Power Flow Equations: Complexity, Approximations and Relaxations

Pascal Van Hentenryck
Credits

▶ Power systems
  – PSCC’11, PES’12 (2), PSCC’14 (5)
  – IJEPES (to appear)

▶ Operations research
  – IJOC 2014, Mathematical Programming 2015

▶ Submissions
  – complexity results (under submission)

▶ People
  – C. Coffrin, A. Grastien, H. Hijazi, K. Lehmann, T. Mak
  – R. Bent (LANL)
  – D. Hill and I. Hiskens
Outline

- Motivation
  - Power Flow Formulations
  - Complexity
  - The LPAC Approximation
  - Case Studies for LPAC
  - Convex Relaxations
  - Case Studies for Convex Relaxations
  - Beyond Steady States
Power Systems

- Biggest machine on earth
  - 400 billions of electricity
Why fix it?

- New challenges
  - challenging existing assumptions
- New applications
  - requiring new technology
- New enabling technologies
  - enabling new functionalities
Motivation.

- Category 3
  - August 21-28, 2011
- Fatalities
  - 49 direct (+ 7)
- Damages
  - ~ $15 billions
San Diego Blackout

- **Causes**
  - Tripping of a line between Arizona and California
  - Cascading effect (not supposed to happen)

- **Effects**
  - >4 millions people without power, Sept. 8-9, 2011

- **Economic Losses over US$ 100 million**
  - Opportunity losses: $70 million
  - Overtime workers: $20 million
  - Spoiled food: $18 million
Joint Repair and Power Restoration

- The challenge
  - Schedule a fleet of repair crews to repair the grid and minimize the overall size of the blackout after a disaster

- Two fundamental aspects
  - Scheduling the repairs
  - Scheduling the power restoration
  - Both are challenging in their own right

- Assumptions for Last-Mile Restoration
  - Steady state behavior of the power grid
  - Ability to dispatch load and generation continuously
Joint Repair and Power Restoration

- **Power Flow**
- **Time**

**Graph:**
- **Restoration Timeline**
- **Minimize**
- **Increase in served demand**
- **Component repair**

From imagination to impact
Joint Repair and Power Restoration

Routing Aspect

Power Flow Aspect
Joint Repair and Power Restoration

- 2-Step Approach (PSCC’11)
  - Restoration Ordering Problem (ROP)
  - Pickup and Delivery Routing with Precedences
    - Randomized Adaptive Decomposition over LNS over CP
Restoration Ordering Problem (ROP)

- Find the best sequence of restoration $[x_1, x_2, x_3, \ldots, x_n]$ so that the size of the blackout is minimised.
The ROP Problem

- Extremely challenging computationally
  - generalizes transmission switching
Computational Challenge

presolved problem has 1960 variables and 2186 constraints
  510 constraints of type <varbound>
  1475 constraints of type <linear>
  201 constraints of type <logicor>
Presolving Time: 0.52

<table>
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<tr>
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The ROP Problem

- Extremely challenging computationally
  - generalizes transmission switching
- Braess paradox
  - restoring a line may decrease the network flow
Braess Paradox
Joint Repair and Power Restoration
The ROP Problem

Modeling the power system
  - Line capacities (thermal constraints)
  - Constraints on real and reactive power injections
  - Power Flow equations
Power Flow Equations

- **Ohm’s law for each line (i,j)**

\[
p_{ij} = g_{ij} v_i^2 - g_{ij} v_i v_j \cos(\theta_i - \theta_j) - b_{ij} v_i v_j \sin(\theta_i - \theta_j)
\]

\[
q_{ij} = -b_{ij} v_i^2 + b_{ij} v_i v_j \cos(\theta_i - \theta_j) - g_{ij} v_i v_j \sin(\theta_i - \theta_j)
\]

- **Kirchhoff’s current law for each bus i**

\[
p_i = \sum_{(i,j) \in E} p_{ij}
\]

\[
q_i = \sum_{(i,j) \in E} q_{ij}
\]
The Core ROP Subproblem

- Push as much load as possible in the network while satisfying the power flow and side constraints
  - nonlinear
  - nonconvex
  - discrete
The Linear DC Model

conductance is much smaller than susceptance

\[ p_{ij} = \sim \]

\[ \sin(x) \text{ close to } x \text{ when } x \text{ is small} \]

\[ -b_{ij} (\theta_i - \theta_j) \]

Ignore reactive power

Voltage magnitudes are close to 1.0
The Linear DC Model

- Ohm’s law for each line (i,j)
  \[ p_{ij} = -b_{ij} (\theta_i - \theta_j) \]

- Kirchhoff’s current law for each bus i
  \[ p_i = \sum_{(i,j) \in E} p_{ij} \]
In the ROP

- The power equation becomes

\[ p_{ij} = -b_{ij} z_{ij} (\theta_i - \theta_j) \]

- It is nonlinear: can be linearized since \( z_i \) is a 0/1 variable

\[ p_{ij} \leq b_{ij} (\theta_i - \theta_j) + M(1 - z_{ij}) \]
\[ p_{ij} \geq b_{ij} (\theta_i - \theta_j) - M(1 - z_{ij}) \]
Case Studies

- 15 large disaster scenarios
  - with damage size in 50-1000
- Generated using,
  - US Transmission and Transportation Infrastructure
  - State-of-the-art disaster simulation tools (NHC, FEMA)
- Study quality over 0.5 - 8 hours
  - Average of 10 runs
- Comparing with best practices in the field
  - utilisation heuristics + greedy routing
- Lower bound
  - assuming infinitely many crews
Case Studies

Restoration Timeline – BM2 S16

Power Flow

Time

292835.177
Case Studies

Restoration Timeline – BM2 S14

Power Flow

Time

366219.884
Case Studies

Restoration Timeline – BM2 S9

Time

Power Flow

567393.666
Case Studies

Fig. 4. Size of the Blackouts: 97 and 504 Tasks.
Power Restoration
A fundamental open question

- Is this “optimal” restoration plan “feasible” operationally?
A fundamental open question

- Is this “optimal” restoration plan “feasible” operationally?

- These are not “normal operating” conditions
  - “Maddeningly difficult” to find an AC solution in cold start contexts [Overbye et al, 2004]

- The ROP is stressing the network
AC Power/Load Flow

- Seed a power flow study
  - with the optimization results
- Different information for different buses

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<th></th>
<th>P</th>
<th>Q</th>
<th>V</th>
<th>δ</th>
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<td>known</td>
<td>unknown</td>
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<tr>
<td>P-V bus</td>
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Power Restoration
N-3 Contingencies (IEEE-30)
N-10 Contingencies (IEEE-30)

Fig. 3. Histograms of Reactive Injection (left) and Voltage Magnitudes (right) of N–10 Contingencies using the DC-GDP Algorithm.
Expansion Planning

- Simplest problem formulation
  - starting point
- Network design problem
  - add lines to meet the increased load
- Under
  - voltage constraints
  - thermal limits
  - constraints on real and reactive power injection
## Expansion Planning

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<td>9</td>
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<td>14</td>
<td>15 (15)</td>
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<td>57</td>
<td>49 (49)</td>
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<tr>
<td>118</td>
<td>37 (37)</td>
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New Challenges

- Power applications that
  - are **mixed** nonlinear optimisation problems
  - require **accurate** (coupled?) models of the power system
    - congestion

- Observe that
  - the discrete nature precludes some technology
    - integrating discrete optimization and homotopy methods is not easy
  - need for relaxations!
    - performance guarantees
Outline

- Motivation
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Power Flows

- Complex Number Formulation

\[
S_i^g - S_i^d = \sum_{(i,j) \in E \cup E^R} S_{ij} \quad \forall i \in N
\]

\[
S_{ij} = Y_{ij}^* V_i V_i^* - Y_{ij}^* V_i V_j^* \quad (i, j) \in E \cup E^R
\]
Power Flows

- Rectangular formulation

\[
p_i^g - p_i^d = \sum_{(i,j) \in E \cup E^R} p_{ij} \quad \forall i \in N
\]

\[
q_i^g - q_i^d = \sum_{(i,j) \in E \cup E^R} q_{ij} \quad \forall i \in N
\]

\[
p_{ij} = g_{ij} \left( (v_i^R)^2 + (v_i^I)^2 \right) - g_{ij} \left( v_i^R v_j^R + v_i^I v_j^I \right) - b_{ij} \left( v_i^I v_j^R - v_i^R v_j^I \right)
\]

\[
q_{ij} = -b_{ij} \left( (v_i^R)^2 + (v_i^I)^2 \right) + b_{ij} \left( v_i^R v_j^R + v_i^I v_j^I \right) - g_{ij} \left( v_i^I v_j^R - v_i^R v_j^I \right)
\]
Power Flows

- Hybrid formulation

\[
 p_i^g - p_i^d = \sum_{(i,j) \in E \cup E^R} p_{ij} \quad \forall i \in N
\]

\[
 q_i^g - q_i^d = \sum_{(i,j) \in E \cup E^R} q_{ij} \quad \forall i \in N
\]

\[
 p_{ij} = g_{ij} v_i^2 - g_{ij} v_i v_j \cos(\theta_i - \theta_j) - b_{ij} v_i v_j \sin(\theta_i - \theta_j)
\]

\[
 q_{ij} = -b_{ij} v_i^2 + b_{ij} v_i v_j \cos(\theta_i - \theta_j) - g_{ij} v_i v_j \sin(\theta_i - \theta_j)
\]
Power Flows

