Preparation and Compression of Symmetric Pure Quantum States

joint work with Stephan Eidenbenz

Andreas Bärtschi
NSEC/CNLS, baertschi@lanl.gov

ISTI Seminar (LANL)
QMATH (Copenhagen)
QIT (ETH Zürich)
August, 2019

How efficiently can we prepare symmetric quantum states?

Efficient Symmetric State Preparation

Symmetric n-qubit States, e.g. $|\psi\rangle = \frac{-\sqrt{2}|000\rangle + i|011\rangle + i|101\rangle + i|110\rangle}{\sqrt{5}} = \frac{-\sqrt{2}|D_0^3\rangle + \sqrt{3}i|D_2^3\rangle}{\sqrt{5}}$

- Symmetric under permutation of the qubits.
- All terms with the same Hamming Weight must have the same amplitude:
  $\Rightarrow$ Dicke States $|D_k^n\rangle$ are equal superpositions of all HW-k strings.
  $\Rightarrow$ They form an orthonormal basis for symmetric pure states.

Efficient in terms of

- Circuit Model: Deterministic Scheme, Linear Nearest Neighbor architecture.
- Small total number of Gates: $\mathcal{O}(n^2)$ many.
- Shallow Depth: $\mathcal{O}(n)$ steps.
- Few Ancilla Qubits: None.

Outline

1 Context
- How many gates for. . . ?
- Known upper bounds
- Results for Symmetric States

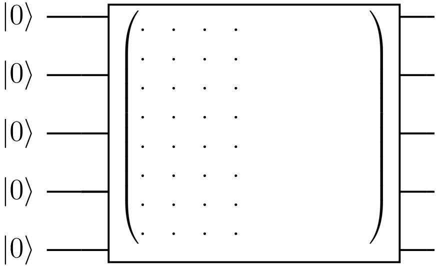
2 Dicke States
- Dicke State Unitaries
- Inductive Approach
- Split & Cyclic Shift Unitaries
- Combining all Ideas

3 Arbitrary Symmetric Pure States
- Preparation of Symmetric Pure States
- Compression of Symmetric Pure States
- Conclusion

Context

How many gates for. . . ?

Given n qubits and a unitary U as a $2^n \times 2^n$ matrix, how many 1- and 2-qubit gates do we need to implement



- all of U,
- the first column of U (state preparation),
- the first K columns of U,

exactly?

$\Theta(K \cdot 2^n)$ gates are sufficient and sometimes necessary.[K95, SBM04]

approximately?

$\widetilde{\Theta}(K \cdot 2^n)$ gates are sufficient and still sometimes necessary!

In fact, the set of states approximately preparable with fewer gates has measure 0.[K95]

Which States can we prepare efficiently?

[K95]: Knill, *Approximation by Quantum Circuits*, 1995
[SBM04]: Shende, Bullock, Markov, *Synthesis of Quantum Logic Circuits*, 2004

Known upper bounds

A state $|\psi\rangle = \sum_{i=0}^{2^n-1} \psi_i |i\rangle$ can be prepared with polynomial resources, if e.g.,

- there are only polynomially many non-zero amplitudes ψ_i,[SBM04]

?

- all ψ_i are easily computed from i and $|\psi_i|^2 \in \mathcal{O}(\frac{1}{2^n})$.[SS04, GR02]

Our states are **"in-between"**: States with

- an intermediate number of non-zero amplitudes,
- a polynomial number of *distinct* amplitudes,
- easily computed amplitudes ψ_i.

[SBM04]: Shende, Bullock, Markov, *Synthesis of Quantum Logic Circuits*, 2004
[SS04]: Soklakov, Schack, *Efficient state preparation for a register of quantum bits*, 2004
[GR02]: Grover, Rudolph, *Creating superpositions [..] efficiently integrable probability distributions*, 2002

Dicke States

The Dicke state $|D^n_k\rangle$ is an equal-weight superposition of all n-qubit computational basis states with Hamming Weight k, e.g.

$$|D^4_2\rangle = \frac{1}{\sqrt{6}}\left(|1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle\right).$$

We show how to prepare $|D^n_k\rangle$ with

- few gates – $\mathcal{O}(kn)$ many,
- low depth – $\mathcal{O}(n)$ steps,
- and no extra ancilla qubits.

