

# Multirate time integration: An overview

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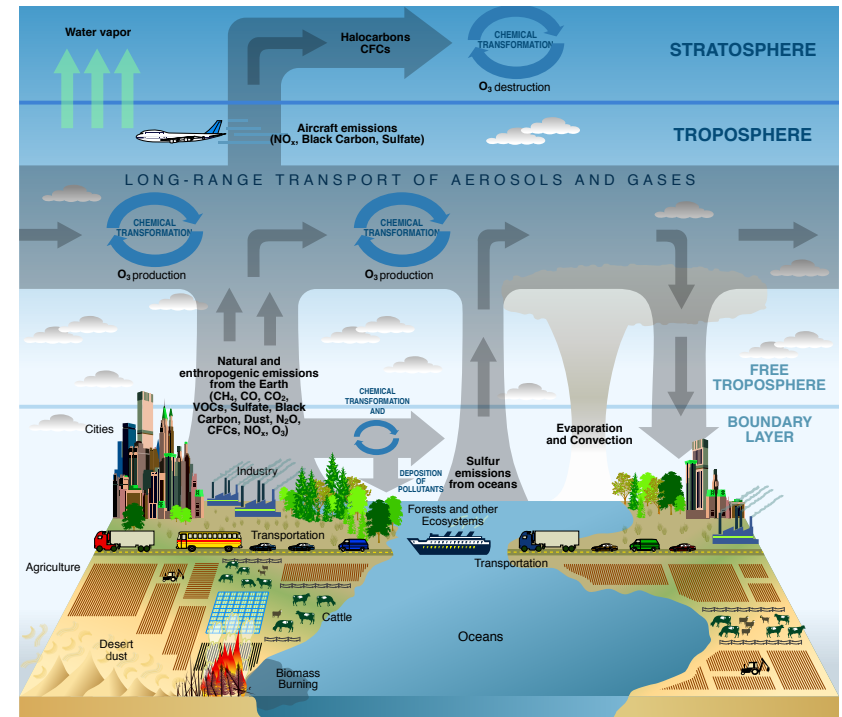
Work supported by AFOSR DDDAS FA9550-17-1-0015, NSF CCF-1613905, NSF ACI-1709727, NSF CDS&E-MSS-1953113, DOE ASCR

DE-SC0021313



# Goal: perform large scale simulations of multiphysics, multiscale systems

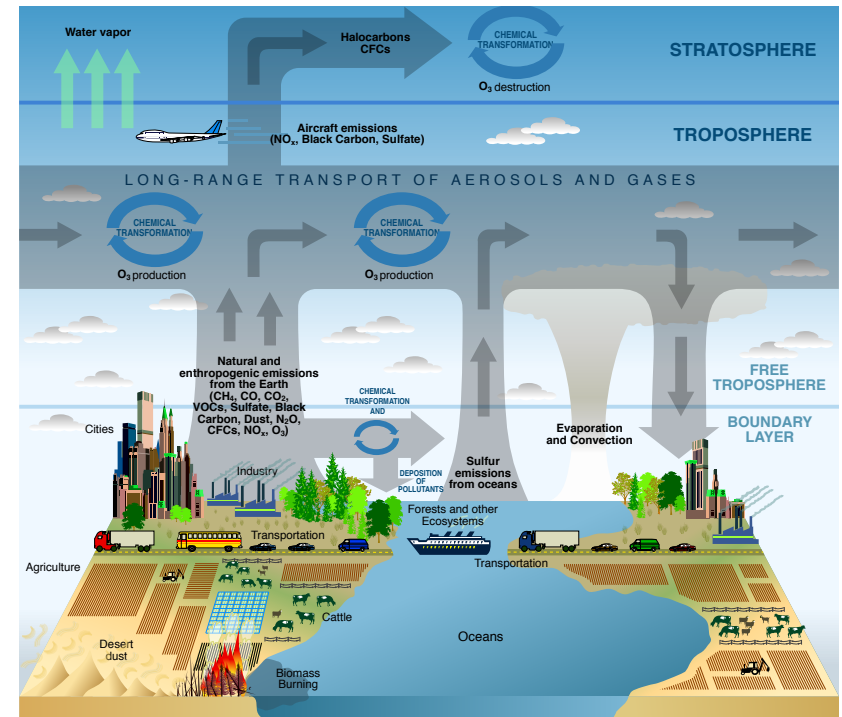
- ▶ “**Multiphysics**” systems are driven by simultaneous processes with vastly different dynamics.
- ▶ “**Multiscale**” systems include components that evolve at vastly different scales.
- ▶ Time stepping: essential ingredient of solution process.



([https://en.wikipedia.org/wiki/Scientific\\_modelling](https://en.wikipedia.org/wiki/Scientific_modelling))

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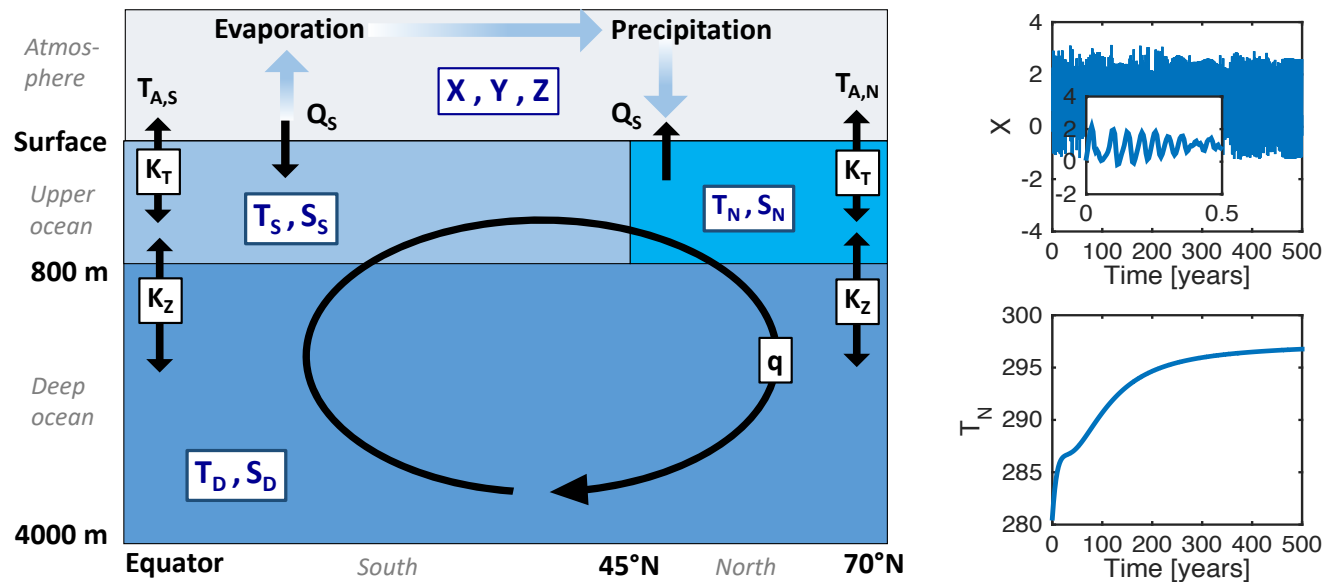
- ▶ **“Multiphysics”** systems are driven by simultaneous processes with vastly different dynamics.
- ▶ **“Multiscale”** systems include components that evolve at vastly different scales.
- ▶ Time stepping: essential ingredient of solution process.



([https://en.wikipedia.org/wiki/Scientific\\_modelling](https://en.wikipedia.org/wiki/Scientific_modelling))

- ▶ The most effective time stepping method for large multiphysics and multiscale systems? None. (Global step, implicit or explicit)
- ▶ **Multimethod** = numerical approach that employs different strategies to solve different components of the system.

# Focus here: physical systems characterized by multiple time scales



**Figure:** Climate dynamics: a prototypical multiscale system (Tardif et al., 2013, Roberts 2017). Efficient time integration? A multimethod approach makes sense: treat fast and slow components differently.

$$\begin{bmatrix} \mathbf{y}^{\{F\}} \\ \mathbf{y}^{\{S\}} \end{bmatrix}' = \begin{bmatrix} \mathbf{f}^{\{F\}}(t, \mathbf{y}^{\{S\}}, \mathbf{y}^{\{F\}}) \\ \mathbf{f}^{\{S\}}(t, \mathbf{y}^{\{S\}}, \mathbf{y}^{\{F\}}) \end{bmatrix}, \quad \begin{bmatrix} \mathbf{y}^{\{F\}}(t_0) \\ \mathbf{y}^{\{S\}}(t_0) \end{bmatrix} = \begin{bmatrix} \mathbf{y}_0^{\{F\}} \\ \mathbf{y}_0^{\{S\}} \end{bmatrix}.$$

# How does one select and apply an appropriate multimethod?

1. Select (form of) the partition:

$$\begin{aligned} \begin{bmatrix} \mathbf{y}^{\{F\}} \\ \mathbf{y}^{\{S\}} \end{bmatrix}' &= \begin{bmatrix} \mathbf{f}^{\{F\}}(t, \mathbf{y}^{\{S\}}, \mathbf{y}^{\{F\}}) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{f}^{\{S\}}(t, \mathbf{y}^{\{S\}}, \mathbf{y}^{\{F\}}) \end{bmatrix} \\ \Leftrightarrow \mathbf{y}' &= \mathbf{f}^{\{F\}}(t, \mathbf{y}) + \mathbf{f}^{\{S\}}(t, \mathbf{y}). \end{aligned}$$

Note: here we assume the partition to be given.

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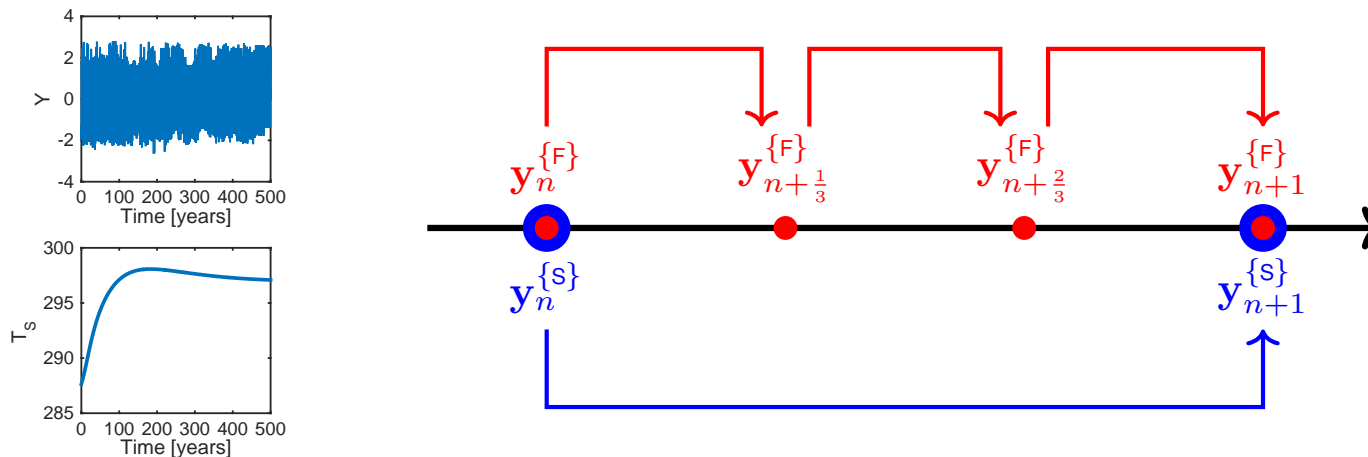
## 2. Select multimethod strategy depending on the goals of the simulation:

- ▶ **Multirate:** We are interested in accurately simulating both the fast and the slow dynamics. Solve each component with appropriate time step.
- ▶ **IMEX:** We are only interested in accurately simulating slow dynamics. Damp out the fast component, such that its impact on slow is captured.
- ▶ **Complex:** Different combinations of discretization schemes and time steps can be selected:
  - ▶ More than two components;
  - ▶ IMIM, MRI-IMEX, IM-LIM-EX, etc.

# Multirate (MR) time integration leverages the existence of multiple scales to gain computational efficiency

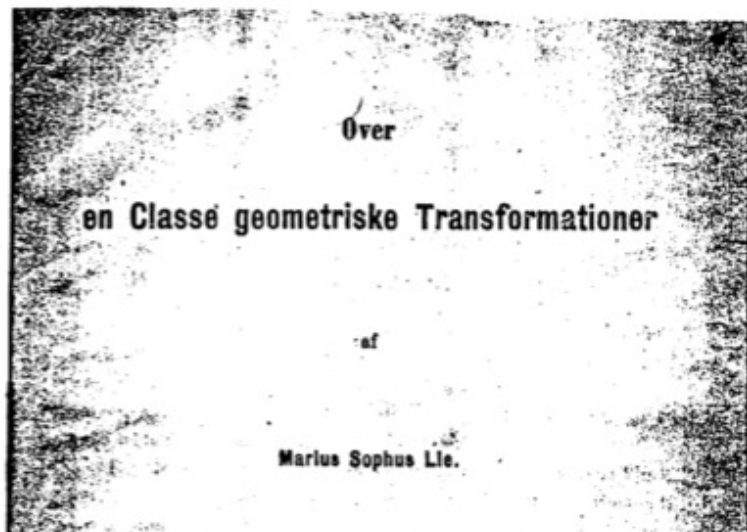
Traditional time step integration has global step restricted by the fastest component of interest.

$$\begin{bmatrix} \mathbf{y}^{\{F\}} \\ \mathbf{y}^{\{S\}} \end{bmatrix}' = \begin{bmatrix} \mathbf{f}^{\{F\}}(t, \mathbf{y}^{\{F\}}, \mathbf{y}^{\{S\}}) \\ \mathbf{f}^{\{S\}}(t, \mathbf{y}^{\{F\}}, \mathbf{y}^{\{S\}}) \end{bmatrix}.$$



**Figure:** Multirate methods: use small step sizes to discretize the fast components, and large step sizes to discretize the slow components.

# Splitting Methods



## ON THE PRODUCT OF SEMI-GROUPS OF OPERATORS<sup>1</sup>

H. F. TROTTER

1. **Introduction.** We consider semi-groups of operators on a Banach space  $X$ , which are of class  $(C_0)$  in the terminology of [3]. Such a semi-group is a family of bounded operators  $T_t$ , defined for all  $t > 0$  and satisfying the *semi-group condition*

$$(1) \quad T_{t+s} = T_t T_s \quad s, t > 0$$

SIAM J. NUMER. ANAL.  
Vol. 5, No. 3, September 1968  
Printed in U.S.A.