- Side constraints

\[
\sqrt{p_{ij}^2 + q_{ij}^2} \leq s_{ij}^u \quad \forall (i, j) \in E \cup E^R
\]

\[
\nu_i^l \leq v_i \leq \nu_i^u \quad i \in N
\]

\[
- \theta_{ij}^\Delta \leq \theta_i - \theta_j \leq \theta_{ij}^\Delta \quad \forall (i, j) \in E
\]
W-Formulation

- Define

\[ W_{ij} = V_i V_j^* \]

- Power Flow becomes

\[
S_i^g - S_i^d = \sum_{(i,j) \in E \cup E^R} S_{ij} \quad \forall i \in N
\]

\[
S_{ij} = Y_{ij}^* W_{ii} - Y_{ij}^* W_{ij} \quad (i, j) \in E \cup E^R
\]

\[ W \geq 0 \]

\[ \text{rank}(W) = 1 \]
W-Formulation

\[ p_i^g - p_i^d = \sum_{(i,j) \in E \cup E^R} p_{ij} \quad \forall i \in N \]

\[ q_i^g - q_i^d = \sum_{(i,j) \in E \cup E^R} q_{ij} \quad \forall i \in N \]

\[ p_{ij} = g_{ij} w_{ii}^R - g_{ij} w_{ij}^R - b_{ij} w_{ij}^I \]

\[ q_{ij} = -b_{ij} w_{ii}^R + b_{ij} w_{ij}^R - g_{ij} w_{ij}^I \]

\[ W \geq 0 \]

\[ \text{rank}(W) = 1 \]
Dist-Flow Relaxation


\[
p_{ij} = \sum_{k:(j,k)\in E} p_{jk} + r_{ij} \frac{p_{ij}^2 + q_{ij}^2}{v_i^2}
\]

\[
q_{ij} = \sum_{k:(j,k)\in E} q_{jk} + x_{ij} \frac{p_{ij}^2 + q_{ij}^2}{v_i^2}
\]

\[
v_j^2 = v_i^2 - 2(r_{ij}p_{ij} + x_{ij}q_{ij}) + (r_{ij}^2 + x_{ij}^2) \frac{p_{ij}^2 + q_{ij}^2}{v_i^2}
\]
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Complexity of AC-Feasibility

- AC-Feasibility is NP-Hard

- What are bananas?
  - small circuits that create discontinuities
Complexity of AC-Feasibility

- AC-Feasibility is NP-Hard
  - Introducing discontinuities

\[
\begin{align*}
\max & \sum_i p_i^d \\
\text{subject to} & \\
& p_i^g - p_i^d = \sum_{(i,j)} p_{ij} \\
& p_{ij} = -b_{ij} \sin(\theta_i - \theta_j) \\
& |\theta_i - \theta_j| \leq \theta^u \\
& p_i^{gl} \leq p_i^g \leq p_i^{gu} \\
& p_i^{dl} \leq p_i^g \leq p_i^{du}
\end{align*}
\]

Only extension to LDC needed to be NP-hard
Complexity of AC-Feasibility

- AC-Feasibility on Acyclic Networks is NP-Hard
  - K. Lehmann, A. Grastien, and P. Van Hentenryck

solve

\[ p_i^g - p_i^d = \sum_{(i,j)} p_{ij} \]
\[ q_i^g - q_i^d = \sum_{(i,j)} q_{ij} \]

\[ p_{ij} = g_{ij}(1 - \cos(\theta_i - \theta_j)) - b_{ij} \sin(\theta_i - \theta_j) \]
\[ q_{ij} = -b_{ij}(1 - \cos(\theta_i - \theta_j)) - g_{ij} \sin(\theta_i - \theta_j) \]

\[ |\theta_i - \theta_j| \leq \theta^u \]
AC-Feasibility on Acyclic Networks

‣ Star Network

‣ Key ideas
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The AC/DC Conundrum

- Find an approximation of AC power flows that
  - is more accurate than the LDC model
  - is useful outside normal operating conditions
  - reasons about voltage magnitudes and reactive power
  - can be embedded in discrete optimization solvers
    - mixed integer programming solvers
Trigonometric Functions

- Two approximations
  - \( \sin(x) \) is replaced by \( x \)
  - \( \cos(x) \) is replaced by its piecewise linear relaxation

\[
\begin{align*}
p_{ij} &= g_{ij} v_i^2 - g_{ij} v_i v_j \tilde{\cos}(\theta_i - \theta_j) - b_{ij} v_i v_j (\theta_i - \theta_j) \\
q_{ij} &= -b_{ij} v_i^2 + b_{ij} v_i v_j \tilde{\cos}(\theta_i - \theta_j) - g_{ij} v_i v_j (\theta_i - \theta_j)
\end{align*}
\]
Trigonometric Functions

Fig. 1. A Piecewise-Linear Approximation of Cosine using 7 Inequalities.
Voltage Magnitudes

- Understanding power flows [Grainger, 94]
  - Phase angle differences determine active power
  - Voltage magnitude differences determine reactive power

- Experiments
  - Per unit system
  - Look at how the equations behave when
    - \( g = 0.2 \) and \( b = 1.0 \)
    - \( v_i = 1.0, v_j \in (0.8, 1.2), (\theta_i - \theta_j) \in (-\pi/6, \pi/6) \)
Voltage Magnitudes

Active Power Field

Reactive Power Field
Voltage Magnitudes

- Key ideas
  - Substitute $\nu = \hat{\nu} + \phi$ into the power flow equations
  - First-order Taylor expansion to remove quadratic terms

\[ p_{nm} = g_{nm} - g_{nm} \cos_{nm} - b_{nm}(\theta_n - \theta_m) \]
\[ q_{nm} = -b_{nm} + b_{nm} \cos_{nm} - g_{nm}(\theta_n - \theta_m) - b_{nm}(\phi_n - \phi_m). \]
\[ \cos_{nm} \in \overline{\cos}(\theta_n - \theta_m). \]
Outline

» Motivation
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» Complexity
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» **Case Studies for LPAC**
» Convex Relaxations
» Case Studies for Convex Relaxations
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Experimental Settings

- **Bonmin (Bonami 2008)**
  - heuristics for solving MINLPs.
  - outer approximation method for convex MINLPs
- **QP and SOCP**
  - CPLEX 12.8 or Gurobi
- **Ipopt (Waechter and Biegler 2006)**
  - NLP
Experimental Results

- Wide variety of IEEE and MATPOWER Benchmarks
  - ieee14, mp24, ieee30, mp30, mp39, ieee57, ieee118, ieedd17, mp300
  - Small benchmarks are easy in general
  - IEEE 118 is also easy
  - All LPAC models solved almost instantly (LPs)

- Comparison with an AC Solver
  - LDC and LPAC solutions versus an AC solution
# Line Active Power

## Table 1: Active Power Flow Accuracy Comparison

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<tr>
<th>Benchmark</th>
<th>Corr</th>
<th>$\mu(\Delta)$</th>
<th>max($\Delta$)</th>
<th>$\delta$(arg max($\Delta$))</th>
<th>$\mu(\delta)$</th>
<th>max($\delta$)</th>
<th>$\Delta$(arg max($\delta$))</th>
<th>approx(%)</th>
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<td>5.787</td>
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Line Active Power

DC Model

LPAC Model
Bus Angles

DC Model

LPAC Model

Bus Phase Angle Correlation (rad)
Importance of cos: Reactive Power

Cold-Start LPAC Model (cos=1)  Cold-Start LPAC Model
Power Restoration

AC Restoration Timeline

AC Power Flow (MW)

Restoration Action

NICTA Copyright 2014 From imagination to impact 69
Power Restoration

AC Restoration Timeline

AC Power Flow (MW)

Restoration Action

LDC–ROP
LPAC–ROP
Power Restoration

DC Restoration Timeline

DC Power Flow (MW)

Restoration Action

LDC–ROP
LPAC–ROP
DC versus LPAC in Restoration

Table 2 Quality of the DC-ROP and LPAC-ROP with a Fixed Restoration Order.

<table>
<thead>
<tr>
<th>Model</th>
<th>Inst. Solved</th>
<th>Models Solved</th>
<th>MLCV</th>
<th>MRIV</th>
<th>MVV</th>
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<tr>
<td>All Instances</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DC</td>
<td>13/18</td>
<td>333/404</td>
<td>301.2</td>
<td>360.6</td>
<td>0.06073</td>
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<tr>
<td>LPAC</td>
<td>18/18</td>
<td>404/404</td>
<td>1.656</td>
<td>13.81</td>
<td>0</td>
</tr>
<tr>
<td>Group A - All E-DC-LPP dispatches are AC-feasible.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DC</td>
<td>13/13</td>
<td>196/196</td>
<td>1</td>
<td>330.8</td>
<td>0.0005223</td>
</tr>
<tr>
<td>LPAC</td>
<td>13/13</td>
<td>196/196</td>
<td>1.261</td>
<td>12.41</td>
<td>0</td>
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<td>Group B - Most DC-LPP dispatches are AC-feasible.</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DC</td>
<td>0/3</td>
<td>75/105</td>
<td>203.3</td>
<td>435.1</td>
<td>0.08339</td>
</tr>
<tr>
<td>LPAC</td>
<td>3/3</td>
<td>105/105</td>
<td>1.508</td>
<td>14.29</td>
<td>0</td>
</tr>
<tr>
<td>Group C - Few DC-LPP dispatches are AC-feasible.</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DC</td>
<td>0/2</td>
<td>62/103</td>
<td>331.3</td>
<td>341.4</td>
<td>0.1522</td>
</tr>
<tr>
<td>LPAC</td>
<td>2/2</td>
<td>103/103</td>
<td>2.561</td>
<td>15.98</td>
<td>0</td>
</tr>
</tbody>
</table>
DC versus LPAC in Restoration

How good is the DC ordering?

