LA-UR-19-27665

Quantum Computing Algorithms

Tricks and Tools

LLNL CASIS Quantum Sensing and Information Processing Series

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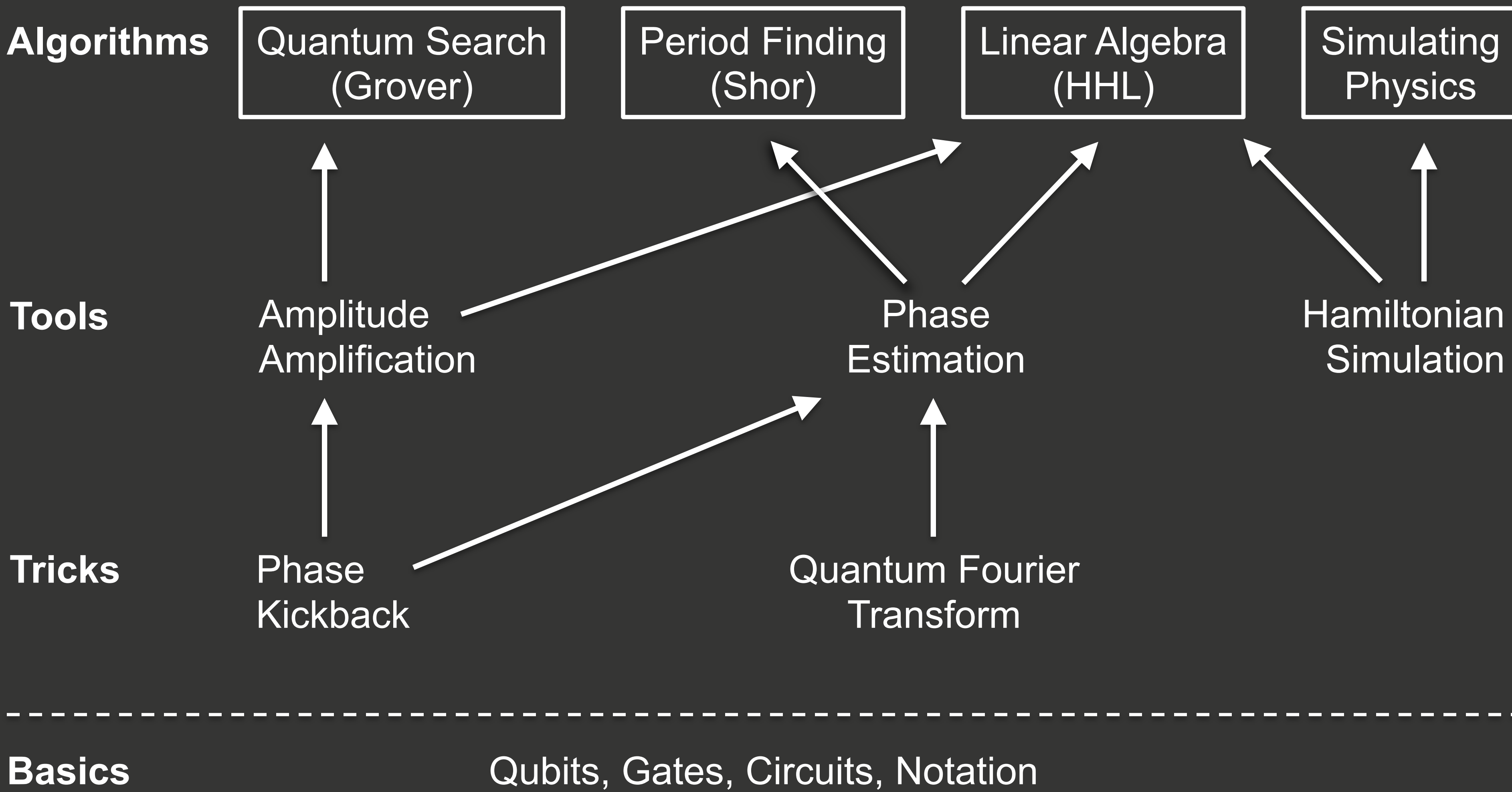
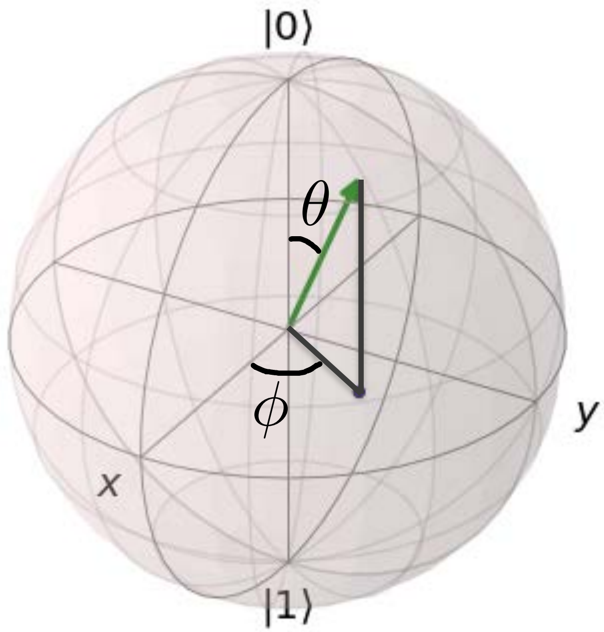

Algorithms
Quantum Search (Grover)
Period Finding (Shor)
Linear Algebra (HHL)
Simulating Physics
Tools
Amplitude Amplification
Phase Estimation
Hamiltonian Simulation
Tricks
Phase Kickback
Quantum Fourier Transform
Basics
Qubits, Gates, Circuits, Notation

Theory (Slides) and Practice (Quirk)

https://cnls.lanl.gov/~baertschi/QCA/

From Bits to Qubits

Bits: Either 0 or 1



Qubits: $\alpha|0\rangle + \beta|1\rangle$ with $\alpha, \beta \in \mathbb{C}$

$$|\alpha|^2 + |\beta|^2 = 1$$

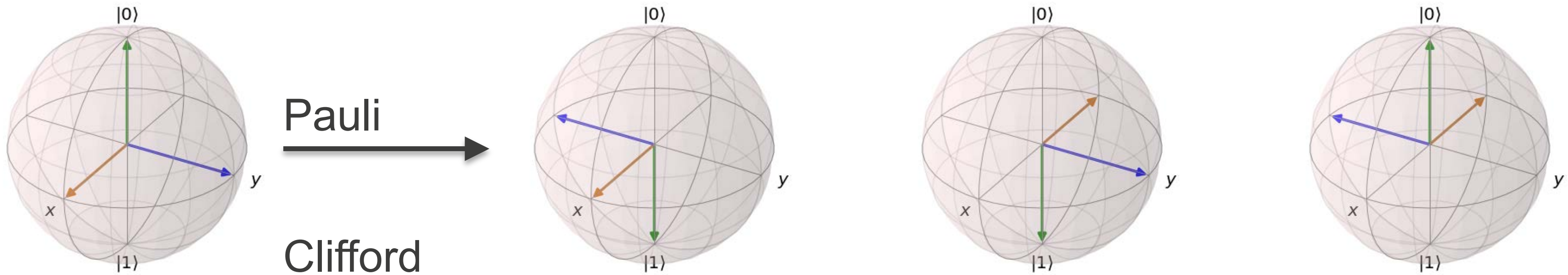
$$\Rightarrow e^{i\delta}\left(\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\right)$$

global phase ($e^{i\delta}$)   relative phase ($e^{i\phi}$)

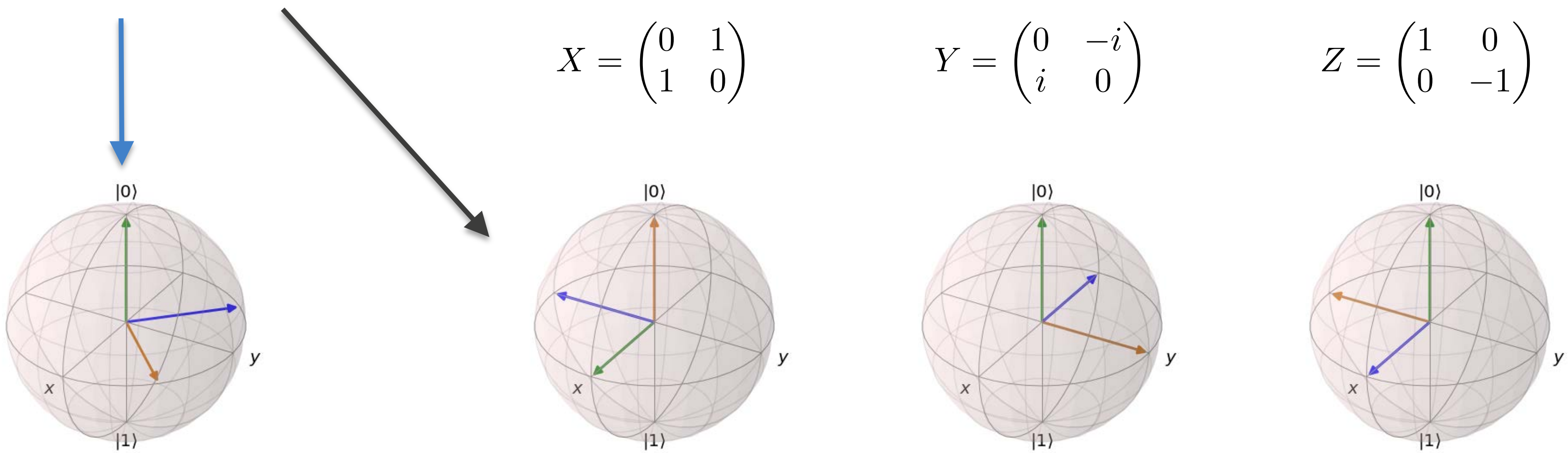
$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$|+\rangle := \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad |-\rangle := \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Single Qubit Gates



$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \qquad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} \qquad S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/2} \end{pmatrix}$$