## ON THE CONSTRUCTION AND COMPARISON OF DIFFERENCE SCHEMES\*

GILBERT STRANG†

In this note we propose a new device for the construction of accurate difference schemes. The most natural applications are to nonlinear initial value problems in two space variables. For these problems, methods which are accurate only to first order are often too crude, and third order methods too complicated. The computations are thus made expensive either

SVAZEK 13 (1968)

APLIKACE MATEMATIKY

ČÍSLO 2

## SOME APPLICATION OF SPLITTING-UP METHODS TO THE SOLUTION OF MATHEMATICAL PHYSICS PROBLEMS

G. I. MARČUK (Г. И. Марчук)

1. Splitting-up methods are becoming a powerful constructive mathematical tool for solving complicated problems in science and engineering. Originally applied to the solution of the simplest two-dimensional elliptic and parabolic equations, these methods are finding increasing use in solving major problems. Naturally, the application of the splitting-up method to the solution of such problems involves formulation of a number of new theoretical problems which develop at different stages of formulation of a general algorithm.



# Multirate Runge-Kutta Methods

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics  
Vol. 64B, No. 3, July–September 1960

## Split Runge-Kutta Method for Simultaneous Equations

John R. Rice<sup>1</sup>

(March 31, 1960)

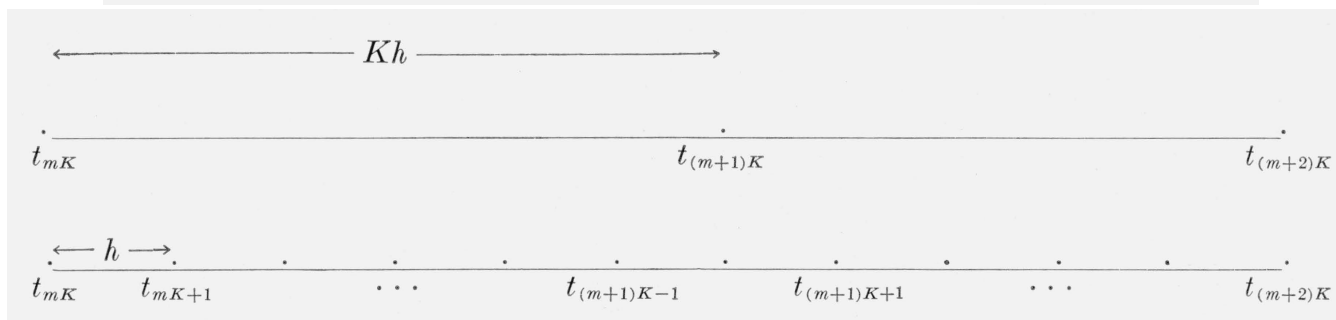
Consider two simultaneous first order differential equations  $x'(t) = F(x, y, t)$ ,  $y'(t) = G(x, y, t)$ . Runge-Kutta type integration methods are developed which allow different integration steps to be used for these equations. These methods retain the desirable properties of Runge-Kutta methods, namely the self-starting property and ease of change of integration step. Two different approaches are considered and extensive experimental work is reported upon. Experiments are done both in situations where these methods are advantageous and where they are not. It is seen that these methods are more efficient than the normal Runge-Kutta methods if they are at all applicable and in ideal situations they give the same accuracy with 90 percent less computation. These methods are applicable to six degree of freedom missile simulations, for which they were developed.

### 1. Introduction

Consider two simultaneous first order differential equations:

$$\frac{dx}{dt} = F(x, y, t) \quad x(t_0) = x_0 \quad (1.1)$$

$$\frac{dy}{dt} = G(x, y, t) \quad y(t_0) = y_0 \quad (1.2)$$



# Multirate Linear Multistep Methods

BIT 24 (1984), 484-502

## MULTIRATE LINEAR MULTISTEP METHODS<sup>1</sup>

C. W. GEAR and D. R. WELLS<sup>2</sup>

*Department of Computer Science,  
University of Illinois at Urbana-Champaign,  
1304 West Springfield Avenue,  
Urbana, Illinois 61801, U.S.A.*

*IBM Corporation  
DSC Division  
Neighborhood Road  
Kingston, NY 12401, U.S.A.*

*Dedicated to Professor Germund Dahlquist on the occasion of his 60th birthday*

### **Abstract**

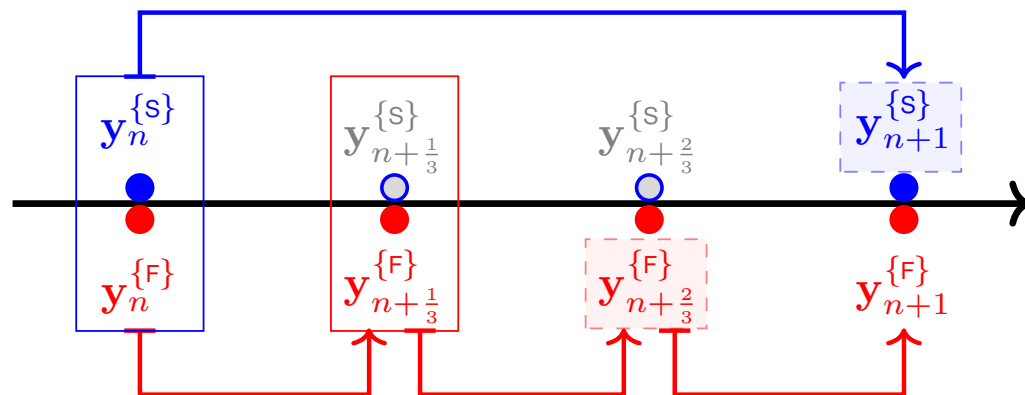
The design of a code which uses different stepsizes for different components of a system of ordinary differential equations is discussed. Methods are suggested which achieve moderate efficiency for problems having some components with a much slower rate of variation than others. Techniques for estimating errors in the different components are analyzed and applied to automatic stepsize and order control. Difficulties, absent from non-multirate methods, arise in the automatic selection of stepsizes, leading to a suggested organization of the code that is counter-intuitive. An experimental code and some initial experiments are described.

# Multirate time integration: a good idea with a long history

“Multirate methods are like the Sleeping Beauty: waiting for a kiss to wake them up.” (P. Rentrop, quoted by M. Günther)

- ▶ **MR Runge-Kutta** {Rice, 1960; Andrus, 1979, 1993; Kværno and Rentrop, 1999; Günther and Rentrop, 1994, 2001; Constantinescu and Sandu, 2007; Günther and Sandu, 2015}, (many others!)
- ▶ **MR Linear multistep methods** {Gear, 1984; Kato et al., 1999; Constantinescu and Sandu, 2009}, (many others!)
- ▶ **MR Rosenbrock-W** {Günther, 1997}, (many others!)
- ▶ **MR extrapolation** {Engstler and Lubich, 1997; Constantinescu and Sandu, 2008},
- ▶ **MR Galerkin time discretizations** {Logg, 2003}.

# Multirate forward Euler: questions and solutions



$$\mathbf{y}^{\{S\}'} = \mathbf{f}^{\{S\}}(t, \mathbf{y}^{\{S\}}, \mathbf{y}^{\{F\}})$$

$$\mathbf{y}^{\{F\}'} = \mathbf{f}^{\{F\}}(t, \mathbf{y}^{\{S\}}, \mathbf{y}^{\{F\}})$$

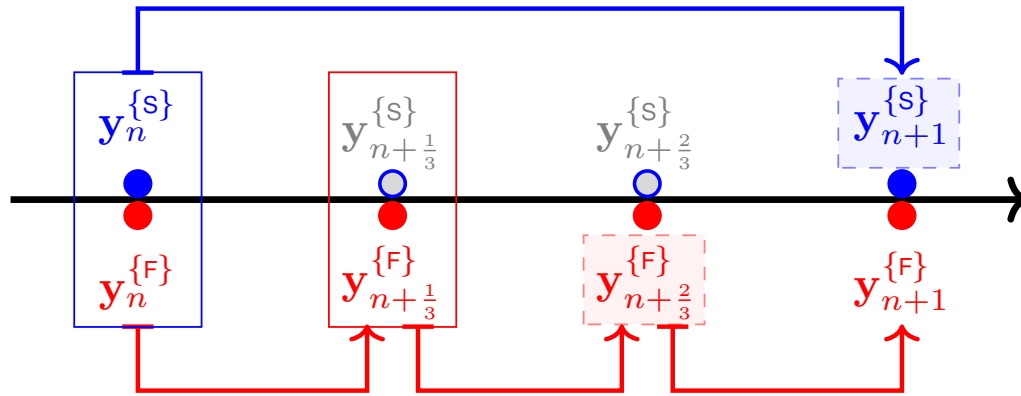
Slow subsystem:

$$\begin{cases} t_{n+1} = t_n + H, \\ \mathbf{y}_{n+1}^{\{S\}} = \mathbf{y}_n^{\{S\}} + H \mathbf{f}^{\{S\}}(t_n, \mathbf{y}_n^{\{S\}}, \mathbf{y}_n^{\{F\}}). \end{cases}$$

Fast subsystem:

$$\begin{cases} h := H/m; & t_{n+l/m} = t_n + l h, \quad l = 0, \dots, m; \\ \mathbf{y}_{n+(l+1)/m}^{\{F\}} = \mathbf{y}_{n+l/m}^{\{F\}} + h \mathbf{f}^{\{F\}}(t_{n+l/m}, \mathbf{y}_{n+l/m}^{\{S\}}, \mathbf{y}_{n+l/m}^{\{F\}}). \end{cases}$$

# Multirate forward Euler: questions and solutions



$$\mathbf{y}^{\{S\}'} = \mathbf{f}^{\{S\}}(t, \mathbf{y}^{\{S\}}, \mathbf{y}^{\{F\}})$$

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Fast subsystem:

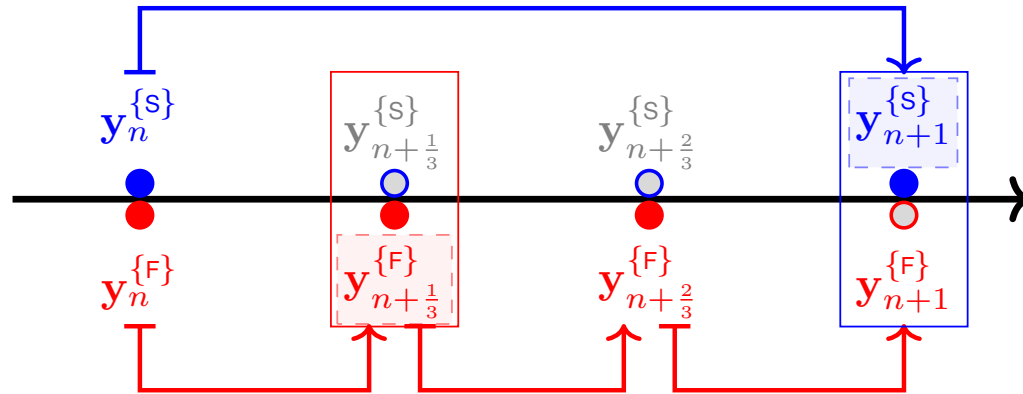
$$\begin{cases} h := H/m; & t_{n+l/m} = t_n + l h, \quad l = 0, \dots, m; \\ \mathbf{y}_{n+(l+1)/m}^{\{F\}} = \mathbf{y}_{n+l/m}^{\{F\}} + h \mathbf{f}^{\{F\}}(t_{n+l/m}, \mathbf{y}_{n+l/m}^{\{S\}}, \mathbf{y}_{n+l/m}^{\{F\}}). \end{cases}$$

Intermediate slow variables  $\mathbf{y}_{n+l/m}^{\{S\}}$  need to be approximated.

Slowest-first:  $\mathbf{y}_{n+l/m}^{\{S\}} = \begin{cases} \mathbf{y}_n^{\{S\}}, & \text{constant,} \\ \frac{m-l}{m} \mathbf{y}_n^{\{S\}} + \frac{l}{m} \mathbf{y}_{n+1}^{\{S\}}, & \text{linear,} \end{cases}$

Fastest-first:  $\mathbf{y}_{n+l/m}^{\{S\}} = \begin{cases} \mathbf{y}_n^{\{S\}}, & \text{constant,} \\ \mathbf{y}_n^{\{S\}} + \frac{l}{m} H \mathbf{f}^{\{S\}}(t_n, \mathbf{y}_n^{\{S\}}, \mathbf{y}_n^{\{F\}}), & \text{linear,} \end{cases}$

# Multirate backward Euler: questions and solutions I



$$\mathbf{y}^{\{S\}'} = \mathbf{f}^{\{S\}}(t, \mathbf{y}^{\{S\}}, \mathbf{y}^{\{F\}})$$

$$\mathbf{y}^{\{F\}'} = \mathbf{f}^{\{F\}}(t, \mathbf{y}^{\{S\}}, \mathbf{y}^{\{F\}})$$

1. Naive Backward Euler steps  $\rightarrow$  fully coupled approach.

$$\begin{aligned} \mathbf{y}_{n+1}^{\{S\}} &= \mathbf{y}_n^{\{S\}} + H \mathbf{f}^{\{S\}}(t_{n+1}, \mathbf{y}_{n+1}^{\{S\}}, \mathbf{y}_{n+1}^{\{F\}}), \\ \mathbf{y}_{n+(\ell+1)/m}^{\{F\}} &= \mathbf{y}_{n+\ell/m}^{\{F\}} + h \mathbf{f}^{\{F\}}(t_{n+(\ell+1)/m}, \mathbf{y}_{n+(\ell+1)/m}^{\{S\}}, \mathbf{y}_{n+(\ell+1)/m}^{\{F\}}), \quad \ell \geq 0; \\ \mathbf{y}_{n+\ell/m}^{\{S\}} &= \begin{cases} \mathbf{y}_n^{\{S\}} \text{ or } \mathbf{y}_{n+1}^{\{S\}}, & \text{constant,} \\ \frac{m-\ell}{m} \mathbf{y}_n^{\{S\}} + \frac{\ell}{m} \mathbf{y}_{n+1}^{\{S\}}, & \text{linear,} \end{cases} \quad \ell = 0, 1, \dots, m-1. \end{aligned}$$

2. Decoupled slowest-first approach:

$$\begin{aligned} \mathbf{y}_{n+1}^{\{S\}} &= \mathbf{y}_n^{\{S\}} + H \mathbf{f}^{\{S\}}(t_{n+1}, \mathbf{y}_{n+1}^{\{S\}}, \mathbf{y}_n^{\{F\}}), \\ \mathbf{y}_{n+(\ell+1)/m}^{\{F\}} &= \mathbf{y}_{n+\ell/m}^{\{F\}} + h \mathbf{f}^{\{F\}}(t_{n+(\ell+1)/m}, \mathbf{y}_{n+(\ell+1)/m}^{\{S\}}, \mathbf{y}_{n+(\ell+1)/m}^{\{F\}}), \quad \ell \geq 0. \end{aligned}$$