Table 3 Comparison of Blackout Sizes Produced by the DC-ROP and LPAC-ROP.

| ID | $|\mathcal{R}|$ | DC-ROP | LPAC-ROP | Relative Difference |
|----|---------|---------|---------|---------------------|
| 1  | 61      | 35,320  | 73,807  | 47,687              | 64.61%              |
| 16 | 53      | 33,104  | 60,655  | 53,138              | 87.61%              |
| 3  | 41      | 12,906  | 31,581  | 22,963              | 72.71%              |
| 13 | 40      | 14,655  | 35,468  | 31,267              | 88.15%              |
| 12 | 36      | 16,954  | 41,442  | 41,415              | 99.94%              |
| 2  | 32      | 6,508   | 20,515  | 13,459              | 65.60%              |
| 4  | 24      | 2,245   | 11,956  | 5,230               | 43.75%              |
Expansion Planning

- Simplest problem formulation
  - starting point

- Network design problem
  - add lines to meet the increased load

- Under
  - voltage constraints
  - thermal limits
  - constraints on real and reactive power injection
Expansion Planning

**Minimize:**
\[
\sum_{(n,m) \in L} c_{nm} (z_{nm} - 1) + \sum_{(n,m) \in L^+} c_{nm} z_{nm}
\]  \hfill (M1.1)

**Subject To:**
\[
\mathfrak{S}(\widetilde{V}_s) = 0
\]  \hfill (M1.2)

\[
|\widetilde{V}_n| \leq |\widetilde{V}_n| \leq |\widetilde{V}_n|; \quad \forall n \in N
\]  \hfill (M1.3)

\[
p_n^g - p_n^l = \sum_{m \in L(n)} z_{nm} p_{nm} \quad \forall n \in N
\]  \hfill (M1.4)

\[
q_n^g - q_n^l = \sum_{m \in L(n)} z_{nm} q_{nm} \quad \forall n \in N
\]  \hfill (M1.5)

\[
1 \leq z_{nm} \quad \forall (n,m) \in L
\]  \hfill (M1.6)

\[
\forall (n,m) \in L \cup L^r \cup L^+ \cup L^+
\]

\[
p_{nm} = g_{nm} |\widetilde{V}_n|^2 - g_{nm} \Re(\widetilde{V}_n \widetilde{V}_m^*) - b_{nm} \mathfrak{S}(\widetilde{V}_n \widetilde{V}_m^*)
\]

\[
q_{nm} = -b_{nm} |\widetilde{V}_n|^2 + b_{nm} \Re(\widetilde{V}_n \widetilde{V}_m^*) - g_{nm} \mathfrak{S}(\widetilde{V}_n \widetilde{V}_m^*)
\]

\[
p_{nm}^2 + q_{nm}^2 \leq |\mathcal{S}_{nm}|
\]  \hfill (M1.9)
Heuristic AC Method (HAC)

- Many proposals

- Destructive heuristics
  - start with all the possible lines (feasible solution)
  - consider each line $l$ in turn for removal
    - if feasible in AC model, remove the line $l$
  - order the line by increasing relative load

- Outperforms the state of the art
  - constructive methods

Case Studies

▷ Traditional benchmarks


▷ New benchmarks

- MathPower benchmarks
- load and generation scaled by a factor of 3
- reaction injection is half of the real injection
- cost is 1
Measuring Accuracy

- Thermal limits

\[
\max\left(\frac{\sqrt{p_{nm}^2 + q_{nm}^2}}{S_{nm}^u}, \frac{\sqrt{p_{mn}^2 + q_{mn}^2}}{S_{mn}^u}\right)
\]

- Voltage magnitudes

\[
\max\left(0, v_n^u - v_n, v_n - v_n^l\right)
\]
## Core Results

<table>
<thead>
<tr>
<th>Case</th>
<th>HAC-TNEP</th>
<th>DC-TNEP</th>
<th>LPAC-TNEP</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>160 (6)</td>
<td>110 (4)</td>
<td>137.32 (5)</td>
</tr>
<tr>
<td>24</td>
<td>2310 (43)</td>
<td>152 (5)</td>
<td>192.86 (16)</td>
</tr>
<tr>
<td>46</td>
<td>569810 (47)</td>
<td>89889 (9)</td>
<td>113.46 (8)</td>
</tr>
<tr>
<td>9</td>
<td>3 (3)</td>
<td>2 (2)</td>
<td>104.45 (1)</td>
</tr>
<tr>
<td>14</td>
<td>15 (15)</td>
<td>5 (5)</td>
<td>110.84 (5)</td>
</tr>
<tr>
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<td>13 (13)</td>
<td>26 (26)</td>
<td>103.01 (10)</td>
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<td>39</td>
<td>47 (47)</td>
<td>1 (1)</td>
<td>111.15 (2)</td>
</tr>
<tr>
<td>57</td>
<td>49 (49)</td>
<td>5 (5)</td>
<td>116.97 (13)</td>
</tr>
<tr>
<td>118</td>
<td>37 (37)</td>
<td>5 (5)</td>
<td>120.44 (2)</td>
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</table>

AC-PF did not converge
### Constraint Tightening (10%)

<table>
<thead>
<tr>
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<th>HAC-TNEP</th>
<th>DC-TNEP</th>
</tr>
</thead>
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<td>Cost</td>
</tr>
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<tr>
<td>24</td>
<td>2378 (44)</td>
<td>266 (8)</td>
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<td>46</td>
<td>569810 (47)</td>
<td>130110 (13)</td>
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<tr>
<td>9</td>
<td>3 (3)</td>
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<td>7 (7)</td>
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</table>

<table>
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<tr>
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<td></td>
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</tr>
<tr>
<td>24</td>
<td>681 (15)</td>
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<td>34 (34)</td>
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<td>57</td>
<td>42 (42)</td>
</tr>
<tr>
<td>118</td>
<td>11 (11)</td>
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</table>
VAr Compensation

- VAr compensation is cheaper
  - can be used to meet voltage bounds
- Case study: Perfect Voltage Profile (PVP)
  - unlimited VAr compensation at each bus
  - bus becomes synchronous condenser
    * unlimited reactive power injection, voltage set-point at 1.0
  - DC model
    * VAr compensation used in second step (cross-over)
  - LPAC model
    * a single, integrated model

### VAr Compensation

#### Table 1: HAC-PVP-TNEP and DC-PVP-TNEP Results

<table>
<thead>
<tr>
<th>Case</th>
<th>Cost</th>
<th>DC-PVP-TNEP</th>
<th>Vol. vio.</th>
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<td></td>
<td>Capacity vio.</td>
<td>Max (MVA)</td>
</tr>
<tr>
<td></td>
<td>Cost</td>
<td>Max</td>
<td>Avg.</td>
</tr>
<tr>
<td>6</td>
<td>130 (5)</td>
<td>104.65</td>
<td>(4)</td>
</tr>
<tr>
<td>24</td>
<td>573 (10)</td>
<td>111.97</td>
<td>(9)</td>
</tr>
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<td>46</td>
<td>277592 (22)</td>
<td>129.86</td>
<td>(10)</td>
</tr>
<tr>
<td>9</td>
<td>2 (2)</td>
<td>105.86</td>
<td>(4)</td>
</tr>
<tr>
<td>14</td>
<td>2 (2)</td>
<td>113.41</td>
<td>(2)</td>
</tr>
<tr>
<td>30</td>
<td>8 (8)</td>
<td>119.61</td>
<td>(9)</td>
</tr>
<tr>
<td>39</td>
<td>24 (24)</td>
<td>100.41</td>
<td>(7)</td>
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<td>2 (2)</td>
<td>122.95</td>
<td>(3)</td>
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<td>118</td>
<td>2 (2)</td>
<td>160.31</td>
<td>(17)</td>
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#### Table 2: LPAC-PVP-TNEP Results

<table>
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<th>Vol. vio.</th>
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</thead>
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<td></td>
<td>Capacity vio.</td>
<td>Vol.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Max (MVA)</td>
<td>Avg. (MVA)</td>
</tr>
<tr>
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<td>130 (5)</td>
<td>0</td>
<td>(0)</td>
</tr>
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<td>(0)</td>
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<td>(1)</td>
</tr>
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<td>9</td>
<td>2 (2)</td>
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<td>(0)</td>
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<tr>
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# VAr Compensation

<table>
<thead>
<tr>
<th>Case</th>
<th>HAC-PVP-TNEP</th>
<th>DC-PVP-TNEP</th>
<th>LPAC-PVP-TNEP</th>
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<tbody>
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<td></td>
<td>P (MW)</td>
<td>Q (MVAr)</td>
<td>P</td>
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</table>
VAR Compensation

![Graphs showing Pareto frontiers for 6 and 24 bus systems, with data points indicating cost and reactive power injection versus the number of lines.]