Multiple Qubits

$$|\Psi\rangle \otimes |\Phi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \otimes \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \\ \psi_2 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \psi_1 \cdot \phi_1 \\ \psi_1 \cdot \phi_2 \\ \psi_2 \cdot \phi_1 \\ \psi_2 \cdot \phi_2 \end{pmatrix}$$

$$|\Psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad |\Phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Example:

$$|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

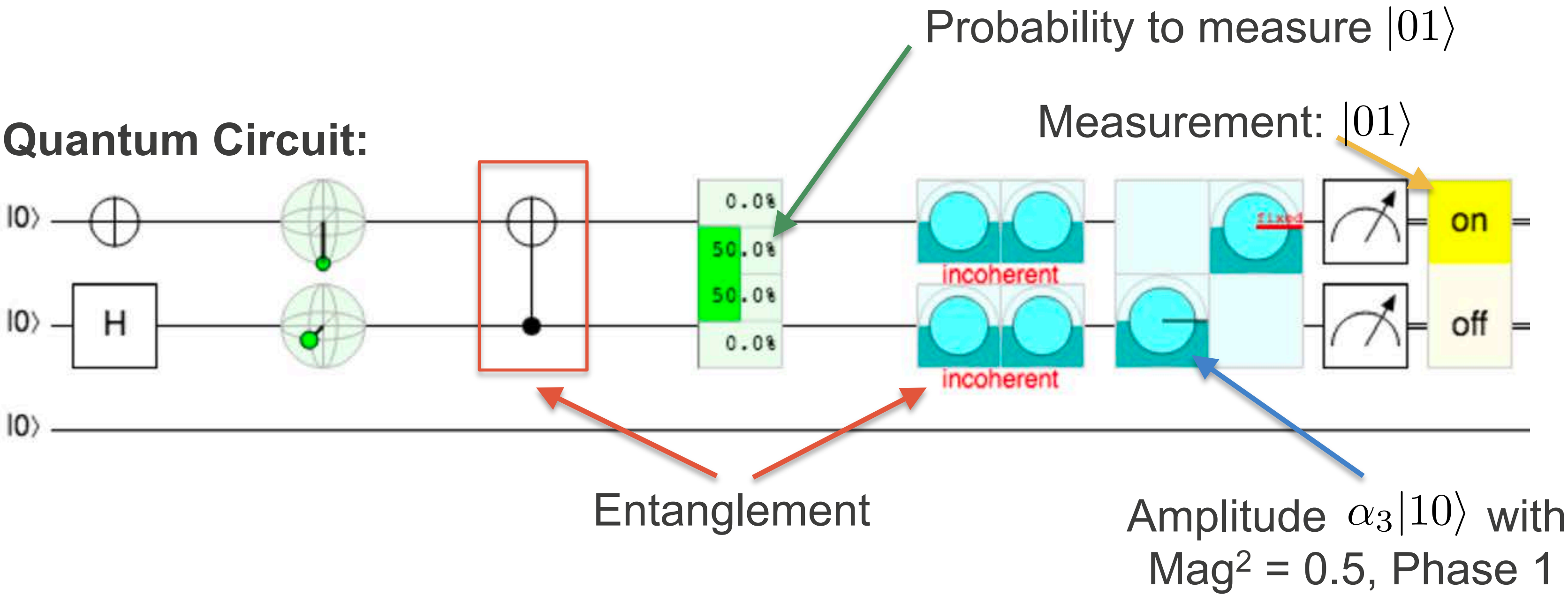
← 3rd Entry: $|3\rangle$ or $|10\rangle$

_n_ Qubits live in a _N_ = 2^n-dimensional Hilbert Space:

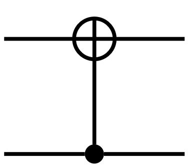
$$\alpha_0|0..00\rangle + \alpha_1|0..01\rangle + \alpha_2|0..10\rangle + \alpha_3|0..11\rangle + \ldots + \alpha_{2^n-1}|1..11\rangle$$

2-Qubit Gates and Entanglement

Quantum Circuit:

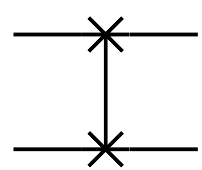


Notable 2-Qubit Gates:



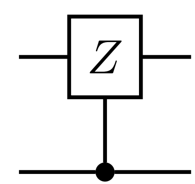
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{array}{lcl} |00\rangle & \rightarrow & |00\rangle \\ |01\rangle & \rightarrow & |01\rangle \\ |10\rangle & \rightarrow & |11\rangle \\ |11\rangle & \rightarrow & |10\rangle \end{array}$$

CNOT



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

SWAP



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Controlled-Z

Algorithms
given by quantum circuits with 1- and 2-qubit gates

Qubits:

- Total Number of Qubits
- Number of Ancillas (“extra qubits”)

Gates:

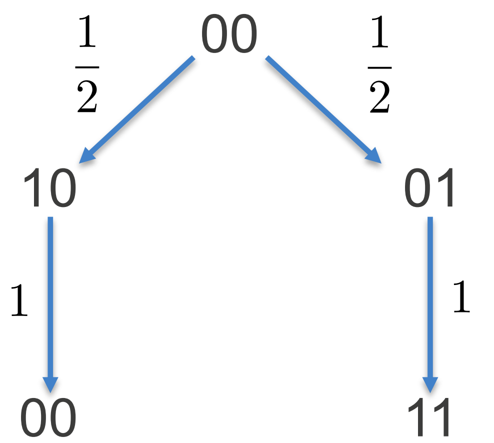
- Total Number of Gates
- Depth of Circuit
- Number of T-Gates

Measurements:

- Total Number of Measurements
- Number of Consecutive Measurements (Error Correction)

Quantum Computing in One Slide

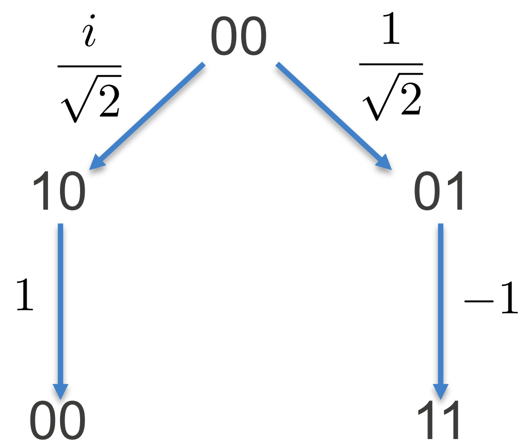
Classical Randomized Algorithm:



$$\Pr[x] = \sum_{\text{paths } p \text{ to } x} \prod_{\text{edges } e \text{ in } p} p_e$$

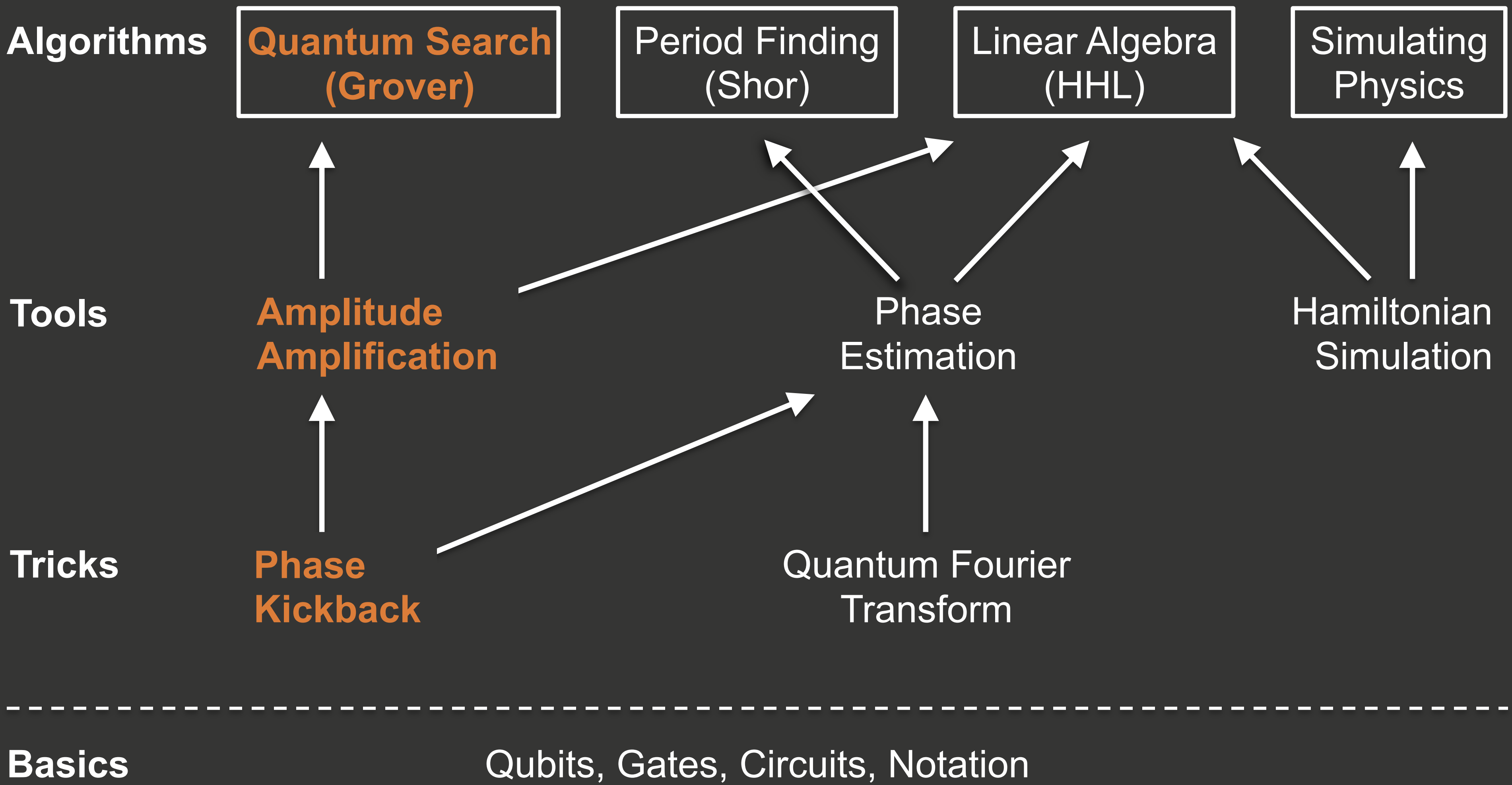
Stochastic Matrices

Quantum Algorithm:



$$\Pr[x] = \left| \sum_{\text{paths } p \text{ to } x} \prod_{\text{edges } e \text{ in } p} \alpha_e \right|^2$$