# Multirate backward Euler: questions and solutions II

## 3. Decoupled fastest-first approach:

$$\begin{aligned} \mathbf{y}_{n+(\ell+1)/m}^{\{F\}} &= \mathbf{y}_{n+\ell/m}^{\{F\}} + h \mathbf{f}^{\{F\}}(t_{n+(\ell+1)/m}, \mathbf{y}_n^{\{S\}}, \mathbf{y}_{n+(\ell+1)/m}^{\{F\}}), \quad \ell \geq 0; \\ \mathbf{y}_{n+1}^{\{S\}} &= \mathbf{y}_n^{\{S\}} + H \mathbf{f}^{\{S\}}(t_{n+1}, \mathbf{y}_{n+1}^{\{S\}}, \mathbf{y}_{n+1}^{\{F\}}). \end{aligned}$$

## 4. Coupled slowest-first predictor approach:

$$\begin{aligned} \mathbf{y}_{n+1}^{\{S\}} &= \mathbf{y}_n^{\{S\}} + H \mathbf{f}^{\{S\}}(t_{n+1}, \mathbf{y}_{n+1}^{\{S\}}, \hat{\mathbf{y}}_{n+1}^{\{F\}}), \\ \hat{\mathbf{y}}_{n+1}^{\{F\}} &= \mathbf{y}_n^{\{F\}} + H \mathbf{f}^{\{F\}}(t_{n+1}, \mathbf{y}_{n+1}^{\{S\}}, \hat{\mathbf{y}}_{n+1}^{\{F\}}), \\ \hat{\mathbf{y}}_{n+1}^{\{F\}} &\Rightarrow \mathbf{y}_{n+(\ell+1)/m}^{\{F\}} = \mathbf{y}_{n+\ell/m}^{\{F\}} + h \mathbf{f}^{\{F\}}_{n+(\ell+1)/m}, \quad \ell \geq 0. \end{aligned}$$

## 5. Coupled first step approach:

$$\begin{aligned} \mathbf{y}_{n+1}^{\{S\}} &= \mathbf{y}_n^{\{S\}} + H \mathbf{f}^{\{S\}}(t_{n+1}, \mathbf{y}_{n+1}^{\{S\}}, \mathbf{y}_{n+1/m}^{\{F\}}), \\ \mathbf{y}_{n+1/m}^{\{F\}} &= \mathbf{y}_n^{\{F\}} + h \mathbf{f}^{\{F\}}(t_{n+1/m}, \mathbf{y}_{n+1}^{\{S\}}, \mathbf{y}_{n+1/m}^{\{F\}}); \\ \mathbf{y}_{n+(\ell+1)/m}^{\{F\}} &= \mathbf{y}_{n+\ell/m}^{\{F\}} + h \mathbf{f}^{\{F\}}(t_{n+(\ell+1)/m}, \mathbf{y}_{n+1}^{\{S\}}, \mathbf{y}_{n+(\ell+1)/m}^{\{F\}}), \quad \ell \geq 1. \end{aligned}$$

# Multirate Euler: Stability analysis

Kværnø (2000) test problem:

$$\begin{bmatrix} \mathbf{y}^{\{S\}} \\ \mathbf{y}^{\{F\}} \end{bmatrix}' = \underbrace{\begin{bmatrix} \lambda^{\{S\}} & \eta^{\{F\}} \\ \eta^{\{S\}} & \lambda^{\{F\}} \end{bmatrix}}_A \begin{bmatrix} \mathbf{y}^{\{S\}} \\ \mathbf{y}^{\{F\}} \end{bmatrix}, \quad \lambda^{\{S\}} < 0, \lambda^{\{F\}} < 0, \eta^{\{S\}}, \eta^{\{F\}} \in \mathbb{R};$$

coupling:  $k = \frac{\eta^{\{F\}} \eta^{\{S\}}}{\lambda^{\{F\}} \lambda^{\{S\}}} < 1.$

- ▶ Multirate Euler solution:

$$\begin{bmatrix} \mathbf{y}_{n+1}^{\{S\}} \\ \mathbf{y}_{n+1}^{\{F\}} \end{bmatrix} = \mathbf{R} \cdot \begin{bmatrix} \mathbf{y}_n^{\{S\}} \\ \mathbf{y}_n^{\{F\}} \end{bmatrix},$$

- ▶ The MRFE scheme is stable iff

$$k \in \left( \frac{\mathbf{R}_{FE}^{\{S\}} \mathbf{R}_{FE}^{\{F\}} - 1}{(1 - \mathbf{R}_{FE}^{\{S\}})(1 - \mathbf{R}_{FE}^{\{F\}})}, \frac{(1 + \mathbf{R}_{FE}^{\{S\}})(1 + \mathbf{R}_{FE}^{\{F\}})}{(1 - \mathbf{R}_{FE}^{\{S\}})(1 - \mathbf{R}_{FE}^{\{F\}})} \right) \cap (-\infty, 1).$$

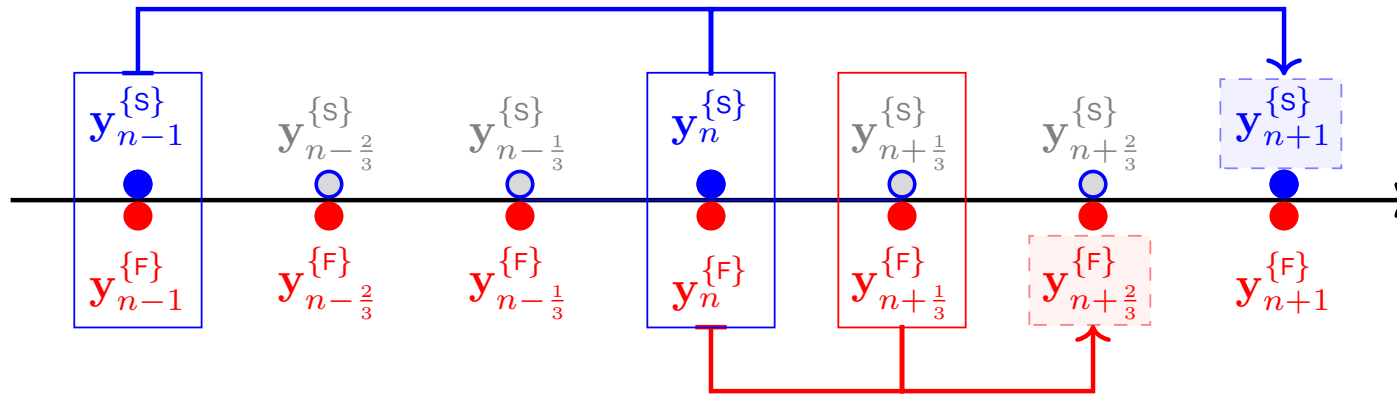
- ▶ The coupled slowest–first MRBE with constant interpolation

$$\mathbf{y}_{n+(\ell+1)/m}^{\{S\}} = \mathbf{y}_{n+1}^{\{S\}}, \text{ is unconditionally stable } (\forall H, h > 0).$$

- ▶ The coupled first step approach is unconditionally stable  $(\forall H, h > 0)$ .



# Multirate linear multistep methods require more accurate coupling between scales



- ▶ Compound coupled LMM step requires inter-/extra-polation:

$$\sum_{r=-1}^k \alpha_r \mathbf{y}_{n-r}^{\{S\}} = H \sum_{r=-1}^k \beta_r \mathbf{f}^{\{S\}}(\mathbf{y}_{n-r}^{\{S\}}, \mathbf{y}_{n-r}^{\{F\}}),$$

$$\sum_{r=-1}^k \alpha_r \mathbf{y}_{n+(\ell-r)/m}^{\{F\}} = h \sum_{r=-1}^k \beta_r \mathbf{f}^{\{F\}}\left(\mathbf{P}_{\text{int}}^{\{S\}}(t_{n+(\ell-r)/m}), \mathbf{y}_{n+(\ell-r)/m}^{\{F\}}\right), \quad \ell = 0, \dots, m-1.$$

- ▶ Base LMM has order  $p$ , use inter-/extra-polation of order  $\geq p-1$ .
- ▶ All variables are coupled. **Dynamic iterations** (Skelboe, 1989).

# Digression: IMEX LMM

{Crouzeix 1978; Ascher et al. 1994}

- ▶ Same step size  $H$  and same solution history  $(\mathbf{y}_{n-r}^{\{S\}}, \mathbf{y}_{n-r}^{\{F\}})$  used for both components
- ▶ Different function coefficients  $\beta_r^{\{S\}}$  for slow and  $\beta_r^{\{F\}}$  for fast components.

$$\sum_{r=-1}^k \alpha_r \mathbf{y}_{n-r}^{\{S\}} = H \sum_{r=0}^k \beta_r^{\{S\}} \mathbf{f}^{\{S\}}(\mathbf{y}_{n-r}^{\{S\}}, \mathbf{y}_{n-r}^{\{F\}}),$$

$$\sum_{r=-1}^k \alpha_r \mathbf{y}_{n-r}^{\{F\}} = H \sum_{r=-1}^k \beta_r^{\{F\}} \mathbf{f}^{\{F\}}(\mathbf{y}_{n-r}^{\{S\}}, \mathbf{y}_{n-r}^{\{F\}}).$$

# Multirate LMM: (de)coupling strategies to allow separate computations on each scale

Semi-implicit fastest-first strategy:

$$\sum_{r=-1}^k \alpha_r \mathbf{y}_{n+(\ell-r)/m}^{\{F\}} = h \sum_{r=-1}^k \beta_r \mathbf{f}^{\{F\}}(\mathbf{P}_{\text{ext}}^{\{S\}}(t_{n+(\ell-r)/m}), \mathbf{y}_{n+(\ell-r)/m}^{\{F\}}), \quad \ell = 0, \dots, m-1$$
$$\sum_{r=-1}^k \alpha_r \mathbf{y}_{n-r}^{\{S\}} = H \sum_{r=-1}^k \beta_r \mathbf{f}^{\{S\}}(\mathbf{y}_{n-r}^{\{S\}}, \mathbf{y}_{n-r}^{\{F\}}).$$

Semi-implicit slowest-first strategy:

$$\sum_{r=-1}^k \alpha_r \mathbf{y}_{n-r}^{\{S\}} = H \left( \sum_{r=0}^k \beta_r \mathbf{f}^{\{S\}}(\mathbf{y}_{n-r}^{\{S\}}, \mathbf{y}_{n-r}^{\{F\}}) + \beta_{-1} \mathbf{f}^{\{S\}}(\mathbf{y}_{n+1}^{\{S\}}, \mathbf{P}_{\text{ext}}^{\{F\}}(t_{n+1})) \right),$$
$$\sum_{r=-1}^k \alpha_r \mathbf{y}_{n+(\ell-r)/m}^{\{F\}} = h \sum_{r=-1}^k \beta_r \mathbf{f}^{\{F\}}(\mathbf{P}_{\text{int}}^{\{S\}}(t_{n+(\ell-r)/m}), \mathbf{y}_{n+(\ell-r)/m}^{\{F\}}), \quad \ell = 0, \dots, m-1.$$

# Multirate LMM Analysis

- ▶ If base multistep scheme has order  $p$ , use extrapolation and interpolation procedures of order at least  $p - 1$ .
- ▶ Linear stability analysis depends on both the LMM scheme and the interpolation polynomials. Useful for numerical results, but too complex to obtain analytical results.

# Multirate extrapolation I

Global error expansion (Gragg, 1964; Hairer, 1984)

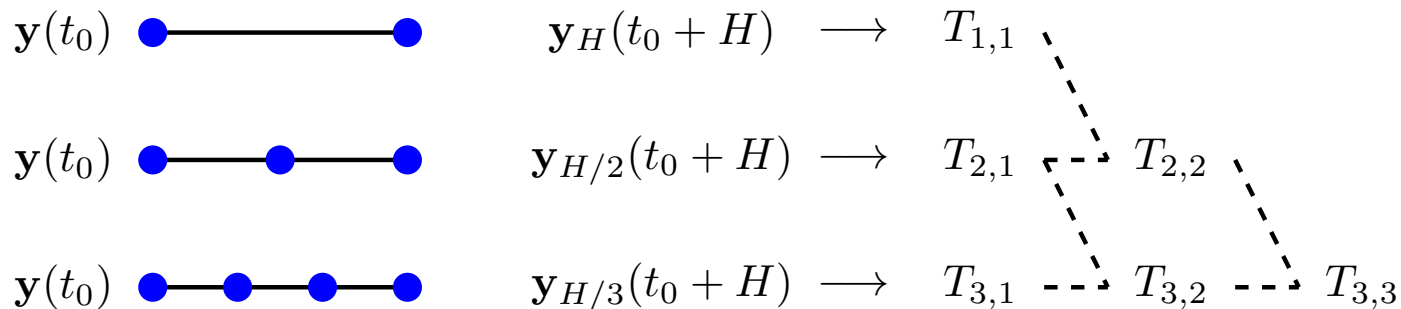
$$\mathbf{y}_h(t+h) = \mathbf{y}_h(t) + h \Phi(t, \mathbf{y}_h(t), h) \xrightarrow{\text{Smooth}}$$

$$\mathbf{y}(t) - \mathbf{y}_h(t) = e_p(t) h^p + \dots + e_N(t) h^N + E_N(t, h) h^{N+1}.$$

Extrapolation table is built from solutions obtained using **multiple time steps**:

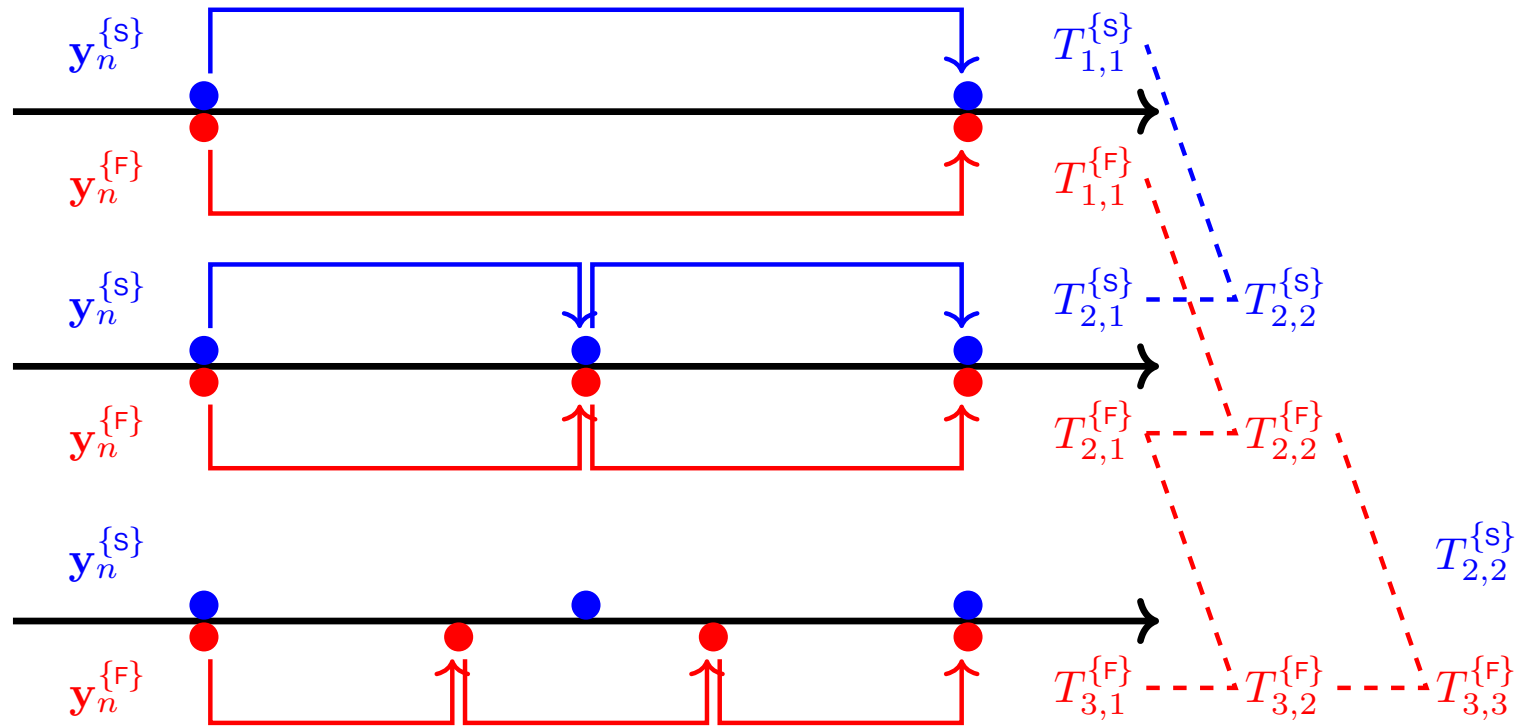
$$T_{j,1} := \mathbf{y}_{h_j}(t_0 + H), \quad h_j = H/n_j \quad [\text{base method}]$$

$$T_{j,k+1} \rightarrow T_{j,k} + \frac{T_{j,k} - T_{j-1,k}}{(n_j/n_{j-k}) - 1}, \quad k < j,$$



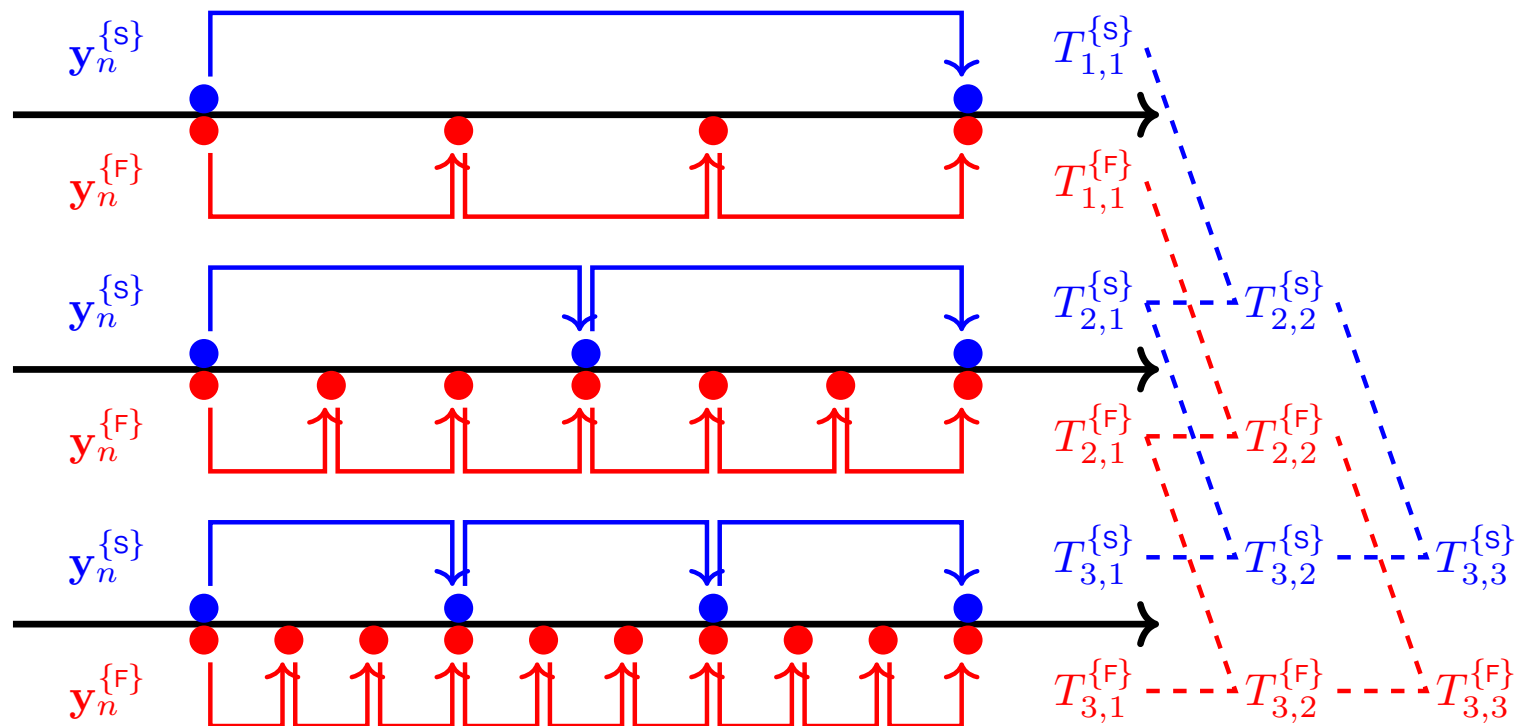
# Multirate extrapolation II

MURX {Engstler and Lubich, 1997} uses adaptive  $m$



# Multirate extrapolation III

MREXTRAP {Constantinescu and Sandu, 2013} with fixed  $m = 3$



# Digression: IMEX Methods by Extrapolation

Perform extrapolation using a LIMEX base scheme

{Constantinescu and Sandu, 2010}:

- ▶ W-IMEX-Euler.

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \left[ \mathbf{I} - h \mathbf{f}_y^{\{F\}}(t_n, \mathbf{y}_n) \right]^{-1} \left( h \mathbf{f}(t_n, \mathbf{y}_n) \right).$$

- ▶ IMEX-Euler.

$$\begin{aligned} \mathbf{y}_{n+1} = \mathbf{y}_n + h \mathbf{f}^{\{S\}}(t_n, \mathbf{y}_n) \\ + \left[ \mathbf{I} - h \mathbf{f}_y^{\{F\}}(t_n, \mathbf{y}_n) \right]^{-1} \left( h \mathbf{f}^{\{F\}}(t_n, \mathbf{y}_n) \right). \end{aligned}$$

- ▶ Split-IMEX-Euler.

$$\begin{aligned} \mathbf{y}_* = \mathbf{y}_n + h \mathbf{f}^{\{S\}}(t_n, \mathbf{y}_n), \\ \mathbf{y}_{n+1} = \mathbf{y}_* + \left[ \mathbf{I} - h \mathbf{f}_y^{\{F\}}(t_n, \mathbf{y}_n) \right]^{-1} \left( h \mathbf{f}^{\{F\}}(t_n, \mathbf{y}_*) \right). \end{aligned}$$



# Solving multiphysics problems with traditional Runge-Kutta

- ▶ One step of a traditional RK scheme:

$$\mathbf{y}' = \mathbf{f}^{\{1\}}(\mathbf{y}) + \cdots + \mathbf{f}^{\{N\}}(\mathbf{y});$$

$$\mathbf{Y}_i = \mathbf{y}_n + h \sum_{j=1}^s a_{i,j} \left( \mathbf{f}^{\{1\}}(\mathbf{Y}_j) + \cdots + \mathbf{f}^{\{N\}}(\mathbf{Y}_j) \right), \quad i = 1, \dots, s,$$

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h \sum_{i=1}^s b_i \left( \mathbf{f}^{\{1\}}(\mathbf{Y}_j) + \cdots + \mathbf{f}^{\{N\}}(\mathbf{Y}_j) \right).$$

- ▶ The corresponding Butcher tableau is

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b} \end{array}.$$

- ▶ One set of RK coefficients for all component functions.
- ▶ Same stage vector arguments for all components.

# Consistent framework multimethods: traditional Partitioned & Additive Runge-Kutta (PRK & ARK)

- ▶ One step of an ARK scheme {Ascher, Ruuth, and Spiteri, 1997; Kennedy and Carpenter, 2003, 2019}

$$\mathbf{y}' = \mathbf{f}^{\{1\}}(\mathbf{y}) + \cdots + \mathbf{f}^{\{N\}}(\mathbf{y});$$

$$\mathbf{Y}_i = \mathbf{y}_n + h \sum_{j=1}^s a_{i,j}^{\{1\}} \mathbf{f}^{\{1\}}(\mathbf{Y}_j) + \cdots + h \sum_{j=1}^s a_{i,j}^{\{N\}} \mathbf{f}^{\{N\}}(\mathbf{Y}_j),$$

$$i = 1, \dots, s,$$

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h \sum_{i=1}^s b_i^{\{1\}} \mathbf{f}^{\{1\}}(\mathbf{Y}_i) + \cdots + h \sum_{i=1}^s b_i^{\{N\}} \mathbf{f}^{\{N\}}(\mathbf{Y}_i).$$

- ▶ Butcher tableau: 
$$\begin{array}{c|ccc} \mathbf{c} & \mathbf{A}^{\{1\}} & \cdots & \mathbf{A}^{\{N\}} \\ \hline & \mathbf{b}^{\{1\}} & \cdots & \mathbf{b}^{\{N\}} \end{array}.$$

- ▶ One set of RK coefficients for each component function.
- ▶ Same stage vector arguments for all components.

# General-structure Additive Runge-Kutta (GARK) methods I

- ▶ One step of a GARK scheme {Sandu & Günther, 2013}:

$$\begin{aligned} \mathbf{y}' &= \mathbf{f}^{\{1\}}(\mathbf{y}) + \cdots + \mathbf{f}^{\{N\}}(\mathbf{y}); \\ \mathbf{Y}_i^{\{1\}} &= \mathbf{y}_n + h \sum_{j=1}^{s^{\{1\}}} a_{i,j}^{\{1,1\}} \mathbf{f}^{\{1\}}(\mathbf{Y}_j^{\{1\}}) + \cdots + h \sum_{j=1}^{s^{\{N\}}} a_{i,j}^{\{1,N\}} \mathbf{f}^{\{N\}}(\mathbf{Y}_j^{\{N\}}), \\ &\vdots \\ \mathbf{Y}_i^{\{N\}} &= \mathbf{y}_n + h \sum_{j=1}^{s^{\{1\}}} a_{i,j}^{\{N,1\}} \mathbf{f}^{\{1\}}(\mathbf{Y}_j^{\{1\}}) + \cdots + h \sum_{j=1}^{s^{\{N\}}} a_{i,j}^{\{N,N\}} \mathbf{f}^{\{N\}}(\mathbf{Y}_j^{\{N\}}) \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + h \sum_{i=1}^{s^{\{1\}}} b_i^{\{1\}} \mathbf{f}^{\{1\}}(\mathbf{Y}_i^{\{1\}}) + \cdots + h \sum_{i=1}^{s^{\{N\}}} b_i^{\{N\}} \mathbf{f}^{\{N\}}(\mathbf{Y}_i^{\{N\}}). \end{aligned}$$

- ▶ Each component function has a different stage vector argument.
- ▶ “Structure-revealing” formulation: individual RKs plus coupling terms.

# General-structure Additive Runge-Kutta (GARK) methods II

- ▶ One set of RK coefficients for each component and stage (flexibility). The Butcher tableau for an internally consistent GARK method:

$$\begin{array}{c|ccc}
 \mathbf{c}^{\{1\}} & \mathbf{A}^{\{1,1\}} & \dots & \mathbf{A}^{\{1,N\}} \\
 \vdots & \vdots & \ddots & \vdots \\
 \mathbf{c}^{\{N\}} & \mathbf{A}^{\{N,1\}} & \dots & \mathbf{A}^{\{N,N\}} \\
 \hline
 & \mathbf{b}^{\{1\}T} & \dots & \mathbf{b}^{\{N\}T}
 \end{array}$$

- ▶ {Traditional ARK}  $\leftrightarrow$  {GARK}.
- ▶ GARK “structure-revealing” formulation of interest in itself.

# Digression: IMEX GARK schemes

- ▶ Two-way partitioned right hand side where  $\mathbf{f}^{\{S\}}$  is non-stiff and  $\mathbf{f}^{\{F\}}$  is stiff

$$\mathbf{y}' = \mathbf{f}^{\{S\}}(\mathbf{y}) + \mathbf{f}^{\{F\}}(\mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0.$$

- ▶ IMEX-GARK scheme:  $(\mathbf{A}^{\{E,E\}}, \mathbf{b}^{\{E\}})$  explicit and  $(\mathbf{A}^{\{I,I\}}, \mathbf{b}^{\{I\}})$  implicit:

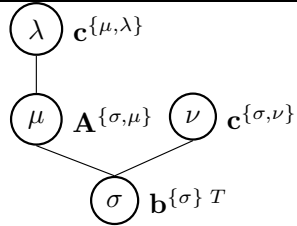
$$\mathbf{Y}_i^{\{S\}} = \mathbf{y}_n + h \sum_{j=1}^{i-1} a_{i,j}^{\{E,E\}} \mathbf{f}^{\{S\}}(\mathbf{Y}_j^{\{S\}}) + h \sum_{j=1}^{i-1} a_{i,j}^{\{E,I\}} \mathbf{f}^{\{F\}}(\mathbf{Y}_j^{\{F\}}), \quad i = 1, \dots, s^{\{E\}},$$

$$\mathbf{Y}_i^{\{F\}} = \mathbf{y}_n + h \sum_{j=1}^i a_{i,j}^{\{I,E\}} \mathbf{f}^{\{S\}}(\mathbf{Y}_j^{\{S\}}) + h \sum_{j=1}^i a_{i,j}^{\{I,I\}} \mathbf{f}^{\{F\}}(\mathbf{Y}_j^{\{F\}}), \quad i = 1, \dots, s^{\{I\}},$$

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h \sum_{i=1}^{s^{\{E\}}} b_i^{\{E\}} \mathbf{f}^{\{S\}}(\mathbf{Y}_i^{\{S\}}) + h \sum_{i=1}^{s^{\{I\}}} b_i^{\{I\}} \mathbf{f}^{\{F\}}(\mathbf{Y}_i^{\{F\}}).$$

# NT-trees, DAT<sub>1</sub>-trees, DAT<sub>2</sub>-trees, and B-series

GARK order conditions canonically extend RK. Method coefficients labelled according to node colors, compatible to matrix multiplication rules.



$$\mathbf{b}^{\{\sigma\}T} \left( (\mathbf{A}^{\{\sigma, \mu\}} \mathbf{c}^{\{\mu, \lambda\}}) \times \mathbf{c}^{\{\sigma, \nu\}} \right) = \frac{1}{8}$$

# NT-trees, DAT<sub>1</sub>-trees, DAT<sub>2</sub>-trees, and B-series

GARK order conditions canonically extend RK. Method coefficients labelled according to node colors, compatible to matrix multiplication rules.