<table>
<thead>
<tr>
<th>λ</th>
<th>Cost (MVar)</th>
<th>Q injection (MVar)</th>
<th>Capacity vio. (MVA)</th>
<th>Vol. vio.</th>
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</thead>
<tbody>
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<td>0</td>
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<td>0.2</td>
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<tr>
<td>0.3</td>
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<td>0.6</td>
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<td>0</td>
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<tr>
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<td>509 (13)</td>
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<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Outline

- Motivation
- Power Flow Formulations
- Complexity
- The LPAC Approximation
- Case Studies for LPAC
- **Convex Relaxations**
- Case Studies for Convex Relaxations
- Beyond Steady States
SDP Relaxation


\[
S_i^g - S_i^d = \sum_{(i,j) \in E \cup E^R} S_{ij} \quad \forall i \in N
\]

\[
S_{ij} = Y_{ij}^*W_{ii} - Y_{ij}^*W_{ij} \quad (i,j) \in E \cup E^R
\]

\[
W \geq 0
\]
SOCP Relaxation


\[
\begin{bmatrix}
  w_{ii} & w_{ij} \\
  w_{ji} & w_{jj}
\end{bmatrix} \succeq 0
\]
SOCP Relaxation


\[
\begin{align*}
    w_{ii} & \geq 0 \\
    w_{ii} w_{jj} & \geq (w_{ij}^R)^2 + (w_{ij}^I)^2
\end{align*}
\]
Dist-Flow Relaxation

\[ p_{ij} = \sum_{k: (j,k) \in E} p_{jk} + r_{ij} \frac{p_{ij}^2 + q_{ij}^2}{v_i^2} \]

\[ q_{ij} = \sum_{k: (j,k) \in E} q_{jk} + x_{ij} \frac{p_{ij}^2 + q_{ij}^2}{v_i^2} \]

\[ v_j^2 = v_i^2 - 2(r_{ij} p_{ij} + x_{ij} q_{ij}) + (r_{ij}^2 + x_{ij}^2) \frac{p_{ij}^2 + q_{ij}^2}{v_i^2} \]

- Define

\[ l_{ij} = \frac{p_{ij}^2 + q_{ij}^2}{v_i v_i} \]
Dist-Flow Relaxation


\[
p_{ij} = \sum_{k: (j,k) \in E} p_{jk} + r_{ij} \ell_{ij}
\]
\[
q_{ij} = \sum_{k: (j,k) \in E} q_{jk} + x_{ij} \ell_{ij}
\]
\[
\nu \nu_j = \nu \nu_i - 2(r_{ij} p_{ij} + x_{ij} q_{ij}) + (r_{ij}^2 + x_{ij}^2) \ell_{ij}
\]
\[
p_{ij}^2 + q_{ij}^2 \leq \ell_{ij} \nu \nu_i
\]
Transcendental Relaxation

Key ideas

- compositional, structural relaxation
- relax trigonometric functions
- exploit the narrow bounds in power systems

\[ \theta^u \leq \pi/2. \]

- dual modelling

Resulting optimization model

- quadratic and convex
Power Flow Equations

- Ohm’s law for each line (i,j)

\[ p_{ij} = g_{ij} v_i^2 - g_{ij} v_i v_j \cos(\theta_i - \theta_j) - b_{ij} v_i v_j \sin(\theta_i - \theta_j) \]
\[ q_{ij} = -b_{ij} v_i^2 + b_{ij} v_i v_j \cos(\theta_i - \theta_j) - g_{ij} v_i v_j \sin(\theta_i - \theta_j) \]

- Kirchhoff’s current law for each bus i

\[ p_i = \sum_{(i,j) \in E} p_{ij} \]
\[ q_i = \sum_{(i,j) \in E} q_{ij} \]
Convex Quadratic Relaxation

- Quadratic convex relaxation of
  - cosine function
  - square function
- Polyhedral relaxation of
  - sine function
- MacCormick relaxation of
  - multi-linear terms
QP Relaxation of Cosine
QP Relaxation of Cosine

Convex quadratic relaxation of cosine

\[
\widetilde{\cos}(\theta) = 1 - \left( \frac{1 - \cos(\theta^u)}{(\theta^u)^2} \right) \theta^2.
\]

Proposition:

\[
\forall \theta \in [-\theta^u, \theta^u] : \widetilde{\cos}(\theta) \geq \cos(\theta).
\]
QP Relaxation of Square
QP Relaxation of Square

- Convex quadratic Relaxation

\[ \tilde{v}^2 \geq v^2 \]

\[ \tilde{v}^2 \leq (v^u + v^l) v - v^u v^l \]
QP Relaxation of Sine
Polynomial Relaxation of Sine

- Polynomial relaxation

\[
\begin{align*}
\tilde{\sin}(\theta) & \leq \cos \left( \frac{\theta^u}{2} \right) \left( \theta - \frac{\theta^u}{2} \right) + \sin \left( \frac{\theta^u}{2} \right) \\
\tilde{\sin}(\theta) & \geq \cos \left( \frac{\theta^u}{2} \right) \left( \theta + \frac{\theta^u}{2} \right) - \sin \left( \frac{\theta^u}{2} \right)
\end{align*}
\]

- Proposition:

\[
\forall \theta \in [-\theta^u, \theta^u] : \cos \left( \frac{\theta^u}{2} \right) \left( \theta - \frac{\theta^u}{2} \right) + \sin \left( \frac{\theta^u}{2} \right) \geq \sin(\theta).
\]

\[
\sin(\theta) \geq \cos \left( \frac{\theta^u}{2} \right) \left( \theta + \frac{\theta^u}{2} \right) - \sin \left( \frac{\theta^u}{2} \right)
\]
MacCormick Relaxations

- Sequential Bilinear Relaxations

\[
\overline{v_i v_j} \geq v_i^l v_j + v_j^l v_i - v_i^l v_j
\]

\[
\overline{v_i v_j} \geq v_i^u v_j + v_j^u v_i - v_i^u v_j
\]

\[
\overline{v_i v_j} \leq v_i^l v_j + v_j^u v_i - v_i^l v_j
\]

\[
\overline{v_i v_j} \leq v_i^u v_j + v_j^l v_i - v_i^u v_j
\]
The Initial QC Model

\[
(QC-\text{Init}) = \begin{cases}
   p_{ij} = g_{ij} \tilde{v}_i - g_{ij} \tilde{w}c_{ij} - b_{ij} \tilde{w}s_{ij} \\
   q_{ij} = -b_{ij} \tilde{v}_i + b_{ij} \tilde{w}c_{ij} - g_{ij} \tilde{w}s_{ij} \\
   \tilde{c}s_{ij} \in \langle \cos(\theta_i - \theta_j) \rangle^R \\
   \tilde{s}_{ij} \in \langle \sin(\theta_i - \theta_j) \rangle^R \\
   \tilde{v}_i \in \langle v^2 \rangle^R \\
   \tilde{w}_{ij} \in \langle v_i, v_j \rangle^M \\
   \tilde{w}c_{ij} \in \langle \tilde{w}_{ij}, \tilde{c}s_{ij} \rangle^M \\
   \tilde{w}s_{ij} \in \langle \tilde{w}_{ij}, \tilde{s}_{ij} \rangle^M
\end{cases}
\]
Dual Modelling

- **Key idea**
  - Use several models of the optimisation problem
  - The solution set is the intersection

- **Heavily used in optimisation for decades**
  - constraint programming (early 1990s)
    - improving propagation
  - mathematical programming

- **In nonlinear optimisation**
  - Formalized by [Liberty, 04] to strengthen the relaxation
  - Generalized reduction constraints [Ruiz & Grossmann, 11]
    - intersecting several formulations based on physical interpretation of the problem
Redundancy in MINLP

\[
\begin{align*}
\text{min } & \quad f(x, y) \\
\text{s.t. } & \quad z_i = g_i(x, y), \ \forall i \in \{1, \ldots, m\}, \\
& \quad x \in \mathbb{R}^n, y \in \mathbb{Z}^m. \\
\end{align*}
\]

\[
\begin{align*}
\text{min } & \quad o \\
\text{s.t. } & \quad o \in f(x, y)^R \\
& \quad z_i \in g_i(x, y)^R, \ \forall i \in \{1, \ldots, m\}, \\
& \quad x \in \mathbb{R}^n, y \in \mathbb{Z}^m. \\
\end{align*}
\]
Redundancy in MINLP

\[
\begin{align*}
\text{min} & \quad f(x, y) \\
\text{s.t.} & \quad z_i = g_i(x, y), \quad \forall i \in \{1, \ldots, m\}, \\
& \quad z_i + z_j = g_i(x, y) + g_j(x, y), \quad \forall (i, j) \in \{1, \ldots, m\}^2, \\
& \quad x \in \mathbb{R}^n, y \in \mathbb{Z}^m.
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad o \\
\text{s.t.} & \quad o \in f(x, y)^R \\
& \quad z_i \in g_i(x, y)^R, \quad \forall i \in \{1, \ldots, m\}, \\
& \quad z_i + z_j \in (g_i(x, y) + g_j(x, y))^R, \quad \forall (i, j) \in \{1, \ldots, m\}^2, \\
& \quad x \in \mathbb{R}^n, y \in \mathbb{Z}^m.
\end{align*}
\]
Power Loss Formulation

- Power loss on line (i,j)

\[
\begin{align*}
p_{ij} + p_{ji} &= r_{ij} \frac{p_{ij}^2 + q_{ij}^2}{v_i^2} \\
q_{ij} + q_{ji} &= x_{ij} \frac{p_{ij}^2 + q_{ij}^2}{v_i^2}
\end{align*}
\]
The QC-NLP Model

- Power loss on line (i,j)

\[
(10) - (17)
\]

(QC-NLP)

How to prove this?