Unitary Matrices $U^{-1} = U^{\dagger}$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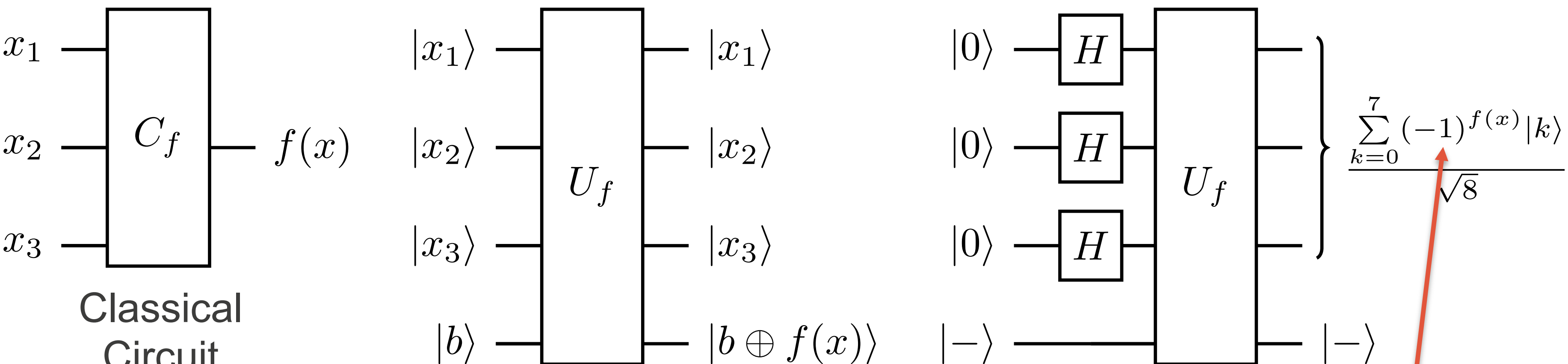
3SAT: hard to solve, easy to verify

$$f(x) = f(x_1, x_2, x_3) = (\overline{x_1}) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2)$$

Only one solution: $(x_1, x_2, x_3) = (0, 1, 1)$



Classical Circuit (irreversible)

Quantum Circuit (reversible)

Phase Kickback

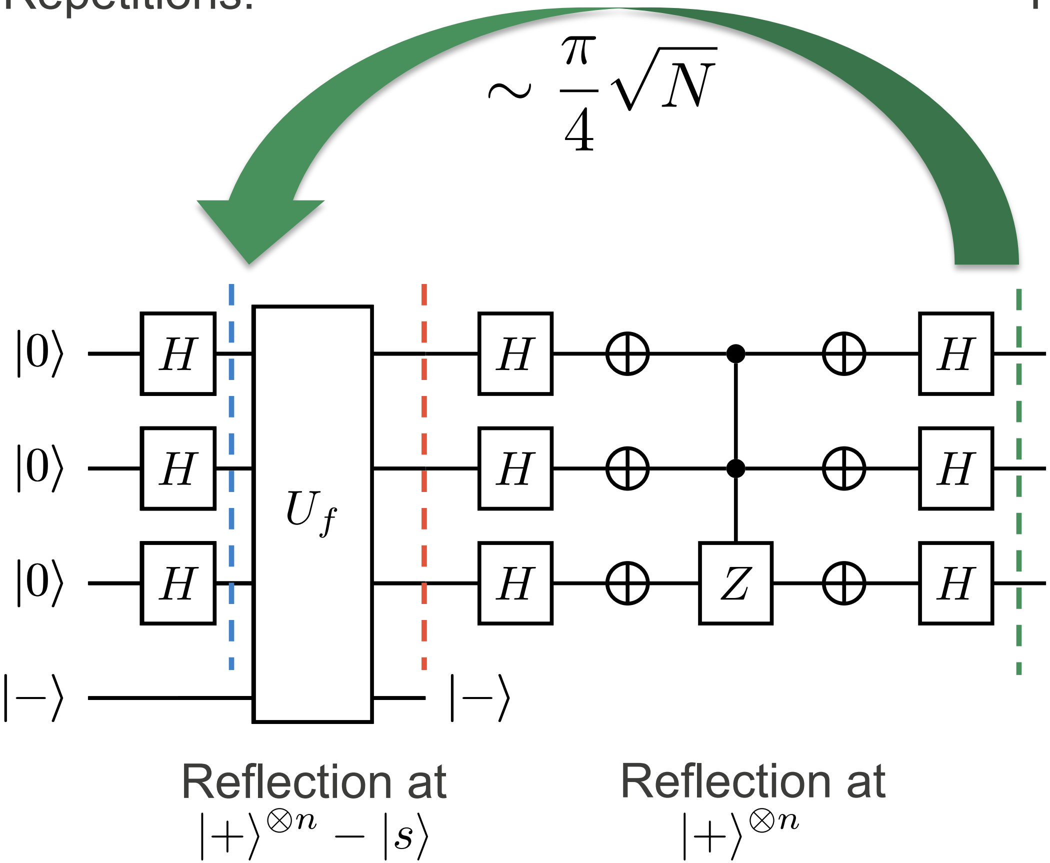
$$\frac{|0 \oplus 1\rangle - |1 \oplus 1\rangle}{\sqrt{2}} = \frac{|1\rangle - |0\rangle}{\sqrt{2}} = -\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

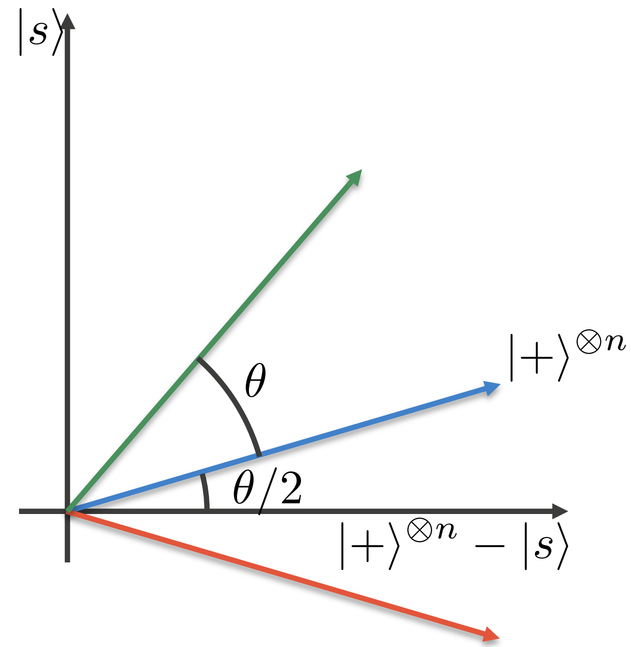
Amplitude Amplification

Repetitions:

Rotation Angle: $\sin(\theta/2) = \frac{1}{\sqrt{N}}$

$\Rightarrow \theta \approx \frac{2}{\sqrt{N}}$





Quantum Search (Grover)

Searching over all Function Inputs with $\mathcal{O}(\sqrt{N})$ Iterations

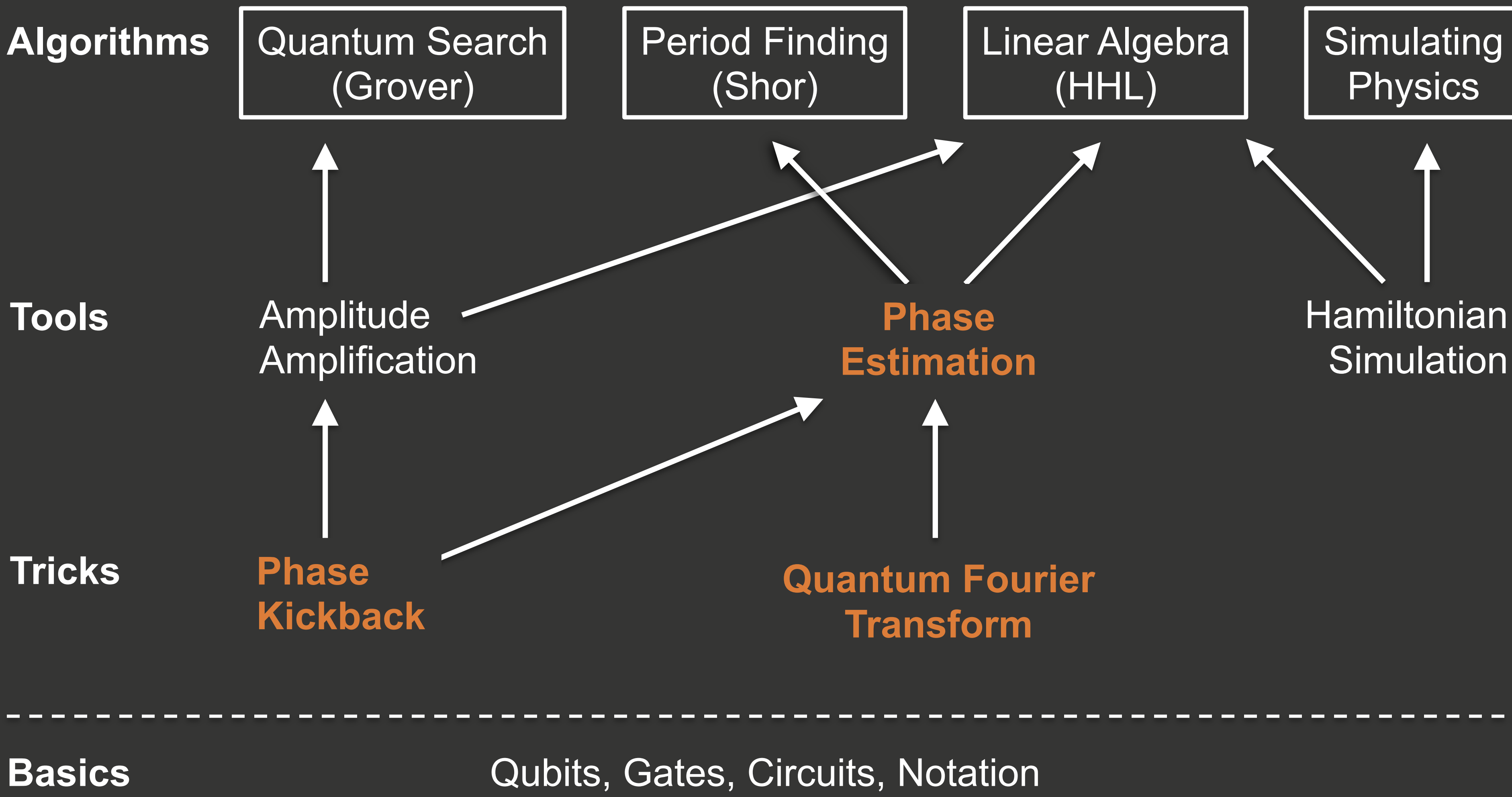
Details

- What when there are M items?
  $\Rightarrow \mathcal{O}(\sqrt{N/M})$ Iterations
- Grover is known to be optimal (in black-box sense).
- 3SAT Grover $\widetilde{\mathcal{O}}(\sqrt{2^n}) \approx \widetilde{\mathcal{O}}(1.41^n)$
  - vs. Schöning Randomized Alg. with $\approx \widetilde{\mathcal{O}}(1.34^n)$ rounds of success probability $\approx \frac{1}{1.34^n}$
  - Use Amplification $\Rightarrow \widetilde{\mathcal{O}}(\sqrt{1.34^n})$!