	$\mathbf{b}^{\{\sigma\}T} \left( (\mathbf{A}^{\{\sigma,\mu\}} \mathbf{c}^{\{\mu,\lambda\}}) \times \mathbf{c}^{\{\sigma,\nu\}} \right) = \frac{1}{8}$
	$\mathbf{b}^{\{z\}T} \mathbf{A}^{\{z,z\}-1} \mathbf{A}^{\{z,x\}} \mathbf{A}^{\{x,x\}} \mathbf{A}^{\{x,z\}} \mathbf{A}^{\{z,z\}-1} \mathbf{c}^{\{z\}} = \frac{1}{6}$
	$\mathbf{b}^{\{x\}T} \mathbf{A}^{\{x,z\}} \mathbf{A}^{\{z,z\}-1} \mathbf{A}^{\{z,x\}} \mathbf{c}^{\{x\}} \times 2 = \frac{1}{12}$

# NT-trees, DAT<sub>1</sub>-trees, DAT<sub>2</sub>-trees, and B-series

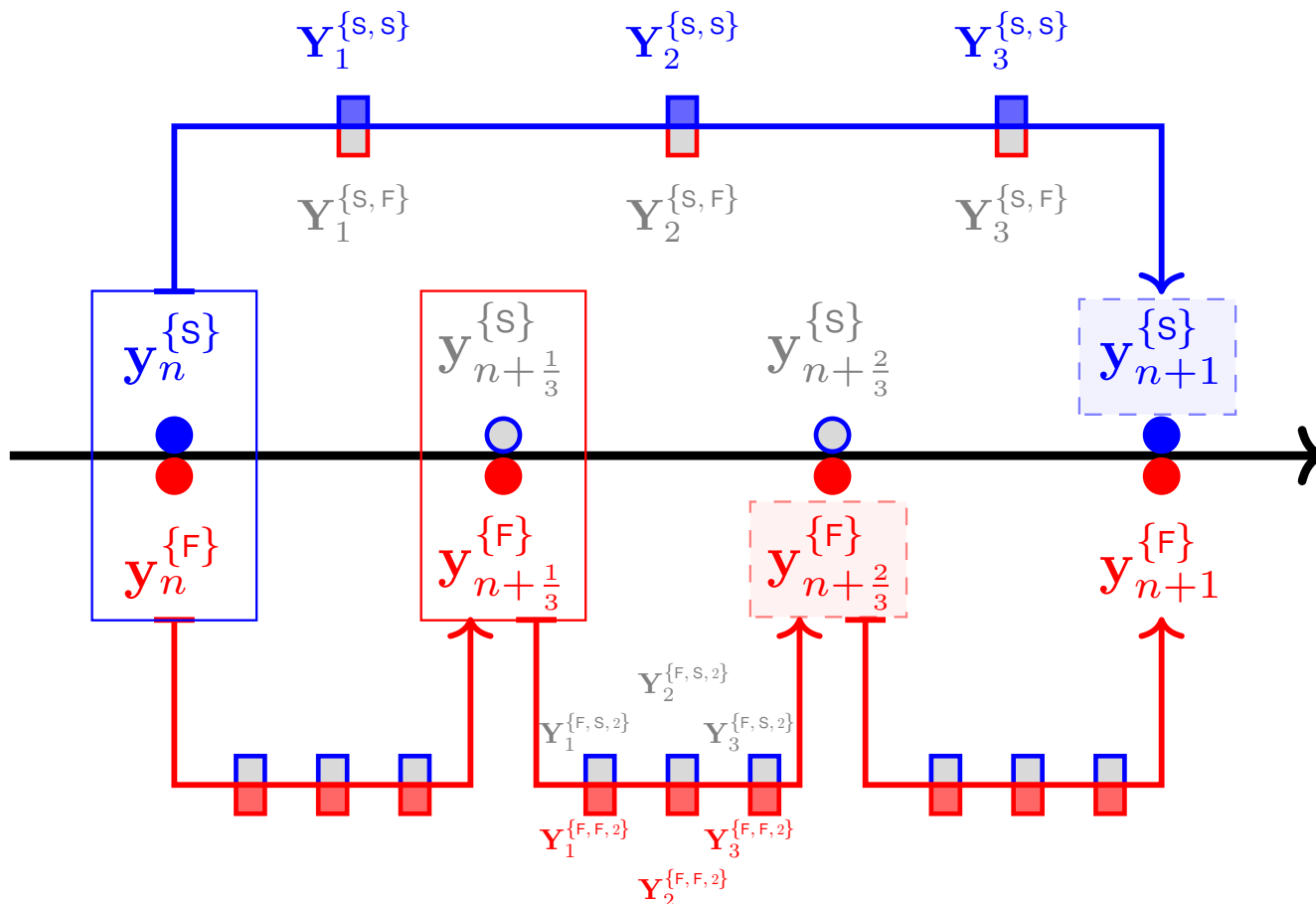
GARK order conditions canonically extend RK. Method coefficients labelled according to node colors, compatible to matrix multiplication rules.

<p>Diagram showing nodes <math>\lambda</math> (white), <math>\mu</math> (white), <math>\nu</math> (black), and <math>\sigma</math> (white). Edges connect <math>\lambda</math> to <math>\mu</math>, <math>\mu</math> to <math>\sigma</math>, and <math>\nu</math> to <math>\sigma</math>. Labels: <math>\lambda</math> <math>\mathbf{c}^{\{\mu,\lambda\}}</math>, <math>\mu</math> <math>\mathbf{A}^{\{\sigma,\mu\}}</math>, <math>\nu</math> <math>\mathbf{c}^{\{\sigma,\nu\}}</math>, <math>\sigma</math> <math>\mathbf{b}^{\{\sigma\}T}</math>.</p>	$\mathbf{b}^{\{\sigma\}T} \left( (\mathbf{A}^{\{\sigma,\mu\}} \mathbf{c}^{\{\mu,\lambda\}}) \times \mathbf{c}^{\{\sigma,\nu\}} \right) = \frac{1}{8}$
<p>Diagram showing a vertical chain of nodes: black, white, black, white, black.</p>	$\mathbf{b}^{\{\mathbf{z}\}T} \mathbf{A}^{\{\mathbf{z},\mathbf{z}\}-1} \mathbf{A}^{\{\mathbf{z},\mathbf{x}\}} \mathbf{A}^{\{\mathbf{x},\mathbf{x}\}} \mathbf{A}^{\{\mathbf{x},\mathbf{z}\}} \mathbf{A}^{\{\mathbf{z},\mathbf{z}\}-1} \mathbf{c}^{\{\mathbf{z}\}} = \frac{1}{6}$
<p>Diagram showing a vertical chain of nodes: black, white, black, white, black. The white nodes have children.</p>	$\mathbf{b}^{\{\mathbf{x}\}T} \mathbf{A}^{\{\mathbf{x},\mathbf{z}\}} \mathbf{A}^{\{\mathbf{z},\mathbf{z}\}-1} \mathbf{A}^{\{\mathbf{z},\mathbf{x}\}} \mathbf{c}^{\{\mathbf{x}\}} \times 2 = \frac{1}{12}$
<p>Diagram showing a vertical chain of nodes: black, white, black, white, black. The white nodes have children.</p>	$\mathbf{b}^{\{\mathbf{z}\}T} \mathbf{A}^{\{\mathbf{z},\mathbf{z}\}-1} (\mathbf{c}^{\{\mathbf{x}\}} \times \mathbf{A}^{\{\mathbf{z},\mathbf{z}\}-1} \mathbf{c}^{\{\mathbf{z}\}} \times 2) = \frac{1}{2}$
<p>Diagram showing a vertical chain of nodes: black, white, black, white, black. The white nodes have children.</p>	$\mathbf{b}^{\{\mathbf{z}\}T} (\mathbf{c}^{\{\mathbf{x}\}} \times \mathbf{A}^{\{\mathbf{z},\mathbf{z}\}-1} \mathbf{c}^{\{\mathbf{z}\}} \times 2) = \frac{2}{3}$



# Multirate GARK framework I

Multirate GARK {Günther & Sandu, 2015} is a natural framework for expressing multirate RK schemes:



# Multirate GARK framework II

Slow component:

$$\mathbf{Y}_i^{\{s,s\}} = \mathbf{y}_n^{\{s\}} + H \sum_{j=1}^{s\{s\}} a_{i,j}^{\{s,s\}} \mathbf{f}^{\{s\}}(\mathbf{Y}_j^{\{s,s\}}, \mathbf{Y}_j^{\{s,F\}})$$

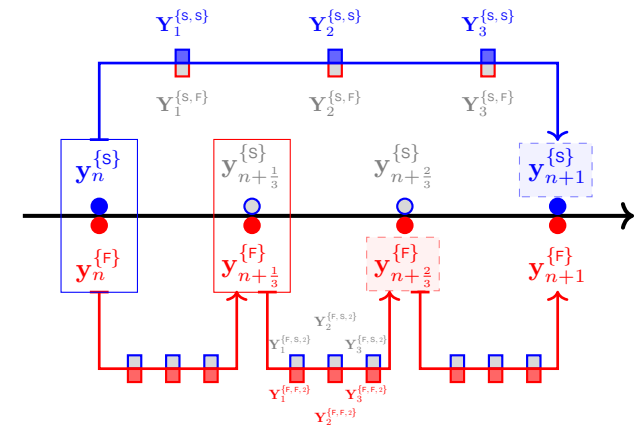
stage  $i$  of slow RK method over  $[t_n, t_n+H]$

$$\mathbf{Y}_i^{\{s,F\}} = \mathbf{y}_n^{\{F\}} + h \sum_{\ell=1}^m \sum_{j=1}^{s\{F\}} a_{i,j}^{\{s,F,\ell\}} \mathbf{f}^{\{F\}}(\mathbf{Y}_j^{\{F,S,\ell\}}, \mathbf{Y}_j^{\{F,F,\ell\}}),$$

fast component value to be paired with slow stage  $i$

$$\mathbf{y}_{n+1}^{\{s\}} = \mathbf{y}_n^{\{s\}} + H \sum_{i=1}^{s\{s\}} b_i^{\{s\}} \mathbf{f}^{\{s\}}(\mathbf{Y}_i^{\{s,s\}}, \mathbf{Y}_i^{\{s,F\}}).$$

solution of slow RK method over  $[t_n, t_n+H]$

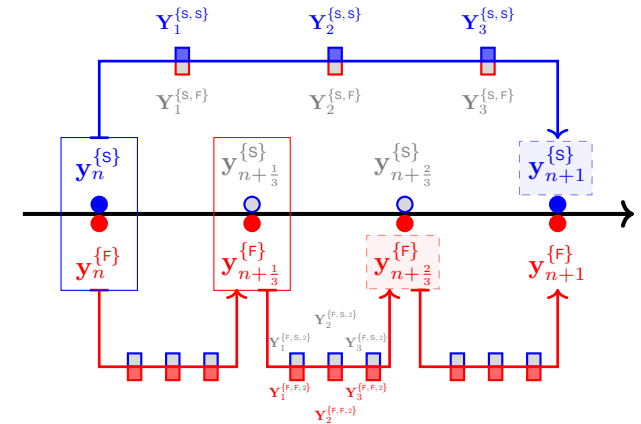


# Multirate GARK framework III

Fast component:

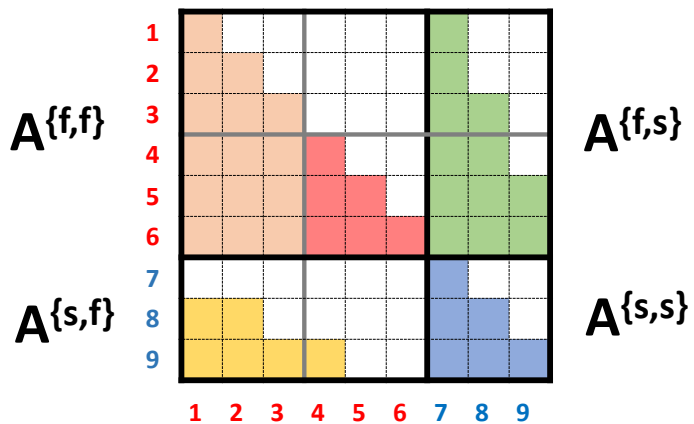
For  $\ell = 1, \dots, m$ :

$$\left\{ \begin{array}{l}
 \tilde{\mathbf{Y}}_i^{\{F, S, \ell\}} = \mathbf{y}_n^{\{S\}} + H \sum_{j=1}^{s^{\{S\}}} a_{i,j}^{\{F, S, \ell\}} \mathbf{f}^{\{S\}}(\mathbf{Y}_j^{\{S, S\}}, \mathbf{Y}_j^{\{S, F\}}) \\
 \text{slow component value to be paired with fast stage } i \\
 \tilde{\mathbf{Y}}_i^{\{F, F, \ell\}} = \tilde{\mathbf{y}}_{n+(\ell-1)/m}^{\{F\}} + h \sum_{j=1}^{s^{\{F\}}} a_{i,j}^{\{F, F\}} \mathbf{f}^{\{F\}}(\mathbf{Y}_j^{\{F, S, \ell\}}, \mathbf{Y}_j^{\{F, F, \ell\}}), \\
 \text{stage } i \text{ of fast RK method over } [t_n + (\ell-1)h, t_n + \ell h] \\
 \tilde{\mathbf{y}}_{n+\ell/m}^{\{F\}} = \tilde{\mathbf{y}}_{n+(\ell-1)/m}^{\{F\}} + h \sum_{i=1}^{s^{\{F\}}} b_i^{\{F\}} \mathbf{f}^{\{F\}}(\mathbf{Y}_i^{\{F, S, \ell\}}, \mathbf{Y}_i^{\{F, F, \ell\}}); \\
 \text{solution of fast RK method over } [t_n + (\ell-1)h, t_n + \ell h] \\
 \mathbf{y}_{n+1}^{\{F\}} = \underbrace{\tilde{\mathbf{y}}_{n+m/m}^{\{F\}}}_{\text{fast component after the last micro-step}}
 \end{array} \right.$$

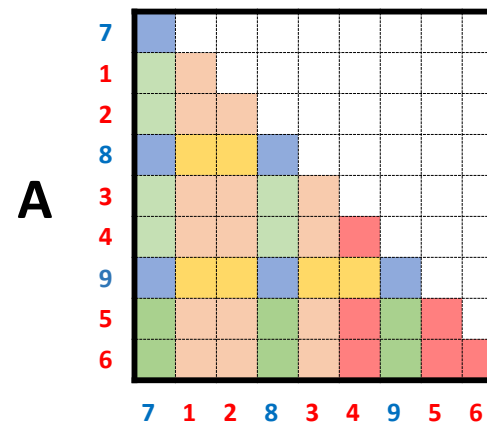


# Multirate GARK framework IV

$$\begin{array}{c|c}
 \mathbf{A}^{\{F,F\}} & \mathbf{A}^{\{F,S\}} \\
 \hline
 \mathbf{A}^{\{S,F\}} & \mathbf{A}^{\{S,S\}} \\
 \hline
 \mathbf{b}^{\{F\}T} & \mathbf{b}^{\{S\}T}
 \end{array}
 :=
 \begin{array}{ccc|c}
 m^{-1}A^{\{F,F\}} & \dots & 0 & A^{\{F,S,1\}} \\
 \vdots & \ddots & & \vdots \\
 m^{-1}\mathbf{1}^{\{F\}} b^{\{F\}T} & \dots & m^{-1}A^{\{F,F\}} & A^{\{F,S,m\}} \\
 \hline
 m^{-1}A^{\{S,F,1\}} & \dots & m^{-1}A^{\{S,F,m\}} & A^{\{S,S\}} \\
 \hline
 m^{-1}b^{\{F\}T} & \dots & m^{-1}b^{\{F\}T} & b^{\{S\}T}
 \end{array}$$



(a) Decoupled MR-GARK. Coupling has complementary sparsity.