Use the perspective \( z f(x/z, y/z) \)
where \( f(x, y) = x^2 + y^2 \)

- Proposition

The function \( f(\mathbb{R}^2 \times \mathbb{R}^*+ \to \mathbb{R}) : f(x, y, z) = \frac{x^2 + y^2}{z} \) is convex
The QC-SOCP Model

- Second-Order Cone Formulation
  - [Farivar, M., Clarke, C., Low, S., Chandy, K. 2011]

\[ l_{ij} \tilde{v}_i \geq p_{ij}^2 + q_{ij}^2 \]  \hspace{1cm} (20a)

\[(\text{QC-SOCP}) = \{ (10) - (19) \} \]

\[ (20a) \]
One/Off Constraints

\[ \min f(x, z) \]

s.t. \( h(x, z) \leq 0, \)

\[ g_k(x) \leq 0 \text{ if } z_k = 1, \quad \forall k \in K \]

\( x \in \mathbb{R}^n, z \in \mathbb{Z}^m. \)

- Can we avoid big-M transformations?
\[ \begin{align*}
\text{min} & \quad f(x, z) \\
\text{s.t.} & \quad h(x, z) \leq 0, \\
(x, z_k) & \in \Gamma_0^k \cup \Gamma_1^k, \quad \forall k \in K \\
\Gamma_0^k & = \{(x, z_k) : z_k = 0, \ l^0 \leq x \leq u^0\} \\
\Gamma_1^k & = \{(x, z_k) : z_k = 1, \ g_k(x) \leq 0, \ l^1 \leq x \leq u^1\}. \\
x & \in \mathbb{R}^n, \quad z \in \mathbb{Z}^m.
\end{align*} \]
On/Off Constraints

\[
\begin{aligned}
\min & \quad h(x, z) \\
\text{s.t.} & \quad g(x, z) \leq 0, \\
& \quad (x, z_k) \in \text{conv}(\Gamma_0^k \cup \Gamma_1^k), \quad \forall k \in \{1, 2, \ldots, K\}, \\
& \quad x \in \mathbb{R}^n, \quad z \in \{0, 1\}^K.
\end{aligned}
\]

- Can I represent the convex hull in the original space of variables?
On/Off Constraints: Point Case

- When $\Gamma_0^k = \{(x, z_k) : z_k = 0, l^0 \leq x \leq u^0\}$ is a point, the convex hull becomes

$$
\Gamma^* = \left\{ (x, z) \in \mathbb{R}^{m+1} : \\
\begin{array}{l}
zg(x/z) \leq 0, \\
zl^1 \leq x \leq zu^1, 0 \leq z < 1
\end{array}
\right\}
$$
On/Off Constraints: quadratic

- Consider the function

\[ g(x) = a_1 x_1^2 + a_2 x_2^2 - a_3 x_3 \]

- The convex hull becomes

\[ \Gamma^* = \left\{ (x, z) \in \mathbb{R}^4 : \begin{array}{l} a_1 x_1^2 + a_2 x_2^2 \leq z(a_3 x_3), \\ zl^1 \leq x \leq zu^1, \ 0 \leq z \leq 1 \end{array} \right\} \]
On/Off Constraints

- The idea can be generalised for intervals and monotone functions: e.g. linear constraints

\[ g(x) = a^T x - b. \]
On/Off Constraints

- The idea can be generalised for intervals and monotone functions: e.g. linear constraints

\[
\begin{align*}
\Gamma^* &= \left\{ (x, z) \in \mathbb{R}^{n+1} : \\
\sum_{i \notin S} a_i x_i &\leq z \left( b - \sum_{i \in S, a_i < 0} a_i u_i^1 - \sum_{i \in S, a_i > 0} a_i l_i^1 \right) \\
+ (1-z) &\left( \sum_{i \notin S, a_i < 0} a_i l_i^0 + \sum_{i \notin S, a_i > 0} a_i u_i^0 \right), \quad \forall S \subseteq \{1, \ldots, m\} \\
zl_i^1 + (1-z)l_i^0 &\leq x_i \leq zl_i^1 + (1-z)u_i^0, \quad \forall i \in \{1, \ldots, m\}, \\
0 &\leq z \leq 1
\right\}
\end{align*}
\]
On/Off Constraints in Power Systems

- When a line is switched off, the phase angle difference must increase.
- We can then apply the above results to each constraint.
On/Off Constraints in Power Systems

- Sine constraint

\[ \Gamma_s^0 = \{ (\tilde{s}, \theta, z) \in \mathbb{R}^3 : -1 \leq \tilde{s} \leq 1, -|E|\theta^u \leq \theta \leq |E|\theta^u, z = 0 \} \]

\[ \Gamma_s^1 = \left\{ (\tilde{s}, \theta, z) \in \mathbb{R}^3 : \begin{align*}
\tilde{s} - \cos(\theta^u/2)\theta &\leq \sin(\theta^u/2) - \cos(\theta^u/2)\theta^u/2, \\
-\tilde{s} + \cos(\theta^u/2)\theta &\leq \sin(\theta^u/2) - \cos(\theta^u/2)\theta^u/2, \\
\sin(-\theta^u) &\leq \tilde{s} \leq \sin(\theta^u), -\theta^u \leq \theta \leq \theta^u, z = 1
\end{align*} \right\} \]
On/Off Constraints in Power Systems

\[ \begin{align*}
(I_s^*) &= \{(\tilde{s}, \theta, z) \in \mathbb{R}^3 : \\
&\quad \tilde{s} - \cos(\theta^u/2)\theta \leq z(\sin(\theta^u/2) - \cos(\theta^u/2)\theta^u/2) \\
&\quad + (1 - z)(\cos(\theta^u/2)|E|\theta^u + 1), \\
&\quad -\tilde{s} + \cos(\theta^u/2)\theta \leq z(\sin(\theta^u/2) - \cos(\theta^u/2)\theta^u/2) \\
&\quad + (1 - z)(\cos(\theta^u/2)|E|\theta^u + 1), \\
&\quad |\tilde{s}| \leq z(\sin(\theta^u/2) + \cos(\theta^u/2)\theta^u/2) + (1 - z), \\
&\quad \cos(\theta^u/2)|\theta| \leq z(\sin(\theta^u/2) - \cos(\theta^u/2)\theta^u/2 + \sin(\theta^u)) \\
&\quad + (1 - z)(\cos(\theta^u/2)|E|\theta^u), \\
&\quad z\sin(-\theta^u) - (1 - z) \leq \tilde{s} \leq z\sin(\theta^u) + (1 - z), \\
&\quad -z\theta^u - (1 - z)|E|\theta^u \leq \theta \leq z\theta^u + (1 - z)|E|\theta^u, \ 0 \leq z \leq 1 \} \end{align*} \]
Outline

- Motivation
- Power Flow Formulations
- Complexity
- The LPAC Approximation
- Case Studies for LPAC
- Convex Relaxations
- Case Studies for Convex Relaxations
- Beyond Steady States
Experimental Settings

- **Instances**: [Matpower 2011]

  ![Table 1: Sizes of Power System Benchmarks.](image)

- **Congested Instances**

  ![Table 2: Congested Instances.](image)
Optimal Power Flows

The problem

- Minimize the cost of generation for a given load
- Intensively studied
- Purest
Optimal Power Flows

\[
\begin{align*}
\min & \quad \sum_{i=1}^{m} c_i'' p_i^2 + c_i' p_i + c_i \\
\text{s.t.} & \quad p_i = \sum_{j=1}^{m} p_{ij}, \quad \forall i \in \{1, \ldots, m\}, \\
& \quad q_i = \sum_{j=1}^{m} q_{ij}, \quad \forall i \in \{1, \ldots, m\}, \\
& \quad (1) - (2) \\
& \quad p_{ij}^2 + q_{ij}^2 \leq s_{ij}^2, \quad \forall (i, j) \in \{1, \ldots, m\}^2, \\
& \quad p_i^l \leq p_i \leq p_i^u, \quad \forall i \in \{1, \ldots, m\}, \\
& \quad q_i^l \leq q_i \leq q_i^u, \quad \forall i \in \{1, \ldots, m\}, \\
& \quad v_i^l \leq v_i \leq v_i^u, \quad \forall i \in \{1, \ldots, m\}, \\
& \quad -\theta_i^u \leq \theta_i - \theta_j \leq \theta_i^u, \quad \forall (i, j) \in \{1, \ldots, m\}^2, \\
& \quad v, \theta, p, q \in \mathbb{R}^n.
\end{align*}
\]

quadratic production cost
# Optimal Power Flows

## Table 3: Optimality Gap for Different Relaxations of the OPF Problem.

<table>
<thead>
<tr>
<th>Benchmark</th>
<th>Quality</th>
<th>Optimality Gap</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AC NLP</td>
<td>QC NLP</td>
</tr>
<tr>
<td>1</td>
<td>3143</td>
<td>0.79%</td>
</tr>
<tr>
<td>2</td>
<td>5296</td>
<td>1.52%</td>
</tr>
<tr>
<td>3</td>
<td>5296</td>
<td>1.52%</td>
</tr>
<tr>
<td>4</td>
<td>8081</td>
<td>4.51%</td>
</tr>
<tr>
<td>5</td>
<td>576</td>
<td>2.03%</td>
</tr>
<tr>
<td>6</td>
<td>576</td>
<td>2.03%</td>
</tr>
<tr>
<td>7</td>
<td>8906</td>
<td>5.25%</td>
</tr>
<tr>
<td>8</td>
<td>41864</td>
<td>1.34%</td>
</tr>
<tr>
<td>9</td>
<td>41737</td>
<td>1.75%</td>
</tr>
<tr>
<td>10</td>
<td>129658</td>
<td>2.48%</td>
</tr>
<tr>
<td>11</td>
<td>722847</td>
<td>1.65%</td>
</tr>
<tr>
<td>12</td>
<td>1898276</td>
<td>6.48%</td>
</tr>
<tr>
<td>13</td>
<td>1308771</td>
<td>2.50%</td>
</tr>
</tbody>
</table>
# Optimal Power Flows