Applications

- Database Search?
  needs Quantum Memory…
- Quantum Speedups for Dynamic Programming
- Distributed QC
  - Sublinear Diameter Computation in CONGEST model

Quantum Speedup: Polynomial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Quantum Fourier Transform

$$QFT: |x\rangle = |x_4x_3x_2x_1\rangle \rightarrow \frac{1}{\sqrt{2^4}} \sum_{k=0}^{2^4-1} e^{2\pi i k x / 2^4} |k\rangle$$

$$= \frac{1}{\sqrt{2^4}} (|0\rangle + e^{2\pi i 0.x_1}|1\rangle) \otimes (|0\rangle + e^{2\pi i 0.x_2x_1}|1\rangle)$$

$$\otimes (|0\rangle + e^{2\pi i 0.x_3x_2x_1}|1\rangle) \otimes (|0\rangle + e^{2\pi i 0.x_4x_3x_2x_1}|1\rangle)$$

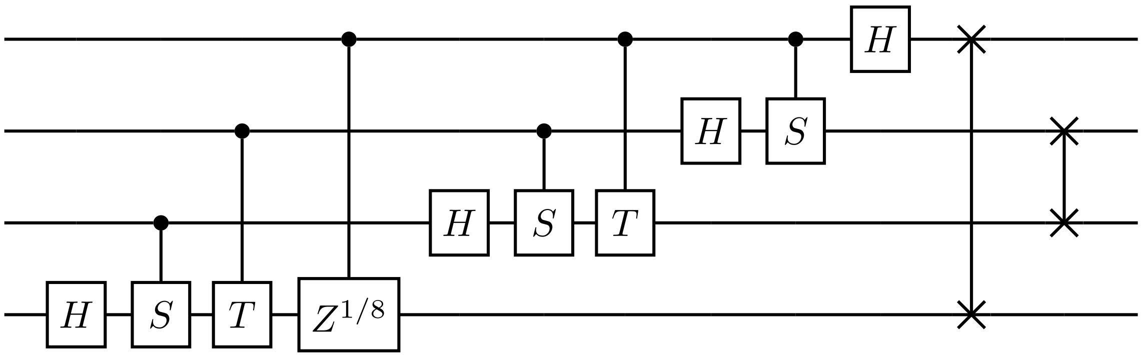
$$0.x_2x_1 = \frac{x_2}{2} + \frac{x_1}{4}$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle \pm e^{\pi i x_3/2} \cdot e^{\pi i x_2/4} \cdot e^{\pi i x_1/8}|1\rangle\right)$$

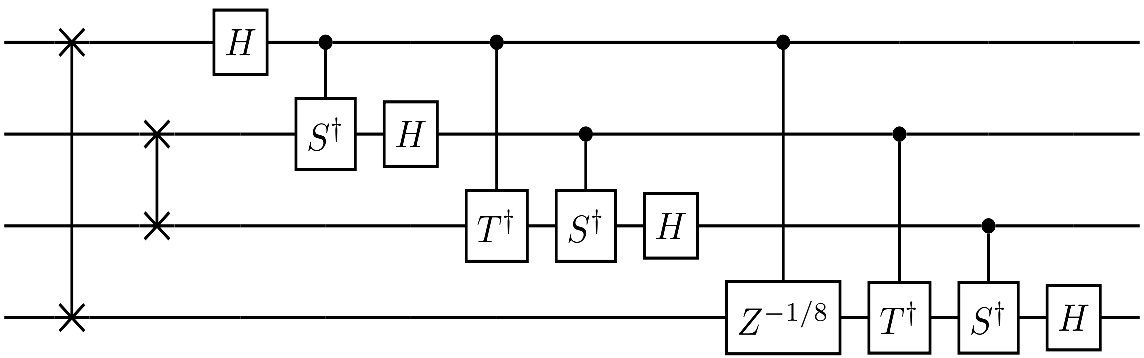
- $\pm$: H
- $e^{\pi i x_3/2}$: $S = Z^{1/2}$
- $e^{\pi i x_2/4}$: $T = Z^{1/4}$
- $e^{\pi i x_1/8}$: $Z^{1/8}$

QFT Circuits

$QFT:$



$QFT^\dagger:$



Quantum Speedup: Exponential

Quantum Phase Estimation

Finding the phase of a Unitary and its Eigenstate

Phase $\phi \in [0, 1)$

For a Unitary U and an Eigenstate $|u\rangle$ we always have:

$$U|u\rangle = e^{2\pi i\phi}|u\rangle = e^{2\pi i\, \underline{0.\phi_n\phi_{n-1}\ldots\phi_2\phi_1}}|u\rangle$$

Terms from QFT!

Note that:

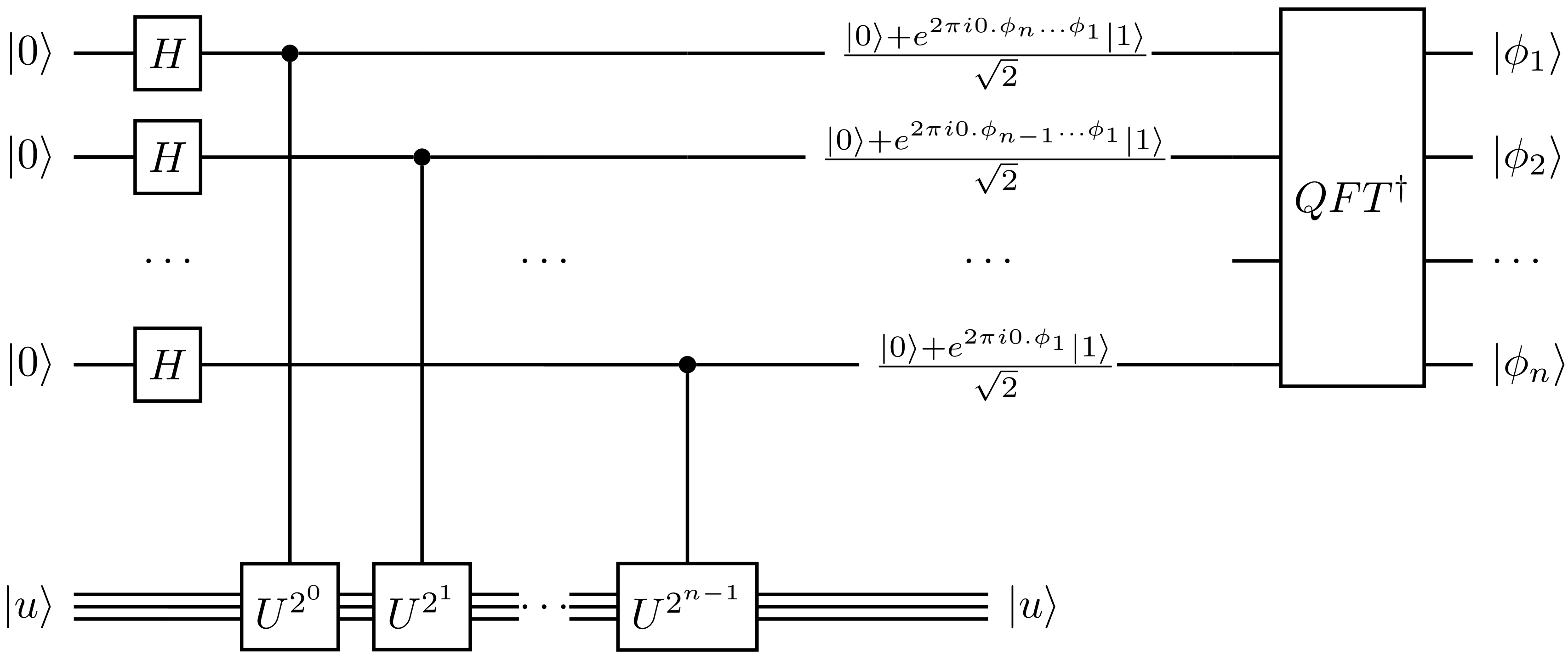
$$U^2|u\rangle = e^{2\pi i\, \phi_n.\phi_{n-1}\ldots\phi_2\phi_1}|u\rangle = e^{2\pi i\, \underline{0.\phi_{n-1}\ldots\phi_2\phi_1}}|u\rangle$$

$$\ldots$$

$$U^{2^{n-1}}|u\rangle = e^{2\pi i\, \phi_n\phi_{n-1}\ldots\phi_2.\phi_1}|u\rangle = e^{2\pi i\, \underline{0.\phi_1}}|u\rangle$$

If only one could kick back the phases into an (ancilla) register and do inverse Quantum Fourier Transform on it…

QPE Circuit


|0⟩
|0⟩
|0⟩
|u⟩
H
H
H
U^{2^0}
U^{2^1}
$U^{2^{n-1}}$
$\frac{|0\rangle + e^{2\pi i 0.\phi_n \cdots \phi_1}|1\rangle}{\sqrt{2}}$
$\frac{|0\rangle + e^{2\pi i 0.\phi_{n-1} \cdots \phi_1}|1\rangle}{\sqrt{2}}$
$\frac{|0\rangle + e^{2\pi i 0.\phi_1}|1\rangle}{\sqrt{2}}$
$QFT^\dagger$
$|\phi_1\rangle$
$|\phi_2\rangle$
$|\phi_n\rangle$
|u⟩

