Formation of Political Networks: Bifurcations, Patterns, and Universality

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I Motivation

II Continuum: Numerics & Scaling

III Discrete: Theory & General Features

with: Paul Krapivsky, Sidney Redner (Boston)
How many political parties?

- Data: CIA world factbook 2002
- 120 countries with multiparty senates
- Average=5.8, Variance=2.9

Simple model?
The Compromise Model

- Opinion measured by continuum variable
  \[-\Delta < x < \Delta\]
- Compromise: via pairwise interactions
  \[(x_1, x_2) \rightarrow \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right)\]
- Conviction: restricted interaction range
  \[|x_1 - x_2| < 1\]
- Initial conditions: uniform distribution
  \[P(x, t = 0) = \begin{cases} 1 & |x| < \Delta, \\ 0 & |x| > \Delta. \end{cases}\]

- Minimal, one parameter model
- Mimics competition between compromise and conviction
Consensus

- Nonlinear rate equations

\[
\frac{\partial P(x, t)}{\partial t} = \int \int_{|x_1 - x_2| < 1} dx_1 dx_2 P(x_1, t)P(x_2, t) \\
\times \left[ 2 \delta \left( x - \frac{x_1 + x_2}{2} \right) - \delta(x - x_1) - \delta(x - x_2) \right]
\]

- Integrable for \( \Delta < 1/2 \):

\[
\langle x^2(t) \rangle = \langle x^2(0) \rangle e^{-\Delta t}
\]

- Final state: localized

\[
P_\infty(x) = 2\Delta \delta(x)
\]

- Time dependence: similarity solution

\[
\Phi(z) = \frac{2\Delta}{\pi} \frac{1}{(1 + z^2)^2} \quad z = \frac{x}{\langle x^2(t) \rangle^{1/2}}
\]

**Generally, what is nature of final state?**
Numerical integration of rate equations
Monte Carlo simulation of random process
Final state:

\[ P_\infty(x) = \sum_{i=1}^{N} m_i \delta(x - x_i) \]

Multiple political networks (parties)
• Periodic sequence of bifurcations

\[ x(\Delta) = x(\Delta + L) \]

• Alternating major-minor pattern
• Clusters are equally spaced
• Period \( \rightarrow \) cluster mass, separation

\[ L = 2.155 \]

Self-similar structure, universality
Cluster masses, bifurcation types

- Masses are periodic as well
  \[ m(\Delta) = m(\Delta + L) \]
- 3 types of bifurcations:
  1. \( \emptyset \to \{-x, x\} \) Nucleation of 2 minor branches
  2. \( \{0\} \to \{-x, x\} \) Nucleation of 2 major branch’s
  3. \( \emptyset \to \{0\} \) Nucleation of central cluster
- Bifurcations occur near origin
- Major: \( M \to 2.15 \), Minor: \( m \to 3 \times 10^{-4} \)

Central cluster may or may not exist
Near critical behavior

- Perturbation theory: $\Delta = 1 + \epsilon$
- Central cluster: mass $M$, $x(\infty) = 0$
- Minor cluster: mass $m$, $x(\infty) = 1 + \epsilon/2$

$$\frac{dm}{dt} = -mM \quad \rightarrow \quad m(t) \sim \epsilon e^{-t}$$

- Process stops when $x \sim e^{-t_f/2} \sim \epsilon$
- Final minor cluster mass

$$m(\infty) \sim m(t_f) \sim \epsilon^3$$

- Argument generalizes to type 3 bifurcations

$$m \sim (\Delta - \Delta_c)^\alpha \quad \alpha = \begin{cases} 
3 & \text{type 1} \\
4 & \text{type 3} 
\end{cases}$$

Masses vanish algebraically near type 1, 3 bif
Discrete Opinions

- Basic process: $(i - 1, i + 1) \rightarrow (i, i)$
- Rate equation:
  \[
  \frac{d}{dt} P_i = 2P_{i-1}P_{i-1} - P_i(P_{i-2} + P_{i+2})
  \]
- Example: 6 states, $P_i = P_{N-i}$
- Initial conditions determine final state
- Isolated fixed points, lines of fixed points
General Features

- Dissipative system: volume contracts
- Lyapunov (energy) function exists $\langle x^2 \rangle$
- No cycles or strange attractors
- Uniform state is unstable: $P_i = 1 + \phi_i$

$$\phi_i + (\phi + a\phi_{xx} + b\phi^2)_{xx} = 0$$

Discrete case yields useful insights
Exponential initial conditions

- Bifurcations induced at the boundary
- Periodic structure
- Two types of bifurcations
  1. Nucleation of major branch
  2. Nucleation of minor branch

Central party is stable
Conclusions

- Networks form via bifurcations
- Periodic structure
- Alternating major-minor pattern
- Central party not always exists
- Power-law behavior near transitions

Outlook

- Role of initial conditions? Classification?
- Role of spatial dimension? Correlations?
- Add disorder, inhomogeneities
- Tiling/Packing in 2D,3D?

G. Weisbuch, cond-mat/0111494.