<table>
<thead>
<tr>
<th>Benchmark</th>
<th>AC NLP</th>
<th>Init.</th>
<th>QC NLP</th>
<th>QC SOCP</th>
<th>[24] SOCP</th>
<th>[27] SDP</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.12</td>
<td>0.23</td>
<td>0.08</td>
<td>0.44</td>
<td>0.05</td>
<td>16.16</td>
</tr>
<tr>
<td>2</td>
<td>0.07</td>
<td>0.06</td>
<td>0.06</td>
<td>0.13</td>
<td>0.04</td>
<td>8.00</td>
</tr>
<tr>
<td>3</td>
<td>0.05</td>
<td>0.10</td>
<td>0.06</td>
<td>0.13</td>
<td>0.03</td>
<td>7.43</td>
</tr>
<tr>
<td>4</td>
<td>0.04</td>
<td>0.18</td>
<td>0.08</td>
<td>0.20</td>
<td>0.09</td>
<td>6.33</td>
</tr>
<tr>
<td>5</td>
<td>0.06</td>
<td>0.09</td>
<td>0.13</td>
<td>0.41</td>
<td>0.15</td>
<td>16.65</td>
</tr>
<tr>
<td>6</td>
<td>0.07</td>
<td>0.18</td>
<td>0.15</td>
<td>0.37</td>
<td>0.18</td>
<td>16.45</td>
</tr>
<tr>
<td>7</td>
<td>0.06</td>
<td>0.22</td>
<td>0.12</td>
<td>0.53</td>
<td>0.15</td>
<td>8.78</td>
</tr>
<tr>
<td>8</td>
<td>0.12</td>
<td>0.20</td>
<td>0.20</td>
<td>0.40</td>
<td>0.22</td>
<td>16.13</td>
</tr>
<tr>
<td>9</td>
<td>0.22</td>
<td>0.18</td>
<td>0.24</td>
<td>0.59</td>
<td>0.32</td>
<td>11.48</td>
</tr>
<tr>
<td>10</td>
<td>0.18</td>
<td>0.51</td>
<td>0.57</td>
<td>8.25</td>
<td>0.82</td>
<td>19.57</td>
</tr>
<tr>
<td>11</td>
<td>0.52</td>
<td>1.60</td>
<td>1.80</td>
<td>5.37</td>
<td>3.59</td>
<td>34.12</td>
</tr>
<tr>
<td>12</td>
<td>9.78</td>
<td>37.51</td>
<td>34.22</td>
<td>1468.19</td>
<td>131.97</td>
<td>2666.00</td>
</tr>
<tr>
<td>13</td>
<td>8.59</td>
<td>39.28</td>
<td>34.25</td>
<td>2707.78</td>
<td>32.48</td>
<td>T.L.</td>
</tr>
</tbody>
</table>

**Table 4** Runtimes for Different Relaxations of the OPF Problem.
Optimal Power Flows

Optimality Gap (%)

Benchmark

QC-Init.
QC-NLP

0 1 2 3 4 5 6

1 2 3 4 5 6 7 8 9 10 11 12 13
## Optimal Power Flows

<table>
<thead>
<tr>
<th>Benchmark</th>
<th>Quality</th>
<th>Optimality Gap</th>
<th>Runtime (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AC NLP</td>
<td>NLP</td>
<td>SOCP</td>
</tr>
<tr>
<td>c-1</td>
<td>1499</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>c-2</td>
<td>22081</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>c-3</td>
<td>47259</td>
<td>2.04%</td>
<td>2.08%</td>
</tr>
<tr>
<td>c-4</td>
<td>1979</td>
<td>14.57%</td>
<td>14.57%</td>
</tr>
<tr>
<td>c-5</td>
<td>107536</td>
<td>0.23%</td>
<td>0.36%</td>
</tr>
<tr>
<td>c-6</td>
<td>137116</td>
<td>1.54%</td>
<td>2.76%</td>
</tr>
<tr>
<td>c-7</td>
<td>2530242</td>
<td>1.69%</td>
<td>2.61%</td>
</tr>
<tr>
<td>c-8</td>
<td>14773480</td>
<td>1.26%</td>
<td>1.66%</td>
</tr>
</tbody>
</table>
Optimal Power Flows: Summary

- Dual modelling is critical
- QC-NLP dominates the SOCP relaxation
  - especially on congested benchmarks
- QC-NLP is orders of magnitude faster than the SDP relaxation with minimal loss in accuracy
- QC-NLP is orders of magnitude faster than QC-SOCP
Outline

- Motivation
- Complexity
- The LPAC Approximation
  - The model
  - Power restoration
- The QC Relaxation
  - The model
  - Optimal power flow
  - Optimal line-switching power flow
  - Capacitor placement
- Conclusion and future work
Optimal Line-Switching Power Flows

- The Problem
  - same as OPF but allows to switch off lines
  - exploit Braess paradox in power systems
  - very hard computationally

- Prior work
  - [Fisher, O’Neil, Ferris, 07], [Bienstock, 12]
  - shows the cost benefits of switching lines off
  - based on the LDC model

- This work
  - First attempt at provable quality bounds
Optimal Line-Switching Power Flows

- The AC Model

\[
\begin{align*}
\min \sum_{i \in G} c_i''(p_i^g)^2 + c_i'(p_i^g) \\
\text{s.t.} \ (3) - (7), (9) \\
pij &= z_{ij}(g_{ij}v_i^2 - g_{ij}v_i v_j \cos(\theta_i - \theta_j) - b_{ij}v_i v_j \sin(\theta_i - \theta_j)), \\
qij &= z_{ij}(-b_{ij}v_i^2 + b_{ij}v_i v_j \cos(\theta_i - \theta_j) - g_{ij}v_i v_j \sin(\theta_i - \theta_j)), \\
-\theta_i^u &\leq z_{ij}(\theta_i - \theta_j) \leq \theta_i^u, \\
z_{ij} &\in \{0, 1\}.
\end{align*}
\]

(AC-MINLP)
Optimal Line-Switching Power Flows

- **AC-MINLP**
  - the AC version

- **DC**

- **QC**
  - QC-NLP
  - QC-SOP
Optimal Line-Switching Power Flows

<table>
<thead>
<tr>
<th>Benchmark</th>
<th>Cost AC</th>
<th>Relative Cost DC</th>
<th>DC to AC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3128</td>
<td>-2.63%</td>
<td>—</td>
</tr>
<tr>
<td>2</td>
<td>5296</td>
<td>-1.52%</td>
<td>1.73%</td>
</tr>
<tr>
<td>3</td>
<td>5296</td>
<td>-1.52%</td>
<td>1.73%</td>
</tr>
<tr>
<td>4</td>
<td>8081</td>
<td>-5.43%</td>
<td>0.85%</td>
</tr>
<tr>
<td>5</td>
<td>573</td>
<td>-1.52%</td>
<td>—</td>
</tr>
<tr>
<td>6</td>
<td>573</td>
<td>-1.52%</td>
<td>—</td>
</tr>
<tr>
<td>7</td>
<td>8906</td>
<td>-6.32%</td>
<td>—</td>
</tr>
<tr>
<td>8</td>
<td>41857</td>
<td>-1.42%</td>
<td>0.23%</td>
</tr>
<tr>
<td>9</td>
<td>41725</td>
<td>-1.72%</td>
<td>—</td>
</tr>
<tr>
<td>10</td>
<td>129444</td>
<td>-2.69%</td>
<td>3.75%</td>
</tr>
<tr>
<td>11</td>
<td>721576</td>
<td>-1.88%</td>
<td>—</td>
</tr>
</tbody>
</table>
### Optimal Line-Switching Power Flows

<table>
<thead>
<tr>
<th>Bench.</th>
<th>Cost AC MINLP</th>
<th>Gap QC(r) Strong NLP</th>
<th>Gap QC(r) Weak SOCP</th>
<th>Cost QC(r) AC MINLP</th>
<th>Gap QC(r) Strong NLP</th>
<th>Gap QC(r) Weak SOCP</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3128</td>
<td>0.16%</td>
<td>0.16%</td>
<td>0.18%</td>
<td>0.12</td>
<td>0.13</td>
</tr>
<tr>
<td>2</td>
<td>5296</td>
<td>0.01%</td>
<td>0.01%</td>
<td>0.09%</td>
<td>0.11</td>
<td>0.06</td>
</tr>
<tr>
<td>3</td>
<td>5296</td>
<td>0.01%</td>
<td>0.01%</td>
<td>0.09%</td>
<td>0.11</td>
<td>0.09</td>
</tr>
<tr>
<td>4</td>
<td>8081</td>
<td>0.12%</td>
<td>0.12%</td>
<td>0.20%</td>
<td>0.43</td>
<td>0.12</td>
</tr>
<tr>
<td>5</td>
<td>573</td>
<td>0.12%</td>
<td>0.12%</td>
<td>0.17%</td>
<td>0.89</td>
<td>0.22</td>
</tr>
<tr>
<td>6</td>
<td>573</td>
<td>0.12%</td>
<td>0.12%</td>
<td>0.17%</td>
<td>0.81</td>
<td>0.25</td>
</tr>
<tr>
<td>7</td>
<td>8906</td>
<td>0.12%</td>
<td>0.12%</td>
<td>0.23%</td>
<td>0.23</td>
<td>0.36</td>
</tr>
<tr>
<td>8</td>
<td>41857</td>
<td>0.02%</td>
<td>0.02%</td>
<td>0.06%</td>
<td>0.75</td>
<td>0.39</td>
</tr>
<tr>
<td>9</td>
<td>41725</td>
<td>0.17%</td>
<td>0.17%</td>
<td>0.21%</td>
<td>15.88</td>
<td>0.58</td>
</tr>
<tr>
<td>10</td>
<td>129444</td>
<td>0.11%</td>
<td>0.11%</td>
<td>0.18%</td>
<td>66.58</td>
<td>0.96</td>
</tr>
<tr>
<td>11</td>
<td>721576</td>
<td>0.47%</td>
<td>0.47%</td>
<td>0.55%</td>
<td>1111.06</td>
<td>3.64</td>
</tr>
</tbody>
</table>

