Jamming and tiling in aggregation of rectangles

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Jamming and tiling in two-dimensions

Pick two neighboring rectangles at random
Merge them if they are compatible

System reaches a jammed state
No two neighboring rectangles are compatible
The jammed state

no two neighbors share a common side
Features of the jammed state

- Local alignment
- **Finite** rectangle density \( \rho = 0.1803 \)
- **Finite** tile density \( T = 0.009949 \)
- **Finite** stick density \( S = 0.1322 \)
- **Finite** square density \( H = 0.02306 \)
- Area distribution of rectangles with width \( w \)
  \[ m_\omega \sim \exp (-\text{const.} \times \omega^2) \]

**No theoretical framework!**
Jamming: mean-field version

- Start with $N$ 1x1 tiles (elementary building blocks)
- Pick two rectangles at random
- Pick an orientation at random (vertical or horizontal)
- Merge rectangles if they are perfectly compatible

\[
(i_1, j) + (i_2, j) \rightarrow (i_1 + i_2, j)
\]
\[
(i, j_1) + (i, j_2) \rightarrow (i, j_1 + j_2)
\]

- System is jammed when $f$ rectangles have:
  $f$ distinct horizontal sizes and $f$ distinct vertical sizes

System reaches a jammed state
An example of a jammed state

- Characterize rectangle by horizontal and vertical size $(i, j)$
- Characterize rectangle by maximal and minimal size $(\omega, \ell)$
- Width = minimal size, Length = maximal length
  \[ \omega = \min(i, j) \quad \ell = \max(i, j) \]
- Ordered widths of $f=13$ rectangles for $N=10,000$
  \{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 7, 9\}

*Width sequence has gaps!*
Number of jammed rectangles

• Average Number of rectangles grows algebraically with $N$

\[ F \sim N^\alpha \]

• Nontrivial exponent

\[ \alpha = 0.229 \pm 0.002 \]

• Typical width of rectangles grows algebraically with $N$

\[ \omega \sim N^\alpha \]

• Area density of rectangles of width $w$ decays as a power law

\[ m_\omega \sim \omega^{-\gamma} \quad \text{with} \quad \gamma = \alpha^{-1} - 2 \]

A single exponent characterizes the jammed state
Rectangles with finite width are macroscopic! Rectangles with width 1,2,3,4,5 contain 95% of area

Still, the area distribution has a broad power-law tail!
Kinetic theory

• Straightforward generalization of ordinary aggregation

\[ \frac{dR_{i,j}}{dt} = \sum_{i_1+i_2=i} R_{i_1,j} R_{i_2,j} - 2R_{i,j} \sum_{k \geq 1} R_{k,j} + \sum_{j_1+j_2=j} R_{i,j_1} R_{i,j_2} - 2R_{i,j} \sum_{k \geq 1} R_{i,k} \]

• Allows calculation of the density of sticks

\[ \frac{dS}{dt} = -S^2 - 2 \sum_{i,j} R_{i,j} R_{i,j} \]

• Simple decay for the stick density and jamming time

\[ S \simeq t^{-1} \quad \Rightarrow \quad \tau \sim N \]

• Jammed state properties give density decay and width growth

\[ \rho \sim t^{\alpha-1} \quad \text{and} \quad w \sim t^\alpha \]

Jarring exponent characterizes the kinetics, too
Numerical validation

Suggest two aggregation modes: elongating and widening
Primary aggregation: elongation

- Aggregation between two rectangles of same width

- Ordinary aggregation equation (example: sticks)

\[
\frac{dR_{1,\ell}}{dt} = \sum_{i+j=\ell} R_{1,i} R_{1,j} - 2SR_{1,\ell} - 2 \left( \sum_i R_{i,\ell} \right) R_{1,\ell}
\]

- Length distribution as in \( d=1 \), length grows linearly \( l \sim t \)

\[
R_{1,\ell} \sim \left( \frac{2}{m_1 t^2} \right) \exp\left( -2\frac{\ell}{m_1 t} \right)
\]

- Behavior extends to all rectangles with finite width

\[
R_{\omega,\ell}(t) \sim t^{-2} \Phi_\omega(\ell t^{-1}) \quad \text{with} \quad \Phi_\omega(x) = \left( \frac{2\omega}{m_\omega} \right) \exp\left( -2\frac{\omega x}{m_\omega} \right)
\]

Finite width: problem reduces to one-dimensional aggregation

However, total mass for each width is not known.
Numerical validation

Exponential scaling function

total mass set by the jammed state
Secondary aggregation: widening

- Aggregation between two rectangles of same length

- The area fraction is coupled to the size distribution

\[
\frac{dm_\omega}{dt} = \frac{1}{2} \sum_{i+j=\omega} \sum_\ell \omega \ell R_i, \ell R_j, \ell - \sum_j \sum_\ell \omega \ell R_j, \ell R_\omega, \ell
\]

- Insights about relaxation toward jammed state

\[
m_\omega(t) - m_\omega(\infty) \approx C_\omega t^{-1} \quad \text{with} \quad C_\omega = -2\omega \sum_{i+j=\omega} \frac{\mu_i \mu_j}{(\mu_i + \mu_j)^2} + 4\omega \sum_j \frac{\mu_\omega \mu_j}{(\mu_\omega + \mu_j)^2}
\]

Closure & theoretical determination of \( \alpha \) remains elusive
Conclusions

• Random aggregation of compatible rectangles
• Process reaches a jammed state where all rectangles are incompatible
• Number of jammed rectangle grows as power-law
• Area distribution decays as a power law
• A single, nontrivial, exponent characterize both the jammed state and the time-dependent behavior
• Primary aggregation: rectangles of same width
• Secondary aggregation: rectangles of same length
• Slow transfer of “mass” from thin to wide rectangles
• Kinetic theory successfully describes primary aggregation process only