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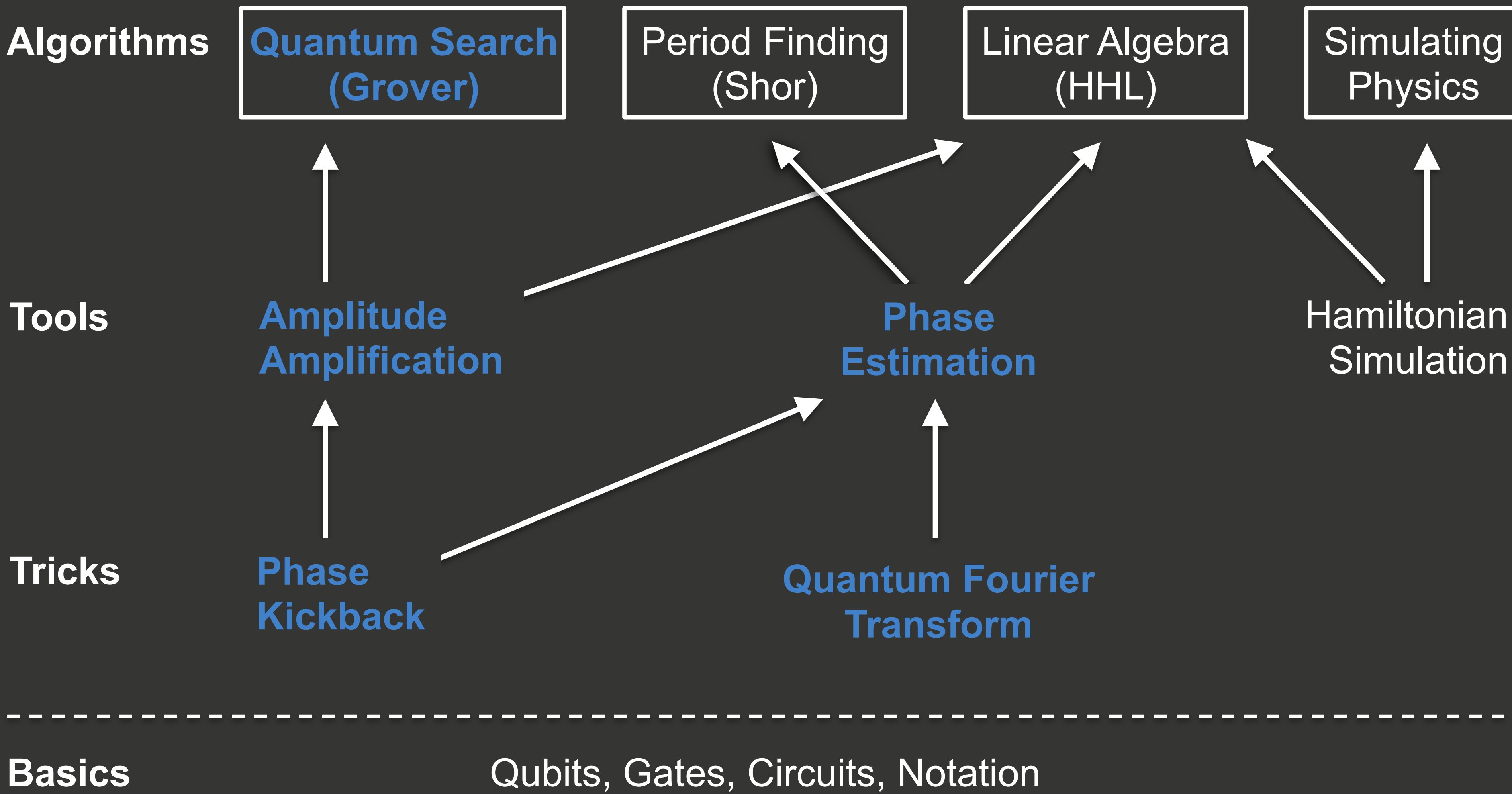
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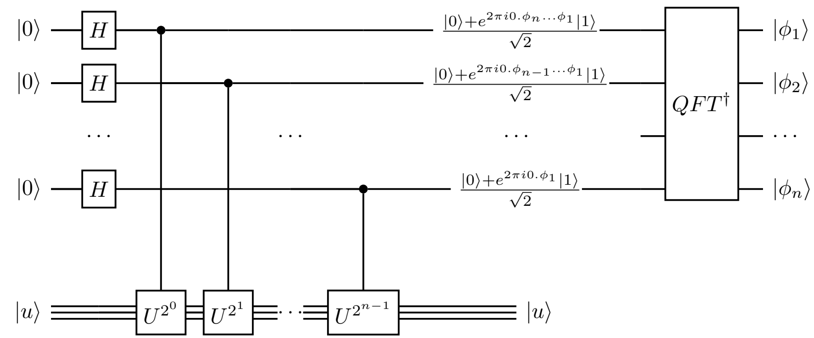
Recap

Amplitude Amplification

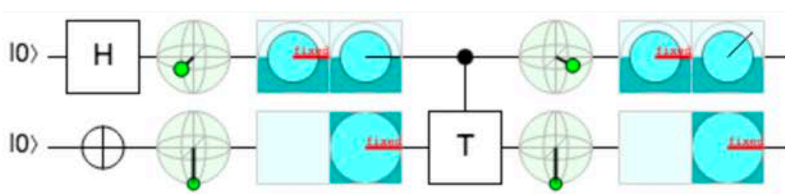
Given algorithm A with success probability *p*, one can do

- **Classical**
  Repeating A for 1/*p* times
- **Quantum**
  Boost amplitudes
  over √1/p rounds

Phase Estimation



Phase Kickback



Quantum Fourier Transform

QFT

$$\bigotimes_{k=n}^{1} |\phi_k\rangle \qquad \bigotimes_{k=1}^{n} \left(|0\rangle + e^{2\pi i\, 0.\phi_k \ldots \phi_1}\, |1\rangle\right)$$

Inverse QFT

Recap Phase Estimation

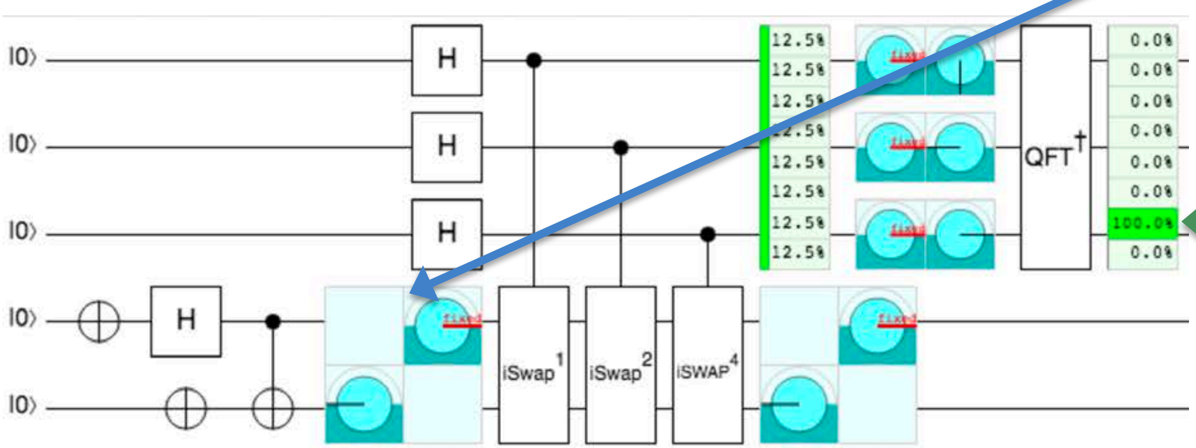
Estimating Phases for the iSWAP gate:

$$
\begin{array}{c} \\ \\ \end{array}
\begin{matrix} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{matrix}
$$

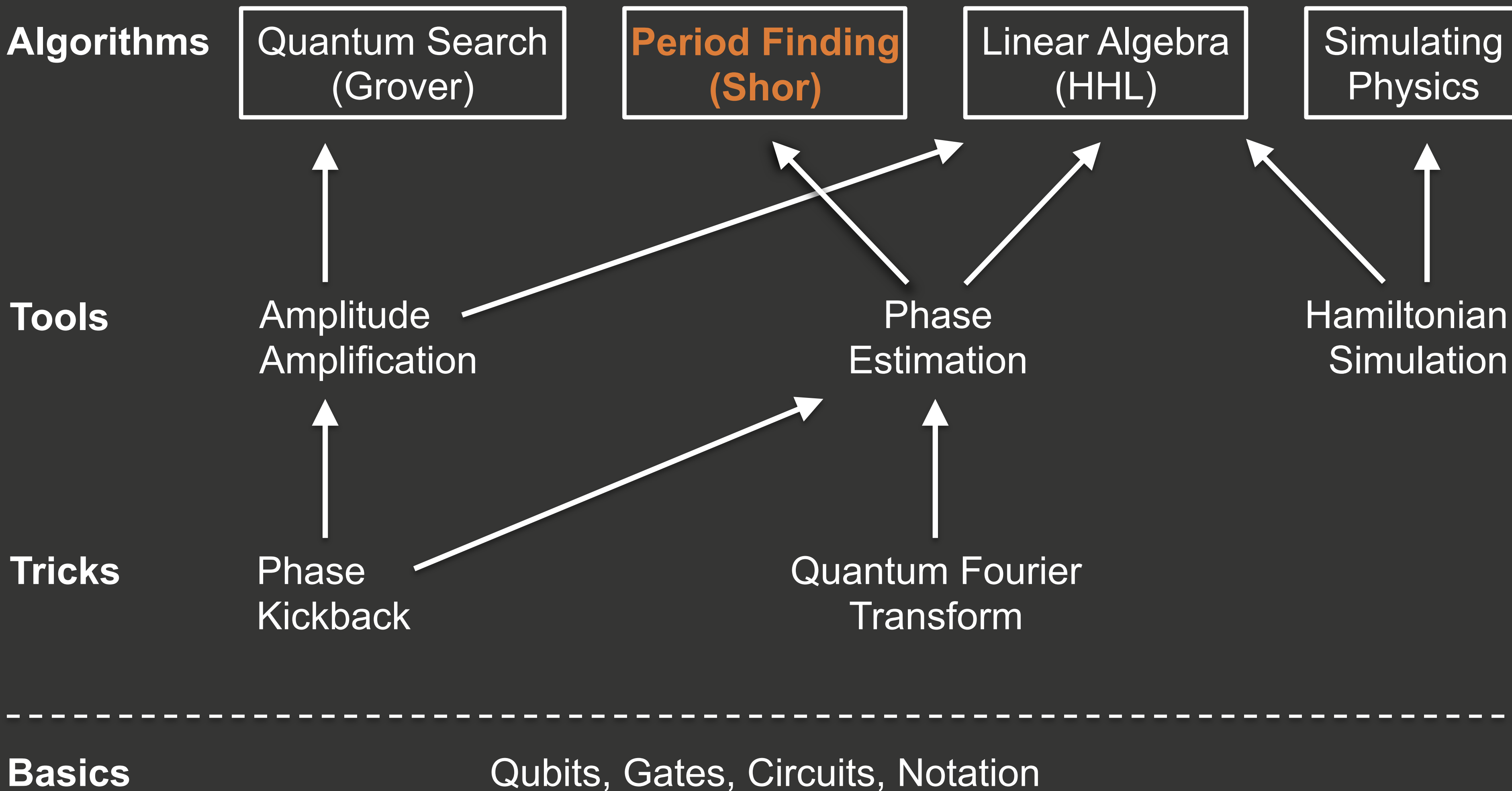
$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}
$$

With Eigenstates and corresponding Eigenvalues:

$\vert 00\rangle$	$\frac{\vert 01\rangle + \vert 10\rangle}{\sqrt{2}}$	$\frac{\vert 01\rangle - \vert 10\rangle}{\sqrt{2}}$	$\vert 11\rangle$
1	i	$-i$	1



$$e^{2\pi i\, 0.110}$$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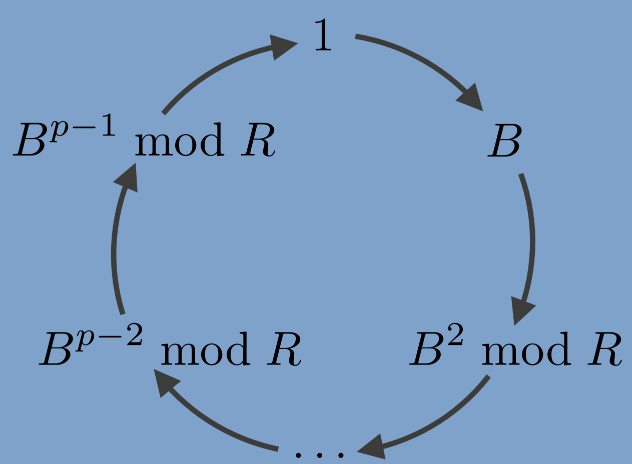
Integer Factorization (Shor)

Given *n*-bit (composite) integer *R*, find a proper factor *f*

Classical Part:
Pre- / Post-Processing, Loop

1. Discard “simple” cases:
   R even, *R* prime power, (*R* prime)
2. Randomly choose 0 < *B* < *R*
   - check $f = \gcd(B, R)$
3. Compute period *p* of *B*
4. If *p* is even:
   - check $f = \gcd(B^{p/2} + 1, R)$
5. Repeat if necessary

Quantum Part:
Find Period *p* of
$B^x \equiv 1 \pmod{R}$



Period Finding I (Shor)

If *B* has period *p* modulo *R*, the unitary operator $\times B \pmod{R}$ has:

- *p* Eigenstates $$|u_s\rangle = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} e^{-2\pi i k s/p} \left| B^k \bmod R \right\rangle$$
- with Eigenvalues $e^{2\pi i s/p}$ Phase Estimation!