Table 7. OTS root-node optimality gap for the QC models on standard benchmarks.
## Optimal Line-Switching Power Flows

### Table 8: OTS root-node optimality gap for the QC models on congested benchmarks.

<table>
<thead>
<tr>
<th>Bench.</th>
<th>Cost</th>
<th>Gap</th>
<th>Runtime (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AC</td>
<td>AC</td>
<td></td>
</tr>
<tr>
<td></td>
<td>MINLP</td>
<td>NLP</td>
<td>SOCP</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Strong</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>QC</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>AC</td>
<td>Strong</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>QC</td>
</tr>
<tr>
<td>c-1</td>
<td>1500</td>
<td>0.05%</td>
<td>0.05%</td>
</tr>
<tr>
<td>c-2</td>
<td>22081</td>
<td>0.36%</td>
<td>0.36%</td>
</tr>
<tr>
<td>c-3</td>
<td>47259</td>
<td>8.06%</td>
<td>8.08%</td>
</tr>
<tr>
<td>c-4</td>
<td>1695</td>
<td>0.76%</td>
<td>0.76%</td>
</tr>
<tr>
<td>c-5</td>
<td>107536</td>
<td>1.21%</td>
<td>1.21%</td>
</tr>
<tr>
<td>c-6</td>
<td>133473</td>
<td>10.29%</td>
<td>10.29%</td>
</tr>
<tr>
<td>c-7</td>
<td>2478127</td>
<td>29.98%</td>
<td>29.98%</td>
</tr>
<tr>
<td>c-8</td>
<td>14745788</td>
<td>13.47%</td>
<td>13.48%</td>
</tr>
</tbody>
</table>
## Optimal Line-Switching Power Flows

<table>
<thead>
<tr>
<th>Bench.</th>
<th>QC&lt;sup&gt;r&lt;/sup&gt;</th>
<th>Gap</th>
<th>Gap</th>
<th>Gap</th>
<th>QC</th>
<th>Runtime (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>NLP</td>
<td>Strong MINLP</td>
<td>Strong SOCP</td>
<td>Weak SOCP</td>
<td>Strong MINLP</td>
<td>Strong SOCP</td>
</tr>
<tr>
<td>c-1</td>
<td>0.05%</td>
<td>0.05%</td>
<td>0.05%</td>
<td>0.05%</td>
<td>0.20</td>
<td>0.24</td>
</tr>
<tr>
<td>c-2</td>
<td>0.36%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.21</td>
<td>0.48</td>
</tr>
<tr>
<td>c-3</td>
<td>8.06%</td>
<td>3.35%</td>
<td>3.36%</td>
<td>3.51%</td>
<td>0.79</td>
<td>3.98</td>
</tr>
<tr>
<td>c-4</td>
<td>0.76%</td>
<td>0.44%</td>
<td>0.44%</td>
<td>0.43%</td>
<td>13.22</td>
<td>16.91</td>
</tr>
<tr>
<td>c-5</td>
<td>1.21%</td>
<td>0.43%</td>
<td>0.45%</td>
<td>0.50%</td>
<td>2.78</td>
<td>5.32</td>
</tr>
<tr>
<td>c-6</td>
<td>10.29%</td>
<td>3.35%</td>
<td>3.44%</td>
<td>5.61%</td>
<td>2367.89</td>
<td>294.74</td>
</tr>
<tr>
<td>c-7</td>
<td>29.98%</td>
<td>3.51%</td>
<td>3.58%</td>
<td>err.</td>
<td>T.L.</td>
<td>15992.07</td>
</tr>
<tr>
<td>c-8</td>
<td>13.47%</td>
<td>4.41%</td>
<td>4.94%</td>
<td>10.54%</td>
<td>T.L.</td>
<td>T.L.</td>
</tr>
</tbody>
</table>

### Table 9
OTS optimality gap for the QC models on congested benchmarks (10 hrs T.L.)
Optimal Line-Switching Power Flows

- The DC model is overly optimistic
  - infeasible configurations
  - suboptimal solutions when crossed-over
- QC bounds shows the quality of heuristic B&B
  - small gaps: 0.13% and 1.95% on traditional benchmarks at the root node
  - gaps can reached 40% on congested benchmarks at the root node
  - gaps can be reduced to 4% by solving the mixed integer version
Outline

- Motivation
- Complexity
- The LPAC Approximation
  - The model
  - Power restoration
- The QC Relaxation
  - The model
  - Optimal power flow
  - Optimal line-switching power flow
    - Capacitor placement
- Conclusion and future work
Capacitor Placement

- Capacitors inject reactive power
- The problem is to place capacitors in a network to improve the voltage profile
  - well-studied problems
  - almost always tackled by heuristics
    - need to reason about reactive power and voltage magnitudes
- The goal is to minimize the number of capacitors
Capacitor Placement

\[ \begin{align*}
\min & \sum_{i=1}^{m} z_i \\
\text{s.t.} & \quad p_i = \sum_{j=1}^{m} p_{ij}, \quad \forall i \in \{1, \ldots, m\}, \\
& \quad q_i = \sum_{j=1}^{m} q_{ij}, \quad \forall i \in \{1, \ldots, m\}, \\
& \quad (1) - (2) \\
& \quad p_{ij}^2 + q_{ij}^2 \leq s_{ij}^2, \quad \forall (i, j) \in \{1, \ldots, m\}^2, \\
& \quad p_i^l \leq p_i \leq p_i^u, \quad \forall i \in \{1, \ldots, m\}, \\
& \quad q_i^l \leq q_i \leq q_i^u + q_i^c z_i, \quad \forall i \in \{1, \ldots, m\}, \\
& \quad v_i^l \leq v_i \leq v_i^u, \quad \forall i \in \{1, \ldots, m\}, \\
& \quad -\theta_i^u \leq \theta_i - \theta_j \leq \theta_i^u, \quad \forall (i, j) \in \{1, \ldots, m\}^2, \\
& \quad v, \theta, p, q \in \mathbb{R}^n, z \in \mathbb{N}^m.
\end{align*} \]
Capacitor Placement

Key Messages

- MINLP heuristics give near optimal solutions
- QC gives tight lower bounds
  - justifying the first message
- The QC model scales well to medium-sized instances

<table>
<thead>
<tr>
<th>id</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
<th>C6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>14</td>
<td>30</td>
<td>30</td>
<td>57</td>
<td>118</td>
<td>300</td>
</tr>
<tr>
<td>$</td>
<td>N</td>
<td>$</td>
<td>14</td>
<td>30</td>
<td>30</td>
<td>57</td>
</tr>
<tr>
<td>$</td>
<td>E</td>
<td>$</td>
<td>20</td>
<td>41</td>
<td>41</td>
<td>78</td>
</tr>
</tbody>
</table>

Table 6  Sizes of Capacitor Placement Benchmarks.
## Capacitor Placement

<table>
<thead>
<tr>
<th>Benchmark</th>
<th>Cost</th>
<th>Gap</th>
<th>Runtime (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AC</td>
<td>QC</td>
<td>AC</td>
</tr>
<tr>
<td></td>
<td>MINLP</td>
<td>MINLP</td>
<td>SOCP</td>
</tr>
<tr>
<td>cpp-1</td>
<td>5</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>cpp-2</td>
<td>8</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>cpp-3</td>
<td>7</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>cpp-4</td>
<td>15</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>cpp-5</td>
<td>67</td>
<td>4.48%</td>
<td>5.97%</td>
</tr>
<tr>
<td>cpp-6</td>
<td>241</td>
<td>5.39%</td>
<td>5.81%</td>
</tr>
</tbody>
</table>

*Table 11*  CPP Runtime and Optimality Gaps using the QC Relaxations with $v^l = 1$. 
Reconfiguration Problems

