quad \alpha = 3
  \]
Pattern selection

- Linear stability analysis

\[ P - 1 \propto e^{i(kx + \omega t)} \implies w(k) = \frac{8}{k} \sin \frac{k}{2} - \frac{2}{k} \sin k - 2 \]

- Fastest growing mode

\[ \frac{dw}{dk} \implies L = \frac{2\pi}{k} = 2.2515 \]

- Traveling wave (FKPP saddle point analysis)

\[ \frac{dw}{dk} = \frac{\text{Im}(w)}{\text{Im}(k)} \implies L = \frac{2\pi}{k} = 2.0375 \]

Patterns induced by wave propagation from boundary
However, emerging period is different

\[ 2.0375 < L < 2.2515 \]

Pattern selection is intrinsically nonlinear
Discrete opinions

- **Compromise process**
  \[(n - 1, n + 1) \rightarrow (n, n)\]

- **Master equation**
  \[
  \frac{dP_n}{dt} = 2P_{n-1}P_{n+1} - P_n(P_{n-2} + P_{n+2})
  \]

- **Simplest example: 6 states**

- **Symmetry + normalization:**

- **Two-dimensional problem**

  Initial condition determines final state

  Isolated fixed points, lines of fixed points
Discrete opinions

- Dissipative system, volume contracts
- Energy (Lyapunov) function exists
- No cycles or strange attractors
- Uniform state is unstable (Cahn-Hilliard)

\[ P_i = 1 + \phi_i \quad \phi_t + (\phi + a \phi_{xx} + b \phi^2)_{xx} \]

Discrete case yields useful insights
Pattern selection

- **Linear stability analysis**
  \[ P - 1 \propto e^{i(kx+wt)} \rightarrow w(k) = 4 \cos k - 4 \cos 2k - 2 \]

- **Fastest growing mode**
  \[ \frac{dw}{dk} \implies L = \frac{2\pi}{k} = 6 \]

- **Traveling wave (FKPP saddle point analysis)**
  \[ \frac{dw}{dk} = \frac{\text{Im}(w)}{\text{Im}(k)} \implies L = \frac{2\pi}{k} = 5.31 \]

Again, linear stability gives useful upper and lower bounds

5.31 < L < 6 while \( L_{\text{select}} = 5.67 \)

Pattern selection is intrinsically nonlinear
I. Restricted averaging: conclusions

• Clusters form via bifurcations
• Periodic structure
• Alternating major-minor pattern
• Central party does not always exist
• Power-law behavior near transitions
• Nonlinear pattern selection
I. Outlook

- Pattern selection criteria
- Gaps
- Role of initial conditions, classification
- Role of spatial dimension, correlations
- Disorder, inhomogeneities
- Tiling/Packing in 2D
- Discord dynamics  (seceder model, Halpin-Heally 03)

Many open questions
II. Restricted averaging with noise
**Diffusion (noise)**

- **Diffusion**: Individuals change opinion spontaneously

\[
 n \xrightarrow{D} n \pm 1
\]

- Adds noise ("temperature")

- Linear process: no interaction

- Mimics unstable, varying opinion

- Influence of environment, news, editorials, events
Rate equations

- **Compromise**: reached through pairwise interactions

\[(n - 1, n + 1) \rightarrow (n, n)\]

- Conserved quantities: total population, average opinion
- Probability distribution \(P_n(t)\)
- Kinetic theory: nonlinear rate equations

\[
\frac{dP_n}{dt} = 2P_{n-1}P_{n+1} - P_n(P_{n-2} + P_{n+2}) + D(P_{n-1} + P_{n+1} - 2P_n)
\]

- Direct Monte Carlo simulations of stochastic process
- Numerical integration of rate equations
Single-party dynamics

- Initial condition: large isolated party
  \[ P_n(0) = m(\delta_n,0 + \delta_n,-1) \]

- Steady-state: compromise and diffusion balance
  \[ DP_n = P_{n-1}P_{n+1} \]

- Core of party: localized to a few opinion states
  \[ P_0 = m \quad P_1 = D \quad P_2 = D^2m^{-1} \]

- Compromise negligible for n>2

Party has a well defined core
The tail

• Diffusion dominates outside the core

\[ \frac{dP_n}{dt} = D(P_{n-1} + P_{n+1} - 2P_n) \quad P \ll D \]