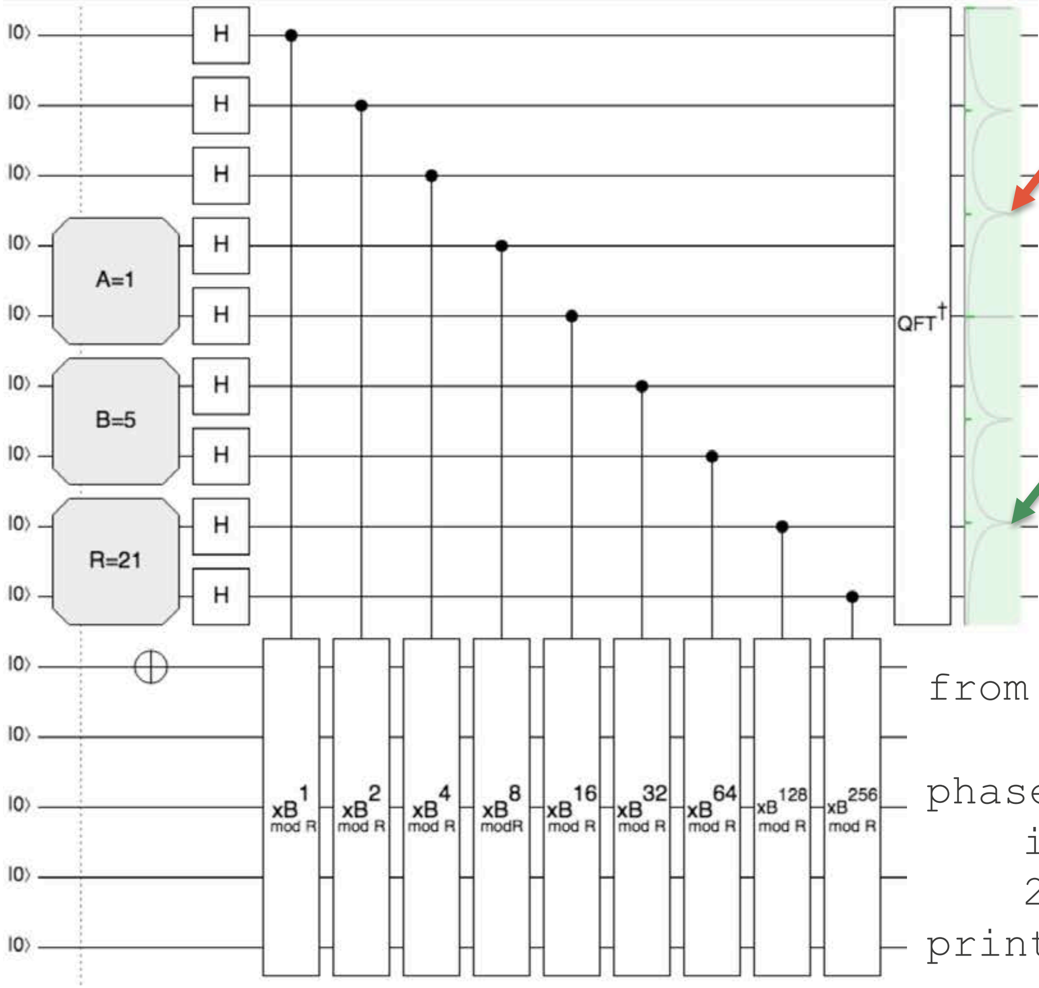
Need phase s/p to determine the period *p*,
but need the period *p* to create $|u_s\rangle$.

Start in $|1\rangle = \frac{1}{\sqrt{p}} \sum_{s=0}^{p-1} |u_s\rangle$ **!!**

$5^k \bmod 21:$

$|5\rangle \to |4\rangle \to |20\rangle \to |16\rangle \to |17\rangle \to |1\rangle$

Period Finding II (Shor)



Phase $s/p \approx 2/6 = 1/3$ gives incorrect period ⇒ fail ⇒ repeat

Phase $427/512 \approx 5/6$

Use continued fractions to find the "real fraction"

```
from fractions import Fraction

phase = Fraction(
    int("110101011", 2),
    2**9)
print(phase.limit_denominator(21))
```

Extra square roots (Shor)

If R is composite and not a prime power,

the number 1 has at least four square roots modulo R (not only 1, -1):

$$B^p \equiv 1 \pmod{R}$$

$$B^{p/2} \cdot B^{p/2} \equiv 1 \pmod{R}$$

$$(B^{p/2} - 1) \cdot (B^{p/2} + 1) \equiv 0 \pmod{R}$$

Want / hope for: $B^{p/2} + 1 \not\equiv 0 \pmod{R}$

Integer Factorization (Shor)

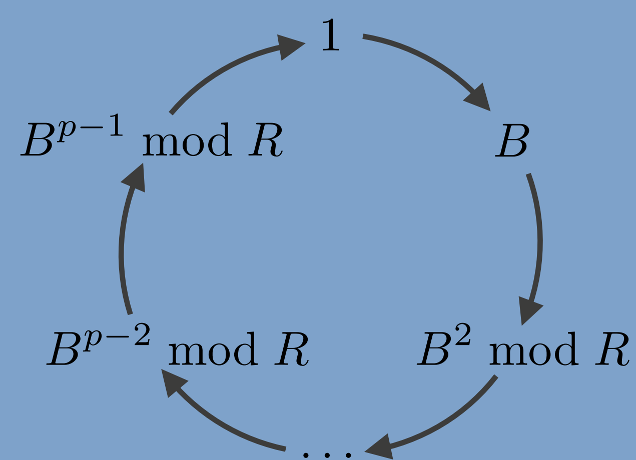
Given *n*-bit integer *R*, find a proper factor *f*

Classical Part:
Pre- / Post-Processing, Loop

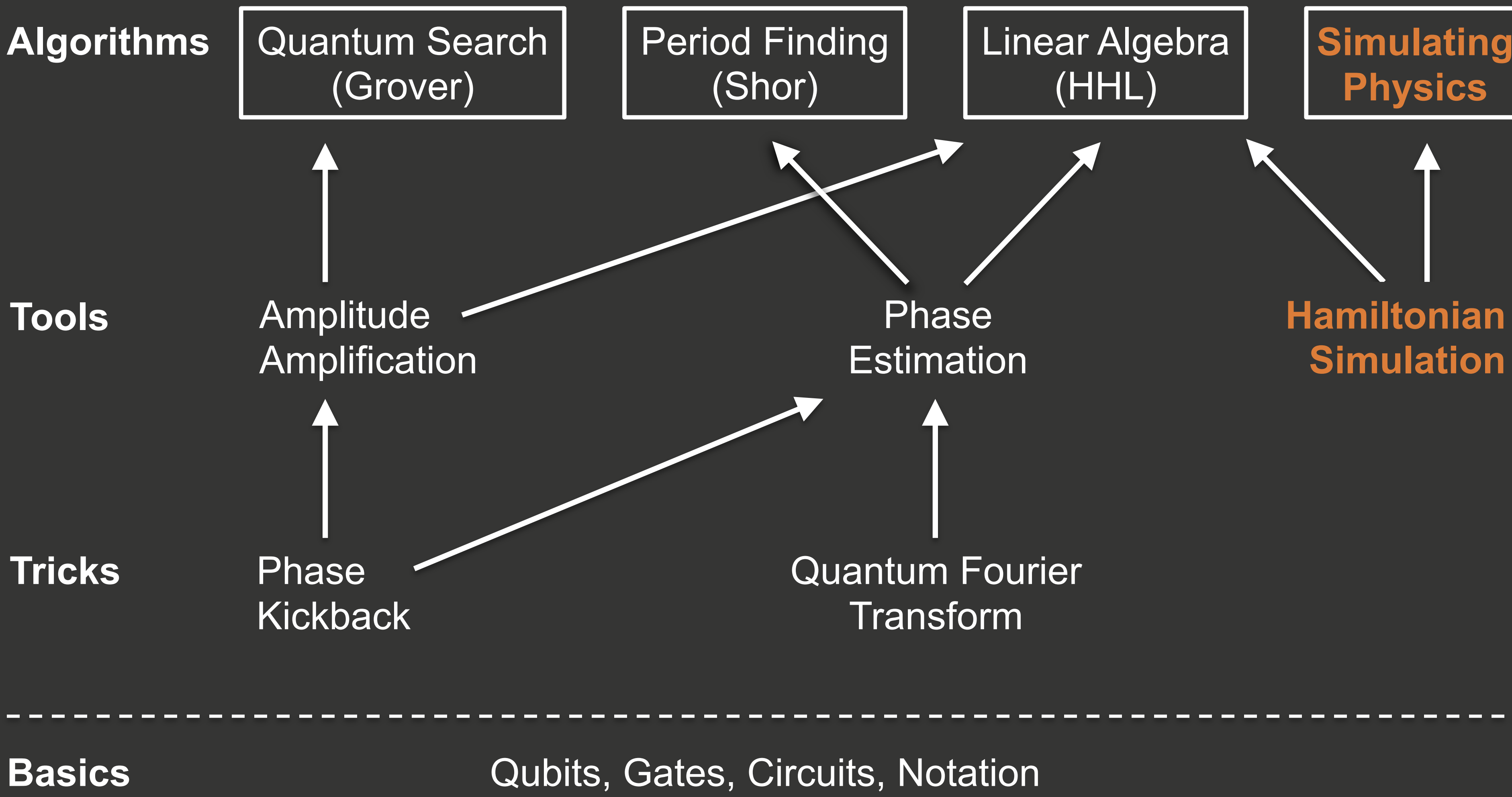
1. Discard "simple" cases:
   R even, *R* prime, *R* prime power
2. Randomly choose 0 < *B* < *R*
   - check $f = \gcd(B, R)$
3. Compute period *p* of *B*
4. If *p* is even:
   - check $f = \gcd(B^{p/2} + 1, R)$
5. Repeat if necessary (~ prob. 0.75)

Quantum Part:
Find Period *p* of

$B^x \equiv 1 \pmod{R}$



Everything $poly(n)$ vs. classical $e^{\sqrt[3]{n}}$: **Quantum Speedup: Superpolynomial**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Hamiltonian Simulation
Problem Definition

Schrödinger Equation:

$$i \frac{d\left|\Psi(t)\right\rangle}{dt} = H\left|\Psi(t)\right\rangle$$

Time-independent Hamiltonian:

$$\left|\Psi(t)\right\rangle = e^{-itH}\left|\Psi(0)\right\rangle$$

Task:

Find a quantum circuit, that simulates e^{-itH} as close as possible!