Results for Symmetric States

Preparation	Type	Ancillas	Circuit Depth	Number of Gates
[CFGG]	Dicke States	$\mathcal{O}(\log n)$	$\mathcal{O}(n \text{ poly} \log n)$	$\mathcal{O}(n \text{ poly} \log n)$
[C+18]	W States	0	$\mathcal{O}(\log n)$	$\mathcal{O}(n)$
[KM04]	Symmetric States	$\mathcal{O}(\log(n/\varepsilon))$	$\mathcal{O}(n \text{ poly} \log(n/\varepsilon))$	$\mathcal{O}(n \text{ poly} \log(n/\varepsilon))$
Our result	Dicke States	0	$\mathcal{O}(n)$	$\mathcal{O}(kn)$
	Symmetric States	0	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$

[CFGG]: Childs, Farhi, Goldstone, Gutmann, *Finding cliques by quantum adiabatic evolution*, 2000
[C+18]: Cruz et al., *Efficient quantum algorithms for GHZ and W states, [..]*, 2018
[KM04]: Kaye, Mosca, *Quantum Networks for Generating Arbitrary Quantum States*, 2004

Compression	Type	Ancillas	Circuit Depth	Number of Gates
[BCH04]	Schur Transform	$\mathcal{O}(\log(n/\varepsilon))$	$\mathcal{O}(n \text{ poly} \log(n/\varepsilon))$	$\mathcal{O}(n \text{ poly} \log(n/\varepsilon))$
[PB09]	Symmetric States	0	$\mathcal{O}(n^2)$	$\mathcal{O}(n^2)$
Our result	Symmetric States	0	$\mathcal{O}(n \text{ poly} \log n)$	$\mathcal{O}(n^2)$

[BCH04]: Bacon, Chuang, Harrow, *Efficient Quantum Circuits for Schur and Clebsch-Gordan Transforms*, 2004
[PB09]: Plesch, Buzek, *Efficient compression of quantum information*, 2009

Dicke States

Dicke State Unitaries

Definition (Dicke State Unitaries $U_{n,k}$)

A **Dicke State Unitary** $U_{n,k}$ is any unitary which *for all* $\ell \leq k$ implements the mapping

$$U_{n,k}:\ |0\rangle^{\otimes n-\ell}\,|1\rangle^{\otimes \ell} = |\underbrace{0..0}_{n-\ell}\underbrace{1..1}_{\ell}\rangle \longrightarrow |D^n_\ell\rangle\,.$$

"for all $\ell \leq k$":

- allows for an Inductive Approach, constructing $U_{n,k}$ inductively over n,
- allows for Symmetric State Preparation, as $U_{n,n}$ can be used to construct *all* Dicke States.

Inductive Approach

Write Dicke States inductively as a superposition involving "smaller Dicke states", grouping terms by the last qubit being $|1\rangle$ or $|0\rangle$:

$$|D^n_\ell\rangle = \binom{n}{\ell}^{-\frac{1}{2}} \sum_{\substack{x \text{ has } n \text{ qubits} \\ \text{with exactly } \ell \text{ 1's}}} |x\rangle$$

$$= \sqrt{\tfrac{\ell}{n}}\, |D^{n-1}_{\ell-1}\rangle \otimes |1\rangle + \sqrt{\tfrac{n-\ell}{n}}\, |D^{n-1}_{\ell}\rangle \otimes |0\rangle \,.$$

Assume we know how to design $U_{n-1,k}$ but do not know how to design $U_{n,k}$:

- $|D^n_\ell\rangle$ *should* be prepared by $U_{n,k}$ from $|0\rangle^{\otimes n-\ell}|1\rangle^{\otimes \ell}$,
- $|D^{n-1}_{\ell-1}\rangle$ *can* be prepared by $U_{n-1,k}$ from $|0\rangle^{\otimes n-\ell}|1\rangle^{\otimes \ell-1}$,
- $|D^{n-1}_{\ell}\rangle$ *can* be prepared by $U_{n-1,k}$ from $|0\rangle^{\otimes n-\ell-1}|1\rangle^{\otimes \ell}$.

Observation: The only thing missing is a unitary which implements the mapping

$$|0\rangle^{\otimes n-\ell-1}|0\rangle|1\rangle^{\otimes \ell} \longrightarrow \sqrt{\tfrac{\ell}{n}}\, |0\rangle^{\otimes n-\ell-1}|0\rangle|1\rangle^{\otimes \ell-1}|1\rangle + \sqrt{\tfrac{n-\ell}{n}}\, |0\rangle^{\otimes n-\ell-1}|1\rangle^{\otimes \ell}|0\rangle \,.$$

Split & Cyclic Shift Unitaries

Definition (Split & Cyclic Shift Unitaries $SCS_{n,k}$)