• Standard problem of diffusion with source

\[ P_n \sim m^{-1} \Psi(n \, t^{-1/2}) \]

• Tail mass

\[ M_{\text{tail}} \sim m^{-1} \, t^{1/2} \]

• Party dissolves when

\[ M_{\text{tail}} \sim m \quad \implies \quad \tau \sim m^4 \]

Party lifetime grows dramatically with its size
Core versus tail

\[ m = 10^3 \]

\[ mP_n \]

\[ m \]

\[ m^{-1} \]

*Party height = m*

*Party depth \( \sim m^{-1} \)*

*Self-similar shape*

*Gaussian tail*
Qualitative features

- Exists in a quasi-steady state
- Tight core localized to a few sites
- Random opinion changes of members do not affect party position
- Party lifetime grows very fast with size
- Ultimate fate of a party: demise
- Its remnant: a diffusive cloud
- Depth inversely proportional to size, the larger the party the more stable
Two party dynamics

- Initial condition: two large isolated parties
  \[ P_n(0) = m> (\delta_{n,0} + \delta_{n,-1}) + m< (\delta_{n,l} + \delta_{n,l+1}) \]
- Interaction between parties mediated by diffusion
  \[ 0 = P_{n-1} + P_{n+1} - 2P_n \]
- Boundary conditions set by parties depths
  \[ P_0 = \frac{1}{m>} \quad P_l = \frac{1}{m<} \]
- Steady state: linear profile
  \[ P_n = \frac{1}{m<} + \left( \frac{1}{m<} - \frac{1}{m>} \right) \frac{n}{l} \]
Merger

- Steady flux from small party to larger one
  \[ J \sim \frac{1}{l} \left( \frac{1}{m_<} - \frac{1}{m_>} \right) \sim \frac{1}{lm_<} \]

- Merger time
  \[ T \sim \frac{m_<}{J} \sim lm_<^2 \]

- Lifetime grows with separation ("niche")

- Outcome of interaction is deterministic

- Larger party position remains fixed throughout merger process

Small party absorbed by larger one
Merger: numerical results
Multiple party dynamics

- **Initial condition:** large isolated party
  
  \[ P_n(0) = \text{randomly chosen number in } [1 - \epsilon : 1 + \epsilon] \]

- **Linear stability analysis**
  
  \[ P_n - 1 \sim e^{ikn + \lambda t} \]

- **Growth rate of perturbations**
  
  \[ \lambda(k) = (4 \cos k - 4 \cos 2k - 2) - 2D(1 - \cos 2k) \]

- **Long wavelength perturbations unstable**
  
  \[ k < k_0 \quad \cos k_0 = D/2 \]

\[ P=1 \text{ stable only for strong diffusion } D > D_c = 2 \]
Strong noise \((D > D_c)\)

- Regardless of initial conditions

\[ P_n \to \langle P_n(0) \rangle \]

- Relaxation time

\[ \lambda \approx (D_c - D) k^2 \implies \tau \sim (D - D_c)^{-2} \]

No parties, disorganized political system
Weak noise (D<D_c): Coarsening

- Smaller parties merge into large parties
- Party size grows indefinitely
- Assume a self-similar process, size scale m
- Conservation of populations implies separation
  \[ l \sim m \]
- Use merger time to estimate size scale
  \[ t \sim lm^2 \sim m^3 \implies m \sim t^{1/3} \]
- Self-similar size distribution
  \[ P_m \sim t^{-1/3} F(m t^{-1/3}) \]

Lifshitz-Slyozov coarsening
Coarsening: numerical results

- Parties are static throughout process
- A small party with a large niche may still outlast a larger neighbor!
Three scenarios

\[ D = 0 \]

\[ D < D_c \]

\[ D > D_c \]
II. Restricted averaging with noise: conclusions

- **Isolated parties**
  - Tight, immobile core and diffusive tail
  - Lifetime grows fast with size

- **Interaction between two parties**
  - Large party grows at expense of small one
  - Deterministic outcome, steady flux

- **Multiple parties**
  - Strong noise: disorganized political system, no parties
  - Weak noise: parties form, coarsening mosaic
  - No noise: stable parties, pattern formation
Publications


“I can calculate the motions of heavenly bodies, but not the madness of people.”

Isaac Newton