(b) Permuted Butcher tableau has lower diagonal structure

# Adaptive MR-GARK methods control errors by adjusting both $H$ and $m$

- ▶ Local truncation error has three components:

$$\text{LTE} = \underbrace{\text{LTE}^{\{S\}}}_{\sim C^{\{S\}} H^{p+1}} + \underbrace{\text{LTE}^{\{F\}}}_{\sim C^{\{F\}} m^{-p} H^{p+1}} + \underbrace{\text{LTE}^{\{c\}}}_{\sim (\sum_{i=0}^{p-1} C_i^{\{c\}} m^{-i}) H^{p+1}}.$$

- ▶ Idea: Use **multiple embedded methods**, such as to obtain additional information about the structure of the local truncation error.

$$\begin{aligned} \text{LTE}_{n+1} &\approx \mathbf{y}_{n+1} - \hat{\mathbf{y}}_{n+1}, & \text{LTE}_{n+1}^{\{S\}} &\approx \mathbf{y}_{n+1} - \hat{\mathbf{y}}_{n+1}^{\{S\}}, & \text{LTE}_{n+1}^{\{F\}} &\approx \mathbf{y}_{n+1} - \hat{\mathbf{y}}_{n+1}^{\{F\}}, \\ \text{LTE}_{n+1}^{\{c\}} &\approx \hat{\mathbf{y}}_{n+1}^{\{S\}} + \hat{\mathbf{y}}_{n+1}^{\{F\}} - \hat{\mathbf{y}}_{n+1} - \mathbf{y}_{n+1}. \end{aligned}$$

- ▶ **Adaptation policy:** Balancing errors

$$\|\text{LTE}_{n+1}\| \Rightarrow H_{\text{new}}; \quad \|\text{LTE}_{n+1}^{\{S\}}\| = \|\text{LTE}_{n+1}^{\{F\}}\| \Rightarrow m_{\text{new}}.$$

- ▶ **Adaptation policy:** Efficiency optimization ( $t^{\{S\}}$ ,  $t^{\{F\}}$  are slow/fast step costs)

$$\min_{H_{\text{new}}, m_{\text{new}}} \frac{t^{\{S\}} + m_{\text{new}} t^{\{F\}}}{H_{\text{new}}} \quad \text{subject to} \quad \|\text{LTE}_{n+1}\| = 1.$$

# Overview of practical MR-GARK methods

{Sarshar et al, SISC 2019; Roberts et al, JSC 2021; Hachtel et al, JCAM 2021}

- ▶ Sparsity patterns of the coupling matrices  $\mathbf{A}^{\{S, F\}}$  and  $\mathbf{A}^{\{F, S\}}$ , determine the F/S coupling patterns and implementation complexity.
- ▶ Multiple embedded schemes reveal different parts of the LTE (fast, slow, coupling errors).
- ▶ Control errors by adapting both  $H$  and  $m$ .

Order	Fast Method	Slow Method	Stiff Accuracy	FSAL FSAL	$H$ -Adaptive	Naturally Adaptive
2	Ralston's ERK2(1)2 {Ralston, 1962}	Ralston's ERK2(1)2 {Ralston, 1962}			✓	✓
2	SDIRK2(1)2 {Alexander, 1977}	Ralston's ERK2(1)2 {Ralston, 1962}	✓		✓	
2	Ralston's ERK2(1)2 {Ralston, 1962}	SDIRK2(1)2 {Alexander, 1977}	✓		✓	
3	Ralston's ERK3(2)3 {Ralston, 1962}	Ralston's ERK3(2)3 {Ralston, 1962}			✓	
3	SDIRK3(2)3 {Alexander, 1977}	Ralston's ERK3(2)3 {Ralston, 1962}	✓		✓	
3	Ralston's ERK3(2)3 {Ralston, 1962}	SDIRK3(2)3 {Alexander, 1977}	✓		✓	
3	Custom ERK3(2)3	Custom ERK3(2)3			✓	✓
4	ERK4(3)5 {Sofroniou and Spaletta, 2004}	ERK4(3)5 {Sofroniou and Spaletta, 2004}		✓	✓	
4	Fehlberg's ERK4(5)6 {Fehlberg, 1969}	SDIRK4(3)5 {Kennedy and Carpenter, 2016}	✓		✓	
4	ERK4(3)4 {Sofroniou and Spaletta, 2004}	Custom SDIRK4(3)6	✓		✓	

# Solving complex systems with Rosenbrock(-W) methods

- ▶ One step of a traditional ROS(-W) scheme:

$$\mathbf{y}' = \mathbf{f}^{\{1\}}(\mathbf{y}) + \cdots + \mathbf{f}^{\{N\}}(\mathbf{y}) \in \mathbb{R}^d;$$

$$\mathbf{k} = h \left( \mathbf{f}^{\{1\}} + \cdots + \mathbf{f}^{\{N\}} \right) (\mathbf{1}_s \otimes \mathbf{y}_n + \boldsymbol{\alpha} \otimes \mathbf{k}) \\ + \left( \mathbf{I}_{s \times s} \otimes h \left( \mathbf{J}_n^{\{1\}} + \cdots + \mathbf{J}_n^{\{N\}} \right) \right) (\boldsymbol{\gamma} \otimes \mathbf{k}),$$

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \mathbf{b}^T \otimes \mathbf{k}.$$

- ▶ One set of ROS(-W) coefficients for all component functions.
- ▶ Same increment vectors  $\mathbf{k} \in \mathbb{R}^d$  for all components.
- ▶ ROS linear algebra uses the total Jacobian  $\mathbf{J}_n^{\{1\}} + \cdots + \mathbf{J}_n^{\{N\}}$ .
- ▶ ROW linear algebra uses an approximation  $\mathbf{L}_n^{\{1\}} + \cdots + \mathbf{L}_n^{\{N\}}$ .

# Linearly implicit GARK schemes I

{Sandu et al, 2021} GARK-ROS(-W) scheme:

$$\mathbf{y}' = \mathbf{f}^{\{1\}}(\mathbf{y}) + \cdots + \mathbf{f}^{\{N\}}(\mathbf{y}) \in \mathbb{R}^d;$$

$$\mathbf{k}^{\{1\}} = h \mathbf{f}^{\{1\}} \left( (\mathbf{1}^{\{q\}} \otimes \mathbf{y}_n + \boldsymbol{\alpha}^{\{1,1\}} \otimes \mathbf{k}^{\{1\}} + \sum_{m=2}^N \boldsymbol{\alpha}^{\{1,m\}} \otimes \mathbf{k}^{\{m\}}) \right) \\ + \left( \mathbf{I}_{s^{\{1\}}} \otimes h \mathbf{J}_n^{\{1\}} \right) \left( \boldsymbol{\gamma}^{\{1,1\}} \otimes \mathbf{k}^{\{1\}} + \sum_{m=2}^N \boldsymbol{\gamma}^{\{1,m\}} \otimes \mathbf{k}^{\{m\}} \right),$$

⋮

$$\mathbf{k}^{\{N\}} = h \mathbf{f}^{\{N\}} \left( (\mathbf{1}^{\{q\}} \otimes \mathbf{y}_n + \sum_{m=1}^{N-1} \boldsymbol{\alpha}^{\{N,m\}} \otimes \mathbf{k}^{\{m\}} + \boldsymbol{\alpha}^{\{N,N\}} \otimes \mathbf{k}^{\{N\}}) \right) \\ + \left( \mathbf{I}_{s^{\{q\}}} \otimes h \mathbf{J}_n^{\{N\}} \right) \left( \sum_{m=1}^{N-1} \boldsymbol{\gamma}^{\{N,m\}} \otimes \mathbf{k}^{\{m\}} + \boldsymbol{\gamma}^{\{N,N\}} \otimes \mathbf{k}^{\{N\}} \right),$$

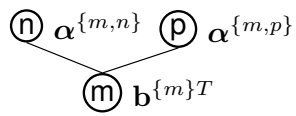
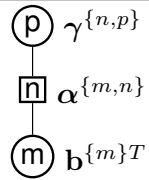
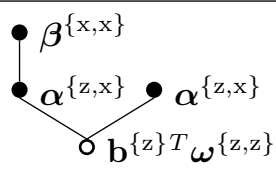
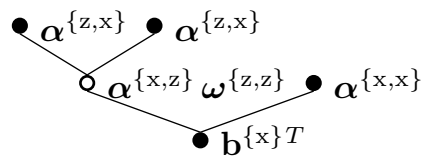
$$\mathbf{y}_{n+1} = \mathbf{y}_n + \mathbf{b}^{\{1\}T} \otimes \mathbf{k}^{\{1\}} + \cdots + \mathbf{b}^{\{N\}T} \otimes \mathbf{k}^{\{N\}}.$$



# Linearly implicit GARK schemes II

► Structure-revealing formulation.

GARK-ROS(-W) order conditions canonically extend GARK-ROS(-W). Method coefficients labelled according to node colors, compatible to matrix multiplication rules.

	$\mathbf{f}_{\mathbf{y},\mathbf{y}}^{\{m\}} (\mathbf{f}^{\{n\}}, \mathbf{f}^{\{p\}})$	$\mathbf{b}^{\{m\}T} ((\boldsymbol{\alpha}^{\{m,n\}} \mathbf{1}^{\{n\}}) \times (\boldsymbol{\alpha}^{\{m,p\}} \mathbf{1}^{\{p\}})) = \frac{1}{3}$
	$\mathbf{f}_{\mathbf{y}}^{\{m\}} \mathbf{L}^{\{n\}} \mathbf{f}^{\{p\}}$	$\mathbf{b}^{\{m\}T} \boldsymbol{\alpha}^{\{m,n\}} \cdot \boldsymbol{\gamma}^{\{n,p\}} \mathbf{1}^{\{p\}} = 0$
	$\mathbf{b}^{\{z\}T} \boldsymbol{\omega}^{\{z,z\}} ((\boldsymbol{\alpha}^{\{z,x\}} \mathbf{e}^{\{x,x\}}) \times \mathbf{c}^{\{z,x\}}) = \frac{1}{2}$	
	$\mathbf{b}^{\{x\}T} (\mathbf{c}^{\{x,x\}} \times (\boldsymbol{\alpha}^{\{x,z\}} \boldsymbol{\omega}^{\{z,z\}} \mathbf{c}^{\{z,x\}} \times 2)) = \frac{1}{4}$	

# Digression: LIMEX GARK ROS(-W) schemes

$$\mathbf{y}' = \mathbf{f}^{\{S\}}(\mathbf{y}) + \mathbf{f}^{\{F\}}(\mathbf{y}).$$

LIMEX GARK ROS(-W) scheme:

$$k_i^{\{S\}} = h \mathbf{f}^{\{S\}} \left( \mathbf{y}_n + \sum_{j=1}^{i-1} \alpha_{i,j}^{\{E,E\}} k_j^{\{S\}} + \sum_{j=1}^{i-1} \alpha_{i,j}^{\{E,I\}} k_j^{\{F\}} \right),$$
$$k_i^{\{F\}} = h \mathbf{f}^{\{F\}} \left( \mathbf{y}_n + \sum_{j=1}^i \alpha_{i,j}^{\{E,E\}} k_j^{\{S\}} + \sum_{j=1}^{i-1} \alpha_{i,j}^{\{I,I\}} k_j^{\{F\}} \right)$$
$$+ h \mathbf{L}^{\{F\}} \left( \sum_{j=1}^i \gamma_{i,j}^{\{E,E\}} k_j^{\{S\}} + \sum_{j=1}^i \gamma_{i,j}^{\{I,I\}} k_j^{\{F\}} \right),$$
$$\mathbf{y}_{n+1} = \mathbf{y}_n + \sum_{i=1}^{s^{\{S\}}} b_i^{\{S\}} k_i^{\{S\}} + \sum_{i=1}^{s^{\{F\}}} b_i^{\{F\}} k_i^{\{F\}}.$$

# Multirate linearly implicit schemes for multiple scales: MR-GARK-ROS(-W) I

{ Günther and Sandu, BIT 2022 }

$$\begin{aligned}
 (m \times h) \left\{ \begin{aligned}
 \mathbf{k}^{\{F, \ell\}} &= h \mathbf{f}^{\{F\}} \left( \mathbf{1}_{s^{\{F\}}} \otimes \mathbf{y}_{n-1+\frac{\ell-1}{m}} + \boldsymbol{\alpha}^{\{F, F\}} \otimes \mathbf{k}^{\{F, \ell\}} + \boldsymbol{\alpha}^{\{F, S, \ell\}} \otimes \mathbf{k}^{\{S\}} \right) \\
 &+ \left( \mathbf{I}_{s^{\{F\}} \times s^{\{F\}}} \otimes h \mathbf{L}^{\{F\}} \right) \left( \boldsymbol{\gamma}^{\{F, F\}} \otimes \mathbf{k}^{\{F, \ell\}} + \boldsymbol{\gamma}^{\{F, S, \ell\}} \otimes \mathbf{k}^{\{S\}} \right), \\
 \mathbf{y}_{n-1+\frac{\ell}{m}} &= \mathbf{y}_{n-1+\frac{\ell-1}{m}} + \mathbf{b}^{\{F, \ell\}T} \otimes \mathbf{k}^{\{F, \ell\}}, \quad \ell = 1, \dots, m;
 \end{aligned} \right. \\
 \\
 (1 \times H) \left\{ \begin{aligned}
 \mathbf{k}^{\{S\}} &= H \mathbf{f}^{\{S\}} \left( \mathbf{1}_{s^{\{S\}}} \otimes \mathbf{y}_{n-1} + \sum_{\ell=1}^m \boldsymbol{\alpha}^{\{S, F, \ell\}} \otimes \mathbf{k}^{\{F, \ell\}} + \boldsymbol{\alpha}^{\{S, S\}} \otimes \mathbf{k}^{\{S\}} \right) \\
 &+ \left( \mathbf{I}_{s^{\{S\}} \times s^{\{S\}}} \otimes H \mathbf{L}^{\{S\}} \right) \left( \sum_{\ell=1}^m \boldsymbol{\gamma}^{\{S, F, \ell\}} \otimes \mathbf{k}^{\{F, \ell\}} + \boldsymbol{\gamma}^{\{S, S\}} \otimes \mathbf{k}^{\{S\}} \right); \\
 \text{(full step)} \quad \mathbf{y}_n &= \mathbf{y}_{n-1} + \sum_{\ell=1}^m \mathbf{b}^{\{F, \ell\}T} \otimes \mathbf{k}^{\{F, \ell\}} + \mathbf{b}^{\{S\}T} \otimes \mathbf{k}^{\{S\}}.
 \end{aligned} \right.
 \end{aligned}$$

Efficient F/S coupling: IMEX, compound-step, step-predictor-corrector.