A **Split & Cyclic Shift Unitary** $SCS_{n,k}$ is any unitary which for all $\ell \in 1, \dots, k$ and $k < n$ implements the mappings

$$\begin{aligned} SCS_{n,k}\colon\ & |0\rangle^{\otimes k+1} && \to |0\rangle^{\otimes k+1}\,, \\ SCS_{n,k}\colon\ & |0\rangle^{\otimes k+1-\ell}\,|1\rangle^{\otimes \ell} && \to \sqrt{\tfrac{\ell}{n}}\,|0\rangle^{\otimes k+1-\ell}\,|1\rangle^{\otimes \ell} + \sqrt{\tfrac{n-\ell}{n}}\,|0\rangle^{\otimes k-\ell}\,|1\rangle^{\otimes \ell}\,|0\rangle\,, \\ SCS_{n,k}\colon\ & |1\rangle^{\otimes k+1} && \to |1\rangle^{\otimes k+1}\,. \end{aligned}$$

$SCS_{n,k}$ unitaries can be built *explicitly*, we now construct $U_{n,k}$ unitaries *inductively*.

Review: Gate Overview

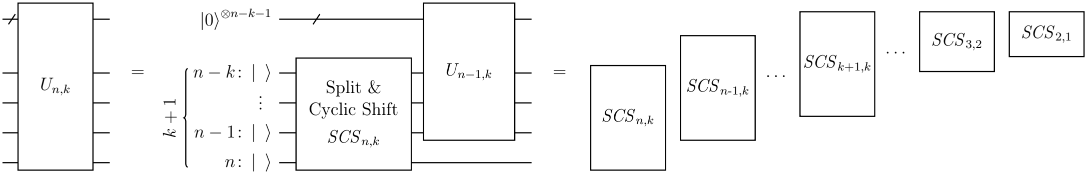
Goal: Build $U_{n,k}$ from
$SCS_{n,k}$ (acting on the last $k+1$ qubits) and $U_{n-1,k}$ (acting on the first $n-1$ qubits):

$$
\begin{aligned}
SCS_{n,k}:\ |0..0\rangle &\mapsto |00..000\rangle \\
|00..001\rangle &\mapsto \sqrt{\tfrac{1}{n}}\,|00..001\rangle + \sqrt{\tfrac{n-1}{n}}\,|00..010\rangle \\
|00..011\rangle &\mapsto \sqrt{\tfrac{2}{n}}\,|00..011\rangle + \sqrt{\tfrac{n-2}{n}}\,|00..110\rangle \\
&\ \ \vdots \\
|01..111\rangle &\mapsto \sqrt{\tfrac{k}{n}}\,|01..111\rangle + \sqrt{\tfrac{n-k}{n}}\,|11..110\rangle \\
|\underbrace{11..111}_{k+1}\rangle &\mapsto |11..111\rangle
\end{aligned}
$$

$$
\begin{aligned}
U_{n-1,k}:\ |0..0\rangle &\mapsto |D_0^{n-1}\rangle \\
|0..000..01\rangle &\mapsto |D_1^{n-1}\rangle \\
|0..000..11\rangle &\mapsto |D_2^{n-1}\rangle \\
&\ \ \vdots \\
|0..001..11\rangle &\mapsto |D_{k-1}^{n-1}\rangle \\
|\underbrace{0..0}_{n-1-k}\underbrace{11..11}_{k}\rangle &\mapsto |D_k^{n-1}\rangle
\end{aligned}
$$

Inductive Construction of $U_{n,k}$

The Split & Cyclic Shift unitaries $SCS_{n,k}$ act non-trivially only on the last $k+1$ of n qubits, and preceding $U_{n-1,k}$ by $SCS_{n,k}$ we get $U_{n,k}$.



Inductively applying this idea, we

- construct $U_{n,k}$ by concatenating $SCS_{n,k}, SCS_{n-1,k}, \ldots, SCS_{k+1,k}$ as a prefix to $U_{k,k}$,
- construct $U_{k,k}$ by concatenating $SCS_{k,k-1}, SCS_{k-1,k-2}, \ldots, SCS_{2,1}$.