Hamiltonian Simulation
Two notable Problem Types

From Physics

- *H* is a sum of Pauli Tensor Product terms: $$H = \sum_{j=1}^{m} H_j$$
- with, e.g., acting on qubits 3, 2, 1, respectively. $$H_j = Z_3 \otimes Y_2 \otimes X_1$$
- Structure from Physics through Jordan-Wigner transform.

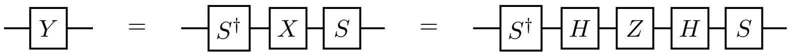
From Maths & CS

- *H* is a *s*-sparse (at most *s* non-zero entries per row and per column)
- Given row *i* and number *x*, there must be a fast way ("Oracle") to get
  - the index *j* and
  - the matrix entry $H_{i,j}$

  of x^{th} non-zero entry in row *i.*

Pauli Tensor Products
can be simulated exactly by **Diagonalization**

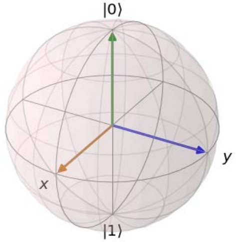
Axis Change



$$Y = S \cdot H \cdot Z \cdot H \cdot S^\dagger$$

$$\Rightarrow e^{-itY} = e^{-it \cdot S \cdot H \cdot Z \cdot H \cdot S^\dagger}$$

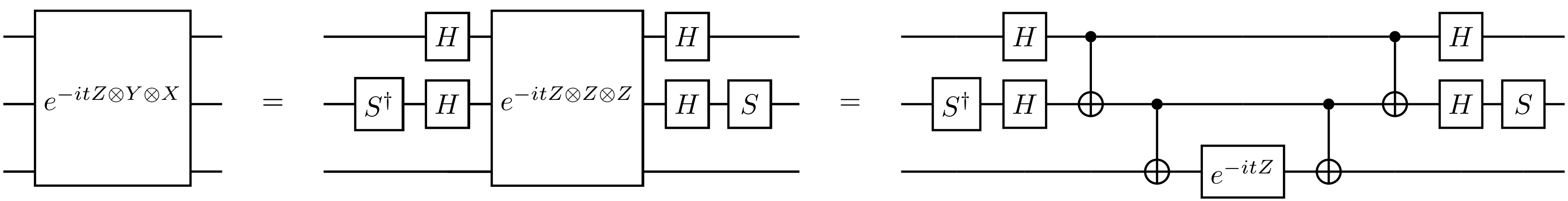
$$= S \cdot H \cdot e^{-it \cdot Z} \cdot H \cdot S^\dagger$$



Parity

$$Z \otimes Z \otimes Z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Diagonalization & Break down



Sum of (non-commuting) H_j

$$H = \sum_j H_j, \quad \text{e.g.} \quad H = X + Z$$

Problem

If the H_j do not commute,

$$X \cdot Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = Z \cdot X$$

then the exponential does not factor:

$$e^{-it\cdot(X+Z)} \neq e^{-itX} \cdot e^{-itZ}$$

Suzuki-Trotter

Small alternating H_j steps

$$H = \sum_{j=1}^{m} H_j = X + Z$$

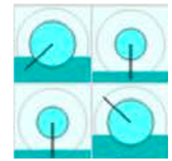
Approximate by:

$$U_1(t,r) = \left(\prod_{j=1}^{m} e^{-i\frac{t}{r}H_j} \right)^r$$

$$U_2(t,r) = \left(\prod_{j=1}^{m} e^{-i\frac{t}{2r}H_j} \prod_{j=m}^{1} e^{-i\frac{t}{2r}H_j} \right)^r$$

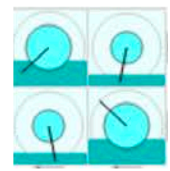
Example, $t = \pi/2$:

- Exact: numpy



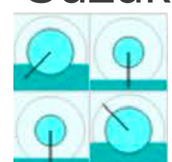
$e^{-i\pi/2(X+Z)}$

- Trotter: 16 gates



$U_1(\pi/2, 8)$

- Suzuki: 9 gates



$U_2(\pi/2, 4)$

$$U_{2k}(t,r) = \left([U_{2k-2}(tp_k, r)]^2 U_{2k-2}((1-4p_k), r) [U_{2k-2}(tp_k, r)]^2 \right)^r$$

$$\text{with } p_k = 1/(4 - 4^{1/(2k-1)})$$

Hamiltonian Simulation Results

Number of "easy" exponentials for error $\leq \varepsilon$ and time *t*

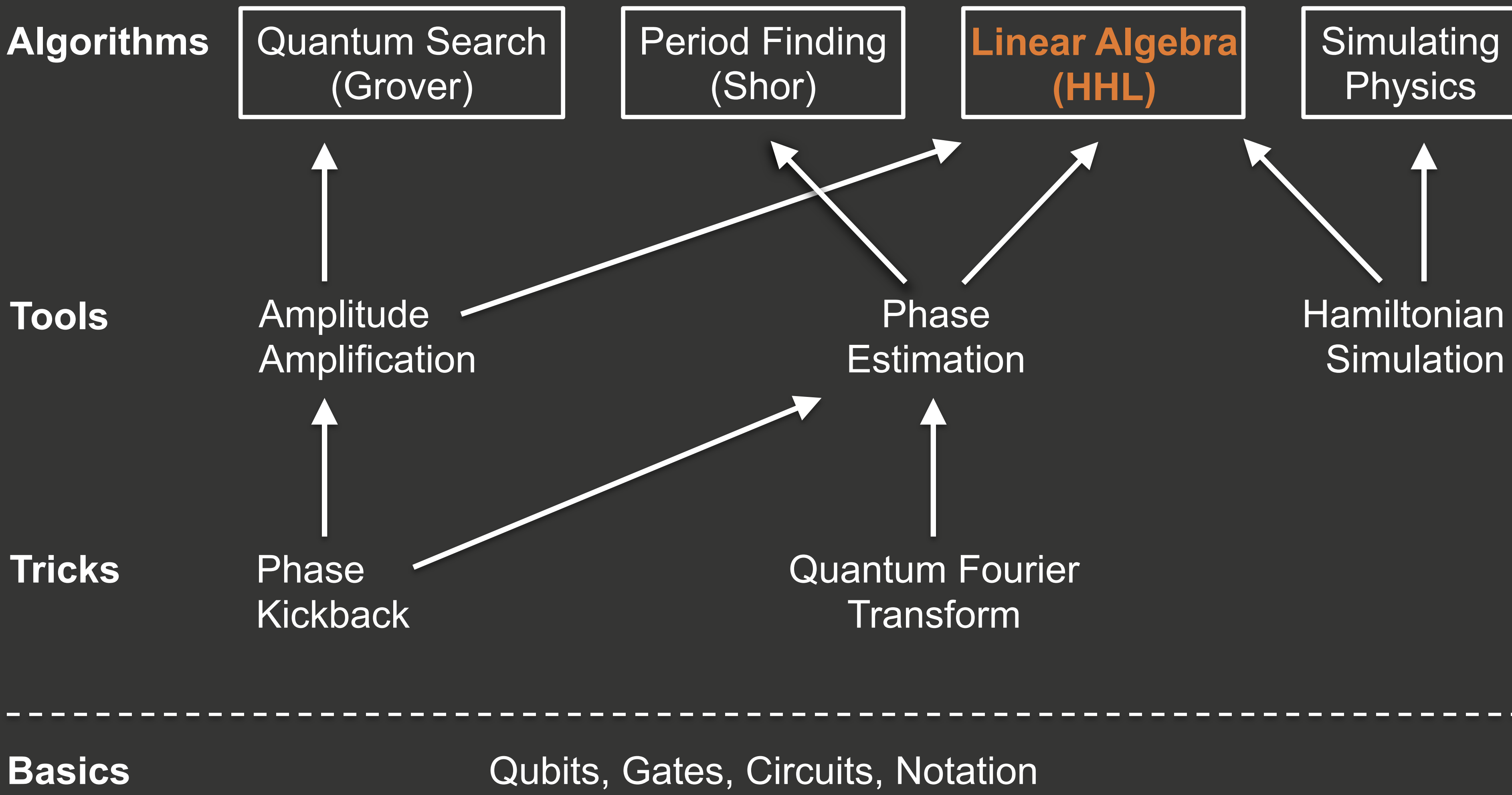
- From Physics: Sum of Pauli Tensor Product terms

$$N_{exp} \leq 5^{2k} 2m(m \cdot t \cdot ||H||)^{1+1/2k} / \varepsilon^{1/2k} \approx \widetilde{\mathcal{O}}(m^2 \cdot t \cdot ||H||)$$

- From Mathematics:
  Decompose *s*-sparse *H* into Sum of $m = 6s^2$ 1-sparse Hamiltonians, then use techniques from above:

$$N_{exp} \approx \widetilde{\mathcal{O}}(s^4 \cdot t \cdot ||H||), \qquad \text{plus } \widetilde{\mathcal{O}}(\log N s^4 \cdot t \cdot ||H||) \text{ extra gates.}$$

Quantum Speedup: Exponential


Algorithms
Quantum Search (Grover)
Period Finding (Shor)
Linear Algebra (HHL)
Simulating Physics
Tools
Amplitude Amplification
Phase Estimation
Hamiltonian Simulation
Tricks
Phase Kickback
Quantum Fourier Transform
Basics
Qubits, Gates, Circuits, Notation