# Letting $m \rightarrow \infty$ : Multirate “infinitesimal step” approach

Seminal paper of Knoth and Wolke {1998} proposed multirate “infinitesimal step”:

1. discretize the slow component with an ERK method, and
2. advance the solution between consecutive stages by solving a modified fast ODE (original plus constant “slow tendency”).

Further developments:

- ▶ Wensch, Knoth, and Galant {2009} extend idea by adding linear combinations of stages to both the initial condition and to the slow tendency term. They aptly named the resulting schemes “**multirate infinitesimal step**” (MIS) methods.
- ▶ Schlegel, Knoth, Wolke, and Arnold {2012} cast MIS schemes as traditional partitioned Runge-Kutta methods. Further work: {Knoth et al., 2014; Schlegel et al., 2009, 2010, 2011}.
- ▶ Günther and Sandu {2013} explained the MIS schemes in the MR-GARK framework.
- ▶ Sandu {2018} and Roberts, Sarshar, Sandu {2018} developed multirate infinitesimal (MRI-GARK) framework.
- ▶ More recent work: Bremicker-Trubelhorn and Ortleb {2017}, Bauer and Knoth {2018}, Reynolds and students {2018–}.

# The concept via MRI-Adams-Bashforth schemes

$$F^{\{S\}}(t_n + \tau H) = \sum_{j=0}^{k-1} \mathbf{f}_{n-j}^{\{S\}} \ell_{n-j}^{[AB]}(\tau), \quad F^{\{F\}}(t_n + \tau H) = \sum_{j=0}^{k-1} \mathbf{f}_{n-j}^{\{F\}} \ell_{n-j}^{[AB]}(\tau);$$

► AB: 
$$\begin{bmatrix} v^{\{S\}'}(\theta) \\ v^{\{F\}'}(\theta) \end{bmatrix} = \begin{bmatrix} F^{\{S\}}(t_n + \theta) \\ F^{\{F\}}(t_n + \theta) \end{bmatrix}, \quad \theta \in [0, H]; \quad \begin{bmatrix} v^{\{S\}}(0) \\ v^{\{F\}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{y}_n^{\{S\}} \\ \mathbf{y}_n^{\{F\}} \end{bmatrix};$$

$$\begin{cases} \begin{bmatrix} \mathbf{y}_{n+1}^{\{S\}} \\ \mathbf{y}_{n+1}^{\{F\}} \end{bmatrix} = \begin{bmatrix} v^{\{S\}}(H) \\ v^{\{F\}}(H) \end{bmatrix} = \begin{bmatrix} \mathbf{y}_n^{\{S\}} \\ \mathbf{y}_n^{\{F\}} \end{bmatrix} + H \sum_{j=0}^{k-1} \underbrace{\left( \int_0^1 \ell_{n-j}^{[AB]}(\tau) d\tau \right)}_{=\beta_j^{[AB]}} \begin{bmatrix} \mathbf{f}_{n-j}^{\{S\}} \\ \mathbf{f}_{n-j}^{\{F\}} \end{bmatrix}. \end{cases}$$

$$\begin{bmatrix} v^{\{S\}'} \\ v^{\{F\}'} \end{bmatrix} = \begin{bmatrix} F^{\{S\}}(t_n + \theta) \\ \mathbf{f}^{\{F\}}(v^{\{S\}}(\theta), v^{\{F\}}(\theta)) \end{bmatrix}, \quad \theta \in [0, H]; \quad \begin{bmatrix} v^{\{S\}}(0) \\ v^{\{F\}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{y}_n^{\{S\}} \\ \mathbf{y}_n^{\{F\}} \end{bmatrix}$$

► MRI-AB: 
$$\begin{cases} v^{\{F\}}(0) = \mathbf{y}_n^{\{F\}}; \quad \mathbf{y}_{n+1}^{\{F\}} = v^{\{F\}}(H), \\ v^{\{F\}}(\theta)' = \mathbf{f}^{\{F\}}\left(t_n + \theta, \mathbf{y}_n^{\{S\}} + H \sum_{j=0}^{k-1} \tilde{\ell}_{n-j}^{[AB]}(\theta/H) \mathbf{f}_{n-j}^{\{S\}}, v^{\{F\}}(\theta)\right), \\ \mathbf{y}_{n+1}^{\{S\}} = \mathbf{y}_n^{\{S\}} + H \sum_{j=0}^{k-1} \beta_j^{[AB]} \mathbf{f}_{n-j}^{\{S\}}. \end{cases}$$

► Properties: Slow AB, Fast modified ODE,  $\mathbf{f}^{\{F\}} \equiv \mathbf{0}$ ,  $\mathbf{f}^{\{S\}} \equiv \mathbf{0}$ .

# MRI-LMM schemes

General MRI-LMM schemes can be developed similarly.

$$v^{\{S\}}(\theta) = - \sum_{j=0}^{k-1} \alpha_j \mathbf{y}_{n-j}^{\{S\}} + H \sum_{j=0}^{k-1} \tilde{p}_{n-j}(\theta/H) \mathbf{f}_{n-j}^{\{S\}}$$

$$v^{\{F\}}(0) = \mathbf{y}_n^{\{F\}} \left( \text{or } - \sum_{j=0}^{k-1} \alpha_j \mathbf{y}_{n-j}^{\{F\}} \right);$$

$$v^{\{F\}}(\theta)' = \mathbf{f}^{\{F\}} \left( t_n + \theta, v^{\{S\}}(\theta), v^{\{F\}}(\theta) \right), \quad \theta \in [0, H];$$

$$\mathbf{y}_{n+1}^{\{F\}} = v^{\{F\}}(H),$$

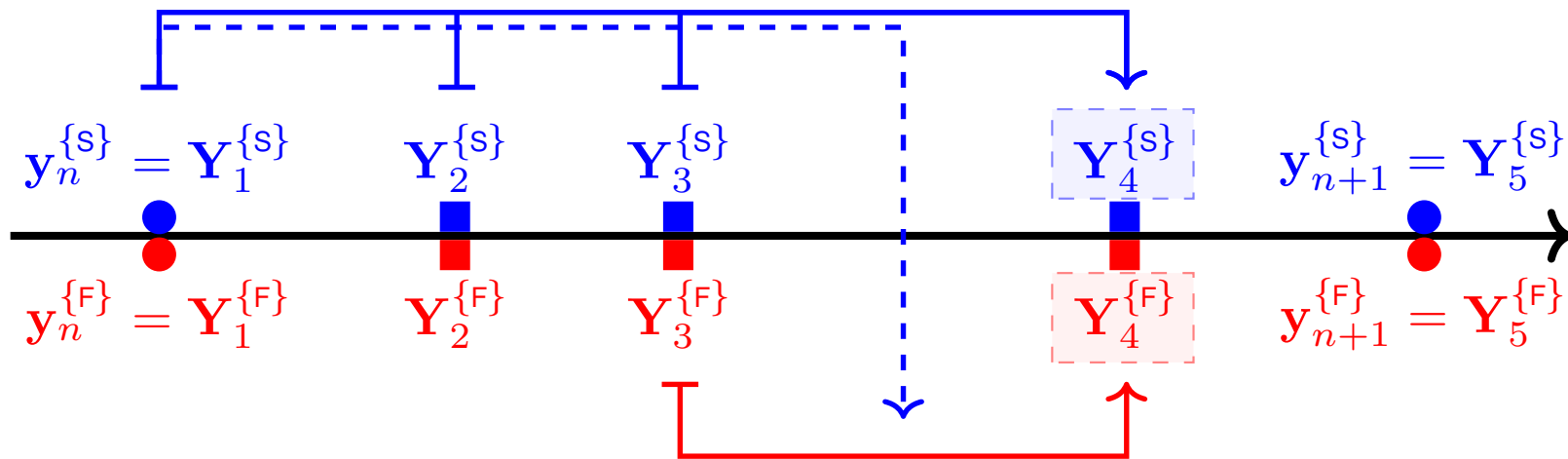
$$\mathbf{y}_{n+1}^{\{S\}} = - \sum_{j=0}^{k-1} \alpha_j \mathbf{y}_{n-j}^{\{S\}} + H \sum_{j=0}^{k-1} \beta_j \mathbf{f}_{n-j}^{\{S\}}.$$

# Multirate infinitesimal (MRI-GARK) schemes I

MIS {Knoth and Wolke, 1998}:  $m \rightarrow \infty$

$\implies$  MRI-GARK {S., SINUM 2019; Roberts et al, SISC 2020}:

$$\begin{bmatrix} \mathbf{y}^{\{S\}} \\ \mathbf{y}^{\{F\}} \end{bmatrix}' = \begin{bmatrix} \mathbf{f}^{\{S\}}(t, \mathbf{y}^{\{S\}}, \mathbf{y}^{\{F\}}) \\ \mathbf{f}^{\{F\}}(t, \mathbf{y}^{\{S\}}, \mathbf{y}^{\{F\}}) \end{bmatrix}$$



$$v_i' = \Delta c_i^{\{S\}} \mathbf{f}^{\{F\}} \left( v_i, \mathbf{Y}_i^{\{S\}} + H \sum_{j=1}^{i+1} \tilde{\gamma}_{i,j}(\theta) \mathbf{f}_j^{\{S\}} \right)$$

# Multirate infinitesimal (MRI-GARK) schemes II

Decoupled MRI-GARK step:

$$\mathbf{Y}_1^{\{S\}} = \mathbf{y}_n^{\{S\}}, \quad \mathbf{Y}_1^{\{F\}} = \mathbf{y}_n^{\{F\}},$$

For each stage  $i = 1, \dots, s^{\{S\}}$  :

$$\left\{ \begin{array}{l} v_i(0) = \mathbf{Y}_i^{\{F\}}, \\ v_i' = \Delta c_i^{\{S\}} \mathbf{f}^{\{F\}} \left( T_i + \Delta c_i^{\{S\}} \theta, v_i, \mathbf{Y}_i + H \sum_{j=1}^{i+1} \boxed{\tilde{\gamma}_{i,j} \left( \frac{\theta}{H} \right)} \mathbf{f}_j^{\{S\}} \right), \quad \theta \in [0, H]; \\ \mathbf{Y}_{i+1}^{\{F\}} = v_i(H); \end{array} \right.$$

$$\left\{ \begin{array}{l} \mathbf{Y}_{i+1}^{\{S\}} = \mathbf{Y}_i^{\{S\}} + H \sum_{j=1}^{i+1} \boxed{\tilde{\gamma}_{i,j}(1)} \mathbf{f}_j^{\{S\}}; \\ \mathbf{f}_{i+1}^{\{S\}} = \mathbf{f}^{\{S\}} \left( T_{i+1}, \mathbf{Y}_{i+1}^{\{F\}}, \mathbf{Y}_{i+1}^{\{S\}} \right); \end{array} \right.$$

$$\mathbf{y}_{n+1}^{\{S\}} = \mathbf{Y}_{s^{\{S\}}+1}, \quad \mathbf{y}_{n+1}^{\{F\}} = \mathbf{Y}_{s^{\{S\}}+1}^{\{F\}}.$$



# Examples of explicit and implicit MRI-GARK methods

$$A_{\text{EMIDP}}^{\{s\}} = \begin{array}{c|cc} 0 & 0 & 0 \\ \hline \frac{1}{2} & \frac{1}{2} & 0 \\ \hline 1 & 0 & 1 \end{array}, \quad A_{\text{ITRAP}}^{\{s\}} = \begin{array}{c|cc} 0 & 0 & 0 \\ \hline 1 & \frac{1}{2} & \frac{1}{2} \\ \hline 1 & \frac{1}{2} & \frac{1}{2} \end{array}.$$

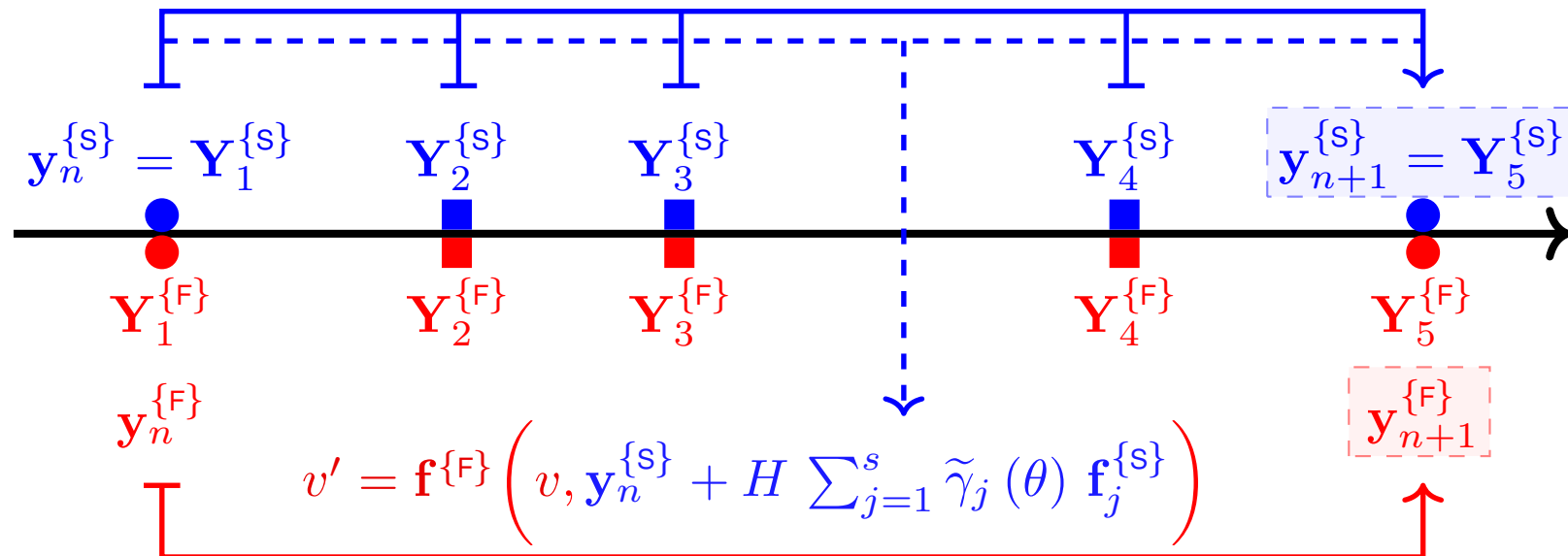
The **explicit midpoint** method:

$$\begin{aligned} v_1(0) &= \mathbf{y}_n; & v_1' &= \frac{1}{2} \mathbf{f}^{\{F\}}(v_1) + \frac{1}{2} \mathbf{f}^{\{s\}}(\mathbf{y}_n), & \theta &\in [0, H]; & \mathbf{Y}_2 &= v_1(H), \\ v_2(0) &= \mathbf{Y}_2; & v_2' &= \frac{1}{2} \mathbf{f}^{\{F\}}(v_2) - \frac{1}{2} \mathbf{f}^{\{s\}}(\mathbf{y}_n) + \mathbf{f}^{\{s\}}(\mathbf{Y}_2), & \theta &\in [0, H], \\ \mathbf{y}_{n+1} &= v_2(H). \end{aligned}$$