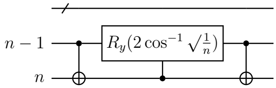
Explicit Construction of $SCS_{n,k}$

Recall:

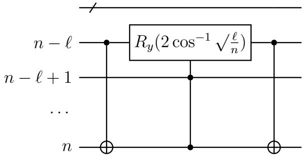
$$SCS_{n,k}:\ |0\rangle^{\otimes k-\ell}\,|0\rangle\,|1\rangle^{\otimes \ell-1}\,|1\rangle \to \sqrt{\tfrac{\ell}{n}}\,|0\rangle^{\otimes k-\ell}\,|0\rangle\,|1\rangle^{\otimes \ell-1}\,|1\rangle + \sqrt{\tfrac{n-\ell}{n}}\,|0\rangle^{\otimes k-\ell}\,|1\rangle\,|1\rangle^{\otimes \ell-1}\,|0\rangle$$

Use blocks of type:

$$(i)\ \ |00\rangle \to |00\rangle\,; \qquad |11\rangle \to |11\rangle$$
$$|01\rangle \to \sqrt{\tfrac{1}{n}}\,|01\rangle + \sqrt{\tfrac{n-1}{n}}\,|10\rangle$$



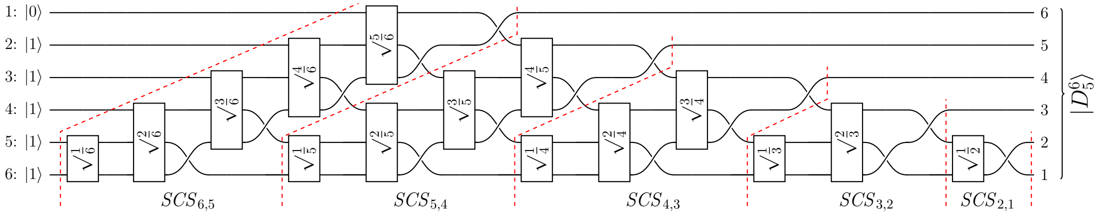
$$(ii)\ \ |000\rangle \to |000\rangle\,; \quad |001\rangle \to |001\rangle$$
$$|010\rangle \to |010\rangle\,; \quad |111\rangle \to |111\rangle$$
$$|011\rangle \to \sqrt{\tfrac{\ell}{n}}\,|011\rangle + \sqrt{\tfrac{n-\ell}{n}}\,|110\rangle$$



with (controlled) Y-rotations $R_y(2\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

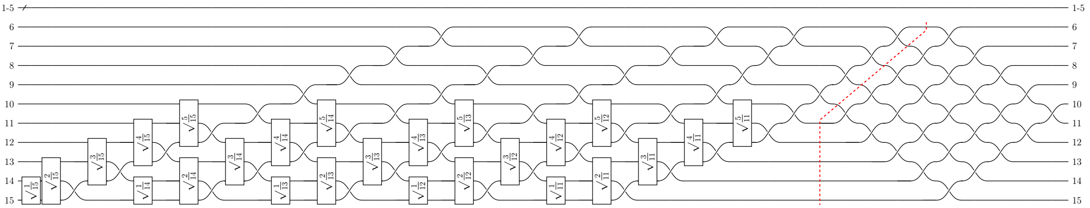
Combining all Ideas

- To build $U_{n,k}$ with $k \in \Theta(n)$:
  Each of the $n-1$ many *SCS* unitaries described above is constructed with $\mathcal{O}(n)$ 3-qubit and $\mathcal{O}(n)$ SWAP-gates in a stair-like shape.
  ⇒ Parallelizes well: $\mathcal{O}(n^2)$ gates in $\mathcal{O}(n)$ steps.
  ⇒ Even when 2-qubit gates are only allowed between neighbours.



Combining all Ideas

- To build $U_{n,k}$ with $k \in \Theta(n)$:
  Each of the $n-1$ many *SCS* unitaries described above is constructed with $\mathcal{O}(n)$ 3-qubit and $\mathcal{O}(n)$ SWAP-gates in a stair-like shape.
  $\Rightarrow$ Parallelizes well: $\mathcal{O}(n^2)$ gates in $\mathcal{O}(n)$ steps.
  $\Rightarrow$ Even when 2-qubit gates are only allowed between neighbours.
- To build $U_{n,k}$ with $k \in o(n)$:
  Make $\mathcal{O}(n/k)$ groups of $\mathcal{O}(k)$ many *SCS* unitaries which are constructed with $\mathcal{O}(k)$ 3-qubit and $\mathcal{O}(k)$ SWAP-gates.
  $\Rightarrow$ Parallelizes well: $\mathcal{O}(kn)$ gates in $\mathcal{O}(n)$ steps.
  $\Rightarrow$ Even when 2-qubit gates are only allowed between neighbours.