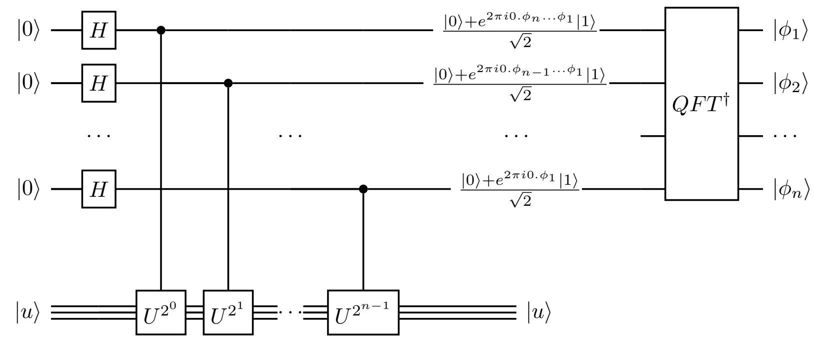
Recall Recap

Amplitude Amplification

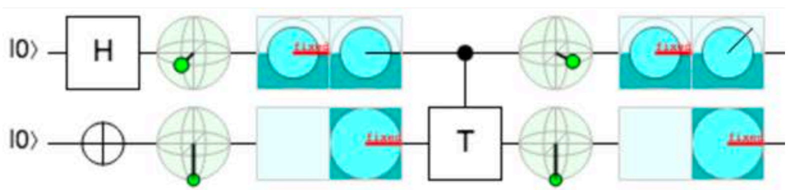
Given algorithm A with success probability *p*, one can do

- **Classical**
  Repeating A for 1/*p* times
- **Quantum**
  Boost amplitudes over √1/p rounds

Phase Estimation



Phase Kickback



Quantum Fourier Transform

QFT

$$\bigotimes_{k=n}^{1} |\phi_k\rangle \quad \bigotimes_{k=1}^{n} \left(|0\rangle + e^{2\pi i\, 0.\phi_k\ldots\phi_1}\,|1\rangle\right)$$

Inverse QFT

Linear System Problems

$$A \cdot \vec{x} = \vec{b}$$

Classical and Quantum (Harrow, Hassidim, Lloyd)

Classical problem

NxN Matrix *A*:

- Condition number *κ*
- Sparsity *s*
- *A* positive semidefinit

Conjugate gradient method: $\mathcal{O}(Ns\sqrt{\kappa}\log(1/\varepsilon))$

Solves $\vec{x} = A^{-1} \cdot \vec{b}$

Quantum problem

Additionally:

- Eigenvalues in [1/*κ*, 1]
- $|b\rangle = \vec{b}/||\vec{b}||$ preparable in time T_B
- Oracle access to *A* in time T_A
- *A* Hermitian, $A = A^\dagger$
- **no interest in all entries of $\vec{x}$**

HHL: $\mathcal{O}(\kappa T_B + \log(N) T_A s^4 \kappa^2/\varepsilon)$

Solves $|x\rangle = \frac{A^{-1}|b\rangle}{||A^{-1}|b\rangle||}$

Quantum Speedup: depends…

Quantum Linear System Problem (HHL)
Example

$$A \cdot \vec{x} = \vec{b}: \qquad \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\vec{x} = A^{-1} \cdot \vec{b} = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$|x\rangle = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Eigenstates	**Eigenvalue**
$\lvert+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	**1**
$\lvert-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	**0.5**

$$|b\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

Quantum Linear System Problem (HHL)
Helpful Facts

Let *A* have Eigenstates $|u_j\rangle$
with Eigenvalues $\lambda_j \in [1/\kappa, 1]$:

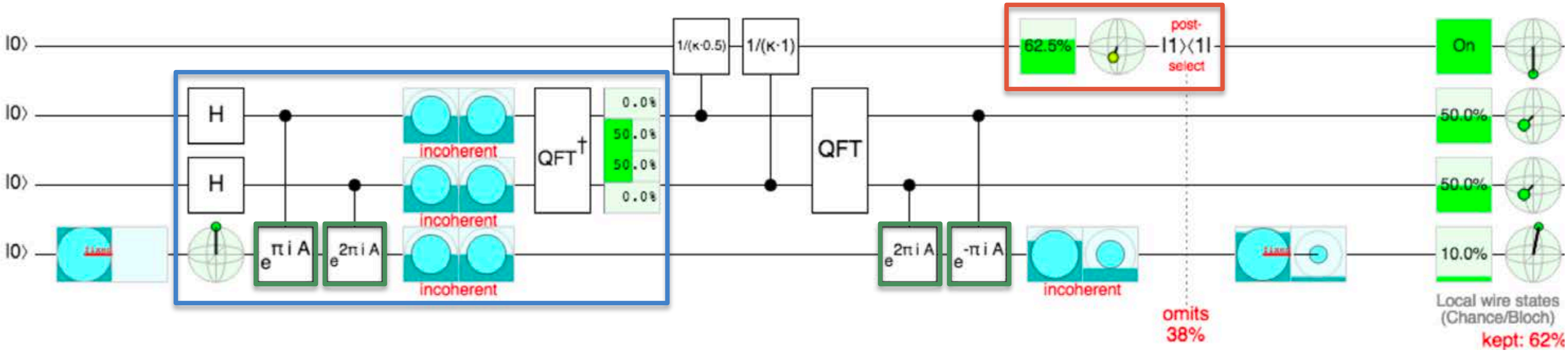
- We can write |b⟩ as
  $\sum \beta_j |u_j\rangle$
- We can write |x⟩ as
  $\approx \sum \frac{\beta_j}{\lambda_j} |u_j\rangle$
- $e^{it\cdot A}$ has Eigenvalues $e^{it\cdot\lambda_j}$
  in particular
  $e^{i\pi A} |u_j\rangle = e^{i\pi\lambda_j} |u_j\rangle = e^{2\pi i\cdot\lambda_j/2} |u_j\rangle$
  has a phase of
  $\lambda_j/2 \in [1/(2\kappa), 1/2] \subset [0, 1)$

Need to find the Eigenvalues!

Using Hamiltonian Simulation we can do Phase Estimation!

Quantum Linear System Problem (HHL)

Hamiltonian Simulation — Phase Estimation — Amplitude Amplification

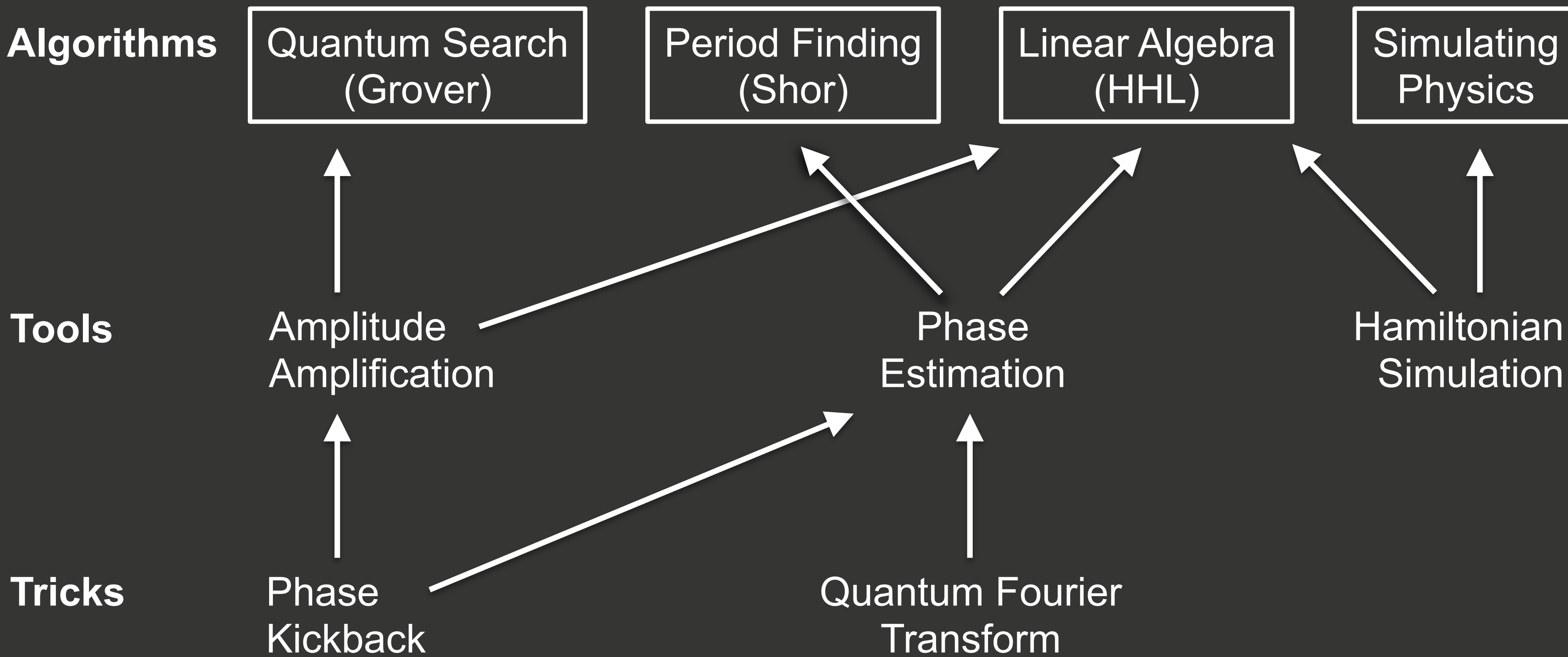


$$|b\rangle = \sum \beta_j |u_j\rangle |0\rangle_r |0\rangle_a \to \sum \beta_j |u_j\rangle |\lambda_j\rangle_r |0\rangle_a \qquad \text{Phase Estimation}$$

$$\to \sum \beta_j |u_j\rangle |\lambda_j\rangle_r \left(\sqrt{1 - \frac{1}{\kappa^2\lambda_j^2}}\,|0\rangle + \frac{1}{\kappa\lambda_j}|1\rangle\right)_a \qquad \text{Conditional Rotation}$$

$$\to \sum \beta_j |u_j\rangle |0\rangle_r \left(\sqrt{1 - \frac{1}{\kappa^2\lambda_j^2}}\,|0\rangle + \frac{1}{\kappa\lambda_j}|1\rangle\right)_a \qquad \text{uncompute}$$

$$\to \sqrt{\frac{\kappa^2}{\sum \beta_j^2/\lambda_j^2}} \sum \frac{\beta_j}{\lambda_j} |u_j\rangle |0\rangle_r |1\rangle_a \approx \sum \frac{\beta_j}{\lambda_j} |u_j\rangle \qquad \text{if we measured } |1\rangle_a$$



Basics Qubits, Gates, Circuits, Notation