The **implicit trapezoidal** method:

$$\begin{aligned} v(0) &= \mathbf{y}_n; & v' &= \mathbf{f}^{\{F\}}(v) + \mathbf{f}^{\{s\}}(\mathbf{y}_n), & \theta &\in [0, H]; & \mathbf{Y}_2 &= v(H), \\ \mathbf{y}_{n+1} &= \mathbf{Y}_2 - \frac{1}{2} \mathbf{f}^{\{s\}}(\mathbf{y}_n) + \frac{1}{2} \mathbf{f}^{\{s\}}(\mathbf{y}_{n+1}). \end{aligned}$$

# Coupled implicit MRI-GARK: Step predictor-corrector



# Scalar linear stability analysis

**Model problem** for additively partitioned systems:

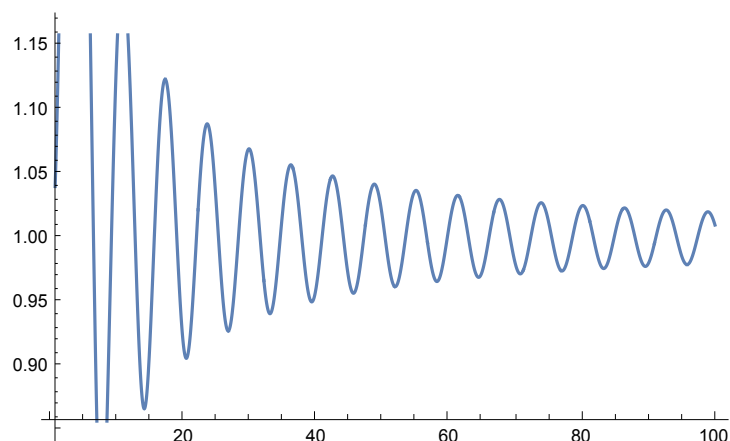
$$y' = \lambda^{\{F\}} y + \lambda^{\{S\}} y, \quad \lambda^{\{F\}}, \lambda^{\{S\}} \in \mathbb{C}^-.$$

**Example:** Stability functions of second order methods.

$$R^{\text{EMIDP}}(z^{\{F\}}, z^{\{S\}}) = \varphi_0(z^{\{F\}}) + \left( \frac{3}{2} \varphi_0\left(\frac{1}{2} z^{\{F\}}\right) - \frac{1}{2} \right) \varphi_1\left(\frac{1}{2} z^{\{F\}}\right) z^{\{S\}} + \frac{1}{2} \varphi_1^2\left(\frac{1}{2} z^{\{F\}}\right) z^{\{S\}^2},$$

$$R^{\text{ITRAP}}(z^{\{F\}}, z^{\{S\}}) = \frac{\varphi_0(z^{\{F\}}) + \left( \varphi_1(z^{\{F\}}) - \frac{1}{2} \right) z^{\{S\}}}{1 - \frac{1}{2} z^{\{S\}}},$$

$$\varphi_0(z) := e^z, \quad \varphi_k(z) := \int_0^1 e^{z(1-t)} t^{k-1} dt.$$



$$|R^{\text{ITRAP}}(z^{\{F\}} = i\omega, z^{\{S\}} \rightarrow -\infty)|$$

# Matrix linear stability analysis

**Model problem** for component partitioned systems:

$$\begin{bmatrix} y^{\{F\}} \\ y^{\{S\}} \end{bmatrix}' = \underbrace{\begin{bmatrix} \lambda^{\{F\}} & -\frac{1-\xi}{\alpha} (\lambda^{\{F\}} - \lambda^{\{S\}}) \\ \alpha \xi (\lambda^{\{F\}} - \lambda^{\{S\}}) & \lambda^{\{S\}} \end{bmatrix}}_{\Omega} \begin{bmatrix} y^{\{F\}} \\ y^{\{S\}} \end{bmatrix},$$

$$\text{eig}(\Omega) = \left\{ \xi \lambda^{\{F\}} + (1 - \xi) \lambda^{\{S\}}, \left[ \frac{1}{\alpha}, 1 \right]^T \right\}, \quad \left\{ (1 - \xi) \lambda^{\{F\}} + \xi \lambda^{\{S\}}, \left[ \frac{1 - \xi}{\alpha}, \xi \right]^T \right\}.$$

**Example:** Stability matrices.

$$\mathbf{M}^{\text{ITRAP}} = \begin{bmatrix} \varphi_0 + w \varphi_2(z^{\{F\}}) & -\frac{1-\xi}{\alpha} (z^{\{F\}} - z^{\{S\}}) (\varphi_1(z^{\{F\}}) + z^{\{F\}} \varphi_2(z^{\{F\}})) \\ \alpha \xi \frac{(z^{\{F\}} - z^{\{S\}})}{2 - z^{\{S\}}} & \frac{2 + z^{\{S\}}}{2 - z^{\{S\}}} \end{bmatrix}.$$

# MRI-GARK schemes as (generalized) exponential methods: integrate exactly a non-linear component

Semi-linear additive system partitioning:

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}) = \underbrace{\mathbf{L}\mathbf{y}}_{\mathbf{f}^{\{f\}}(\mathbf{y})} + \underbrace{\mathbf{f}(\mathbf{y}) - \mathbf{L}\mathbf{y}}_{\mathbf{f}^{\{s\}}(\mathbf{y})=\mathbf{g}(\mathbf{y})}.$$

MRI-GARK leads to an exponential Runge-Kutta method:

$$\mathbf{Y}_0^{\{s\}} := \mathbf{y}_n, \quad c_0^{\{s\}} := 0,$$

$$\mathbf{Y}_i^{\{s\}} = \mathbf{Y}_{i-1}^{\{s\}} + \Delta c_i^{\{s\}} H \varphi_1(\Delta c_i^{\{s\}} H \mathbf{L}) \mathbf{f}(\mathbf{Y}_{i-1}^{\{s\}}) + H \sum_{j=1}^{i-2} \theta_{i,j}(H\mathbf{L}) \mathbf{g}(\mathbf{Y}_j^{\{s\}})$$

$$+ H \psi_{i,i-1}(H\mathbf{L}) \mathbf{g}(\mathbf{Y}_{i-1}^{\{s\}}), \quad i = 1, \dots, s^{\{s\}},$$

$$\mathbf{y}_{n+1} = \mathbf{Y}_{s^{\{s\}}}^{\{s\}},$$

# MRI-GARK-ROS(-W) methods

Base slow scheme  $(\mathbf{b}^{\{s\}}, \boldsymbol{\alpha}^{\{s,s\}}, \boldsymbol{\gamma}^{\{s,s\}})$ . MRI-GARK-ROS(-W) step:

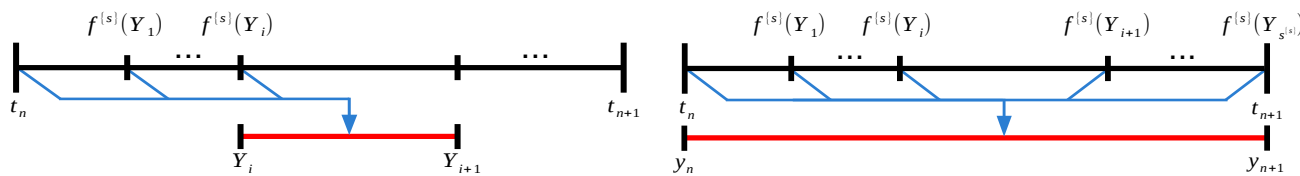
$$\tilde{\mathbf{y}}_{n-1} = \mathbf{y}_{n-1};$$

For  $\ell = 1, \dots, s^{\{s\}}$  :

$$\begin{cases} \mathbf{v}_\ell(0) = \tilde{\mathbf{y}}_{n-1+\frac{\ell-1}{s}}, \\ \mathbf{v}'_\ell(\theta) = \Delta \mathbf{c}_\ell^{\{s,s\}} \mathbf{f}^{\{F\}} \left( \mathbf{v}_\ell(\theta) + \sum_{j=1}^{\ell-1} \mathbf{r}_{\ell,j} \left( \frac{\theta}{H} \right) \mathbf{k}_j^{\{s\}} \right), & \theta \in [0, H], \\ \tilde{\mathbf{y}}_{n-1+\frac{\ell}{s}} = \mathbf{v}_\ell(H); \end{cases}$$

$$\begin{cases} \mathbf{k}_\ell^{\{s\}} = H \mathbf{f}^{\{s\}} \left( \tilde{\mathbf{y}}_{n-1+\frac{\ell}{s}} + \sum_{j=1}^{\ell-1} \boldsymbol{\alpha}_{\ell,j}^{\{s,s\}} \mathbf{k}_j^{\{s\}} \right) \\ \quad + H \mathbf{L}^{\{s\}} \left( \mathbf{q}_{\ell,\ell} \left( \tilde{\mathbf{y}}_{n-1+\frac{\ell}{s}} - \tilde{\mathbf{y}}_{n-1+\frac{\ell-1}{s}} \right) + \sum_{j=1}^{\ell} \boldsymbol{\gamma}_{\ell,j}^{\{s,s\}} \mathbf{k}_j^{\{s\}} \right); \end{cases}$$

$$\mathbf{y}_n = \tilde{\mathbf{y}}_n + \mathbf{b}^{\{s\}T} \otimes \mathbf{k}^{\{s\}}.$$



Coupling coefficients:  $\mathbf{r}_\ell(\theta) = \sum_{k \geq 0} \mathbf{r}_{\ell,k} \theta^k \in \mathbb{R}^{s^{\{s\}}}$ ,  $\mathbf{q}_\ell \in \mathbb{R}^{s^{\{s\}}}$ .

# Step predictor-corrector MRI-GARK-ROS(-W) I

Step predictor:

$$\begin{aligned} \mathbf{k} &= H(\mathbf{f}^{\{F\}} + \mathbf{f}^{\{S\}}) \left( \mathbf{1}_s \otimes \mathbf{y}_{n-1} + \boldsymbol{\alpha}^{\{s,s\}} \otimes \mathbf{k} \right) \\ &\quad + \left( \mathbf{I}_{s \times s} \otimes H(\mathbf{L}^{\{F\}} + \mathbf{L}^{\{S\}}) \right) \left( \boldsymbol{\gamma}^{\{s,s\}} \otimes \mathbf{k} \right), \\ \mathbf{k}^{\{s\}} &= H \mathbf{f}^{\{s\}} \left( \mathbf{1}_{s^{\{s\}}} \otimes \mathbf{y}_{n-1} + \boldsymbol{\alpha}^{\{s,s\}} \otimes \mathbf{k} \right) + \left( \mathbf{I}_{s \times s} \otimes H \mathbf{L}^{\{s\}} \right) \left( \boldsymbol{\gamma}^{\{s,s\}} \otimes \mathbf{k} \right). \end{aligned}$$

Step corrector:

$$\begin{aligned} \mathbf{v}(0) &= \mathbf{y}_{n-1}, \\ \mathbf{v}' &= \mathbf{f}^{\{F\}} \left( \mathbf{v} + \boldsymbol{\mu}^T \left( \frac{\theta}{H} \right) \otimes \mathbf{k}^{\{s\}} \right), \quad 0 \leq \theta \leq H. \\ \mathbf{y}_n &= \mathbf{v}(H) + \mathbf{b}^{\{s\}T} \otimes \mathbf{k}^{\{s\}}. \end{aligned}$$

Multirate infinitesimal RODAS, order 4:  $\boldsymbol{\mu}(t) := \boldsymbol{\mu}_0 + \boldsymbol{\mu}_1 t$ :

$$\boldsymbol{\mu}_0 = \begin{bmatrix} \theta_1 \\ \theta_2 \\ -1.923 \theta_1 + 2.44\mathbf{e-01} \theta_2 + 4.509\mathbf{e-02} \\ 1.405 \theta_1 - 2.181 \theta_2 + 1.289\mathbf{e-01} \\ -4.812\mathbf{e-01} \theta_1 + 9.371\mathbf{e-01} \theta_2 - 1.740\mathbf{e-01} - \theta_3 \\ \theta_3 \end{bmatrix}, \quad \boldsymbol{\mu}_1 = \begin{bmatrix} -2 \theta_1 + 4.061\mathbf{e-01} \\ -2 \theta_2 + 5.932358823451654\mathbf{e-01} \\ 3.847 \theta_1 - 4.892\mathbf{e-01} \theta_2 - 3.657\mathbf{e-01} \\ -2.810 \theta_1 + 4.363 \theta_2 + 5.00\mathbf{e-02} \\ 9.625\mathbf{e-01} \theta_1 - 1.8741 \theta_2 + 3.162\mathbf{e-01} - \theta_4 \\ \theta_4 \end{bmatrix}.$$

# Multirating, closing remarks

- ▶ Long history: MR variants of many families of methods developed
- ▶ *New algorithmic and analysis developments* power modern MR era
- ▶ MRI approach very flexible, promising
- ▶ Toward a wide(r) adoption of MR schemes:
  - ▶ Split the system, dynamically
  - ▶ Telescopic algorithms for any number of levels
  - ▶ Adapt time steps at all levels to control overall error
  - ▶ Ensure special solution properties
  - ▶ Factor codes to work with individual partitions efficiently.
  - ▶ Develop general MR time integration software

## References (hyperlinks):

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