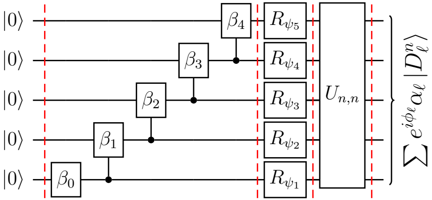
Arbitrary Symmetric Pure States

Preparation of Symmetric Pure States

Every n-qubit symmetric pure state can be written as a superposition of the $n+1$ Dicke states $|D_0^n\rangle, |D_1^n\rangle, \dots, |D_n^n\rangle$ in the form $\sum_\ell e^{i\phi_\ell}\alpha_\ell |D_\ell^n\rangle$,
with magnitudes $\alpha_\ell \in [0,1]$, $\alpha_0^2 + \ldots + \alpha_n^2 = 1$, and phases $\phi_\ell \in [0, 2\pi)$, $\phi_0 = 0$.

Given the unitary $U_{n,n}$, it remains to prepare $\sum_\ell e^{i\phi_\ell}\alpha_\ell |0\rangle^{\otimes n-\ell} |1\rangle^{\otimes \ell}$. To this end, we define

- magnitudes β_ℓ such that $\alpha_\ell = \beta_\ell \cdot \prod_{j=0}^{\ell-1} \sqrt{1-\beta_j^2}$,
- and angles ψ_ℓ such that $\phi_\ell = \sum_{j=0}^{\ell} \psi_j$.



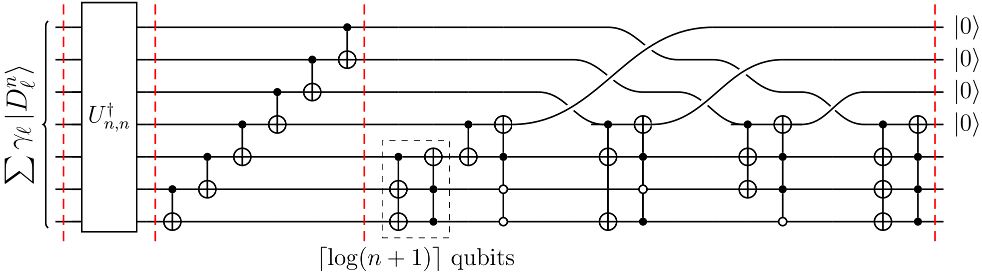
Using Y-rotations $R_y(2\cos^{-1}\beta_\ell)$ and phase-shift gates $R_{\psi_\ell} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\psi_\ell} \end{pmatrix}$ yields:

$$|0\rangle^{\otimes n} \xrightarrow{\beta_0,\dots,\beta_{n-1}} \sum_\ell \alpha_\ell |0\rangle^{\otimes n-\ell} |1\rangle^{\otimes \ell} \xrightarrow{\psi_1,\dots,\psi_n} \sum_\ell e^{i\phi_\ell}\alpha_\ell |0\rangle^{\otimes n-\ell} |1\rangle^{\otimes \ell} \xrightarrow{U_{n,n}} \sum_\ell e^{i\phi_\ell}\alpha_\ell |D_\ell^n\rangle .$$

Compression of Symmetric Pure States

Quasilinear-depth quantum compression circuit:

Compressing a symmetric n-qubit pure state $\sum_\ell \gamma_\ell |D^n_\ell\rangle$ into $\lceil \log(n+1) \rceil$ qubits:



- The inverse unitary $U^\dagger_{n,n}$ maps each Dicke state $|D^n_\ell\rangle$ to $|0\rangle^{\otimes n-\ell} |1\rangle^{\otimes \ell}$,
- which gets mapped to ℓ's one-hot encoding $|0\rangle^{\otimes n-\ell} |1\rangle |0\rangle^{\otimes \ell-1}$,
- and finally to ℓ's binary encoding $|\ell\rangle$ (with padded zeroes).

$$\sum_\ell \gamma_\ell |D^n_\ell\rangle \xrightarrow{U^\dagger_{n,n}} \sum_\ell \gamma_\ell |0\rangle^{\otimes n-\ell} |1\rangle^{\otimes \ell} \xrightarrow[\text{stair}]{\textit{CNOT}} \sum_\ell \gamma_\ell |0\rangle^{\otimes n-\ell} |1\rangle |0\rangle^{\otimes \ell-1} \xrightarrow[\text{change}]{\text{encoding}} \sum_\ell \gamma_\ell |\ell\rangle .$$

Conclusion

We have shown how to prepare Dicke States $|D_k^n\rangle$ – and by extension Symmetric Pure States – deterministically on circuits with Linear Nearest Neighbor architecture with

- $\mathcal{O}(kn)$ many gates,
- $\mathcal{O}(n)$ depth,
- and no ancilla qubits.

This also gives a quasilinear quantum compression circuit for Symmetric Pure States.

Open problems:

- Can we implement Dicke States in polylogarithmic depth (for small k)?
- Is there an upper or lower bound for state preparation of states with a small number of *distinct* non-zero amplitudes?