- Find out the network topology (by opening and closing switches) to
  - minimize losses
  - or maximize balances
- while satisfying operational constraints
  - thermal limits
  - phase angle and voltage magnitude constraints
  - ...
- Relaxation
  - Dist-flow for radial networks, QC for meshed networks
**Loss Minimization**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32-bus</td>
<td>0.01395</td>
<td>0%</td>
<td>2.90</td>
<td>–</td>
<td>32/37</td>
</tr>
<tr>
<td>70-bus</td>
<td>0.03016</td>
<td>0%</td>
<td>7.45</td>
<td>–</td>
<td>68/76</td>
</tr>
<tr>
<td>135-bus</td>
<td>0.28013</td>
<td>0%</td>
<td>24.80</td>
<td>–</td>
<td>135/156</td>
</tr>
<tr>
<td>880-bus</td>
<td>0.45703</td>
<td>0%</td>
<td>2886.94</td>
<td>–</td>
<td>873/900</td>
</tr>
<tr>
<td>Meshed</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32-bus</td>
<td>0.01232</td>
<td>0.55%</td>
<td>6.26</td>
<td>0.10</td>
<td>36/37</td>
</tr>
<tr>
<td>70-bus</td>
<td>0.02977</td>
<td>0.17%</td>
<td>35.42</td>
<td>0.21</td>
<td>75/76</td>
</tr>
<tr>
<td>135-bus</td>
<td>0.27079</td>
<td>3.00%</td>
<td>T.L.</td>
<td>3.42</td>
<td>149/156</td>
</tr>
<tr>
<td>880-bus</td>
<td>0.45175</td>
<td>8.72%</td>
<td>T.L.</td>
<td>T.L.</td>
<td>900/900</td>
</tr>
</tbody>
</table>

**TABLE I. MINIMAL LOSS RESULTS**
# Loss Minimization

<table>
<thead>
<tr>
<th>Network</th>
<th>Val</th>
<th>Time</th>
<th>Val</th>
<th>Time</th>
<th>Val</th>
<th>Time</th>
<th>Val</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>32-bus</td>
<td>0.01395</td>
<td>2.90</td>
<td>0.01395</td>
<td>0.39</td>
<td>0.01395</td>
<td>23.83</td>
<td>0.01395</td>
<td>5.23</td>
</tr>
<tr>
<td>70-bus</td>
<td>0.03016</td>
<td>7.45</td>
<td>0.03016</td>
<td>0.98</td>
<td>0.03016</td>
<td>769.63</td>
<td>0.03016</td>
<td>52.48</td>
</tr>
<tr>
<td>135-bus</td>
<td>0.28013</td>
<td>24.80</td>
<td>0.28013</td>
<td>54.74</td>
<td>0.28661</td>
<td>T.L.</td>
<td>0.29171</td>
<td>T.L.</td>
</tr>
<tr>
<td>880-bus</td>
<td>0.45703</td>
<td>2886.94</td>
<td>0.45704</td>
<td>T.L.</td>
<td>–</td>
<td>T.L.</td>
<td>mem. err.</td>
<td>–</td>
</tr>
</tbody>
</table>
Reconfiguration Problems

Key messages

- Zero-gap relaxation on radial networks
- Tight on meshed networks
  - loses accuracy on the largest one
- Scalability
Outline

- Motivation
- Power Flow Formulations
- Complexity
- The LPAC Approximation
- Case Studies for LPAC
- Convex Relaxations
- Case Studies for Convex Relaxations
- Beyond Steady States
Restoration Ordering Problem (ROP)

- Find the best sequence of restoration
  
  \[ [x_1, x_2, x_3, \ldots, x_n] \]

  so that the size of the blackout is minimised
Key Question

- Can I move from steady states to steady states?
- Capturing the dynamics

\[ \dot{x} = f(x, y) \]
\[ 0 = g(x, y) \]
Can you correct the steady states to ensure transient stability?
  - rotor angle stability
  - Swing equation

\[ \dot{x} = f(x, y) \]

\[ \delta_{it+1} - \delta_{it} - \frac{\Delta}{2} (\omega_{it+1} + \omega_{it}) = 0 \]

\[ \omega_{it+1} - \omega_{it} - \frac{\Delta}{2} (a_{it+1} + a_{it}) = 0 \]

\[ a_{it} - \frac{\omega_0}{2H_i} (p_{i1}^e - p_{it}^e - D_i \omega_{it}) = 0 \]
Transient Analysis

- Can you correct the steady states to ensure transient stability?
  - rotor angle stability
  - Swing equation

$$\dot{x} = f(x, y)$$

3-bus: Generators 1 and 2 Rotor Angle
Transient Stability

- Angle for the center of inertia

\[ \delta^r_t = \frac{\sum_{i \in G} H_i \delta_{it}}{\sum_{i \in G} H_i} \quad (1 \leq t \leq T). \]

The stability constraints then become:

\[ \forall i \in G, 1 \leq t \leq T : -\bar{\delta} \leq \delta_{it} - \delta^r_t \leq \bar{\delta} \]
Transient Stability

- A generator is modeled as an internal bus

\[ p_{it}^e = \frac{v_i^g v_{it}}{X_i} \sin(\theta_{it} - \delta_{it}) \]

\[ q_{i1}^e = \frac{v_{i1}^2}{X_i} - \frac{v_i^g v_{i1}}{X_i} \cos(\theta_{i1} - \delta_{i1}) \]
Network State

- Power flow equations before closing

\[\sum_{j \in N} [v_i v_j (G_{ij}^o \cos(\theta_{ij1}) + B_{ij}^o \sin(\theta_{ij1}))] = \sum_{j \in G(i)} (p_{j1}^e - p_{j}^l l_j) \]

(9)

\[\sum_{j \in N} [v_i v_j (G_{ij}^o \sin(\theta_{ij1}) - B_{ij}^o \cos(\theta_{ij1}))] = \sum_{j \in G(i)} (q_{j1}^e - p_{j}^l l_j) \]

(10)
Network State

- Power flow equations after closing

\[(i \in N, 2 \leq t \leq T):\]

\[
\sum_{j \in N} \left[ v_{it} v_{jt} (G_{ij}^c \cos(\theta_{ijt}) + B_{ij}^c \sin(\theta_{ijt})) \right] = \sum_{j \in G(i)} (p_{jt}^e - p_j^l l_j) \tag{7}
\]

\[
\sum_{j \in N} \left[ v_{it} v_{jt} (G_{ij}^c \sin(\theta_{ijt}) - B_{ij}^c \cos(\theta_{ijt})) \right] = \sum_{j \in G(i)} (q_{jt}^e - p_j^l l_j) \tag{8}
\]
Network State

- Voltage Stability Constraints

\[ v_{it} - \bar{v} \Delta \leq v_{it+1} \leq v_{it} + \bar{v} \Delta. \]
Minimal Transient-Stable Correction

Model 2 The One-Step Line-Closing Model

Inputs:
- \( P_N \) - Power network
- \( Y^o, Y^c \) - Reduced matrix before and after line closing
- \( p_i^T, q_i^T \) - \( i \)th generator active/reactive target dispatch
- \( X_i, H_i, D_i \) - transient reactance, inertia, and damping constant
- \( f_q = \frac{\omega_0}{2\pi}, \Delta, T \) - Grid frequency, integration step, time horizon
- Maximum voltage fluctuations

Variables: (\( \forall t : 1 \leq t \leq T \))
- \( \forall i \in N, \theta_{it} \) - Terminal bus angle
- \( v_{it} \) - Bus voltage
- \( \forall i \in G, v_i^g \) - Internal bus voltage
- \( p_{it}^e, q_{it}^e \) - Generator active and reactive injection
- \( l_i \) - load percentage of load \( i \)
- \( \delta_{it}, \omega_{it}, a_{it} \) - Generator rotor angle, velocity, and acceleration

Minimize
\[
\sum_{i \in G} \left( p_{i1}^e - p_i^T \right)^2 + \left( q_{i1}^e - q_i^T \right)^2
\]

Subject to:
Equations (1) - (17)
Overall Procedure

- Bus node
- Line restored & closed
- Line not-yet restored/opened
- Line to be closed

Steady state

Line closing next step
Overall Procedure

- Bus node
- Line restored & closed
- Line not-yet restored/opened
- Line to be closed

Steady state

Line closing next step
Overall Procedure

1. Solve Restoration Ordering Problem
2. Extract steady state sequence for transient stability enhancement

- Perform **Kron reduction** to further remove all “non-generator” buses
- Kron reduction:
  1. Admittance preserving reduction
  2. Widely used in transient stability to reduce buses not under study
  3. Based on Gaussian Elimination and Kirchhoff’s Current Law

3. Remove islands unrelated to line closing
4. Construct admittance matrices and perform Kron reduction
5. Solve Rotor Stability Optimization Model
Power Restoration with Transient Stability

- Minimize the distance to the steady-state dispatch while constraining the rotor angle

| TABLE I |

| GENERATOR DISPATCH RESULTS TO MAINTAIN STANDARD 90 DEGREE ROTOR SWINGS |

<table>
<thead>
<tr>
<th>6 Bus</th>
<th>14 Bus</th>
<th>30 Bus</th>
<th>39 Bus</th>
<th>57 Bus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gen. Reactance</td>
<td>Max. %</td>
<td>Max. MW</td>
<td>Avg. %</td>
<td>Avg. MW</td>
</tr>
<tr>
<td>0.02</td>
<td>0.00058</td>
<td>0.00066</td>
<td>0.00013</td>
<td>0.00012</td>
</tr>
<tr>
<td>0.06</td>
<td>0.00058</td>
<td>0.00066</td>
<td>0.00013</td>
<td>0.00012</td>
</tr>
<tr>
<td>0.10</td>
<td>0.00046</td>
<td>0.00051</td>
<td>0.00011</td>
<td>0.00010</td>
</tr>
<tr>
<td>0.14</td>
<td>0.00047</td>
<td>0.00051</td>
<td>0.00011</td>
<td>0.00010</td>
</tr>
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<td>0.20</td>
<td>0.00049</td>
<td>0.00054</td>
<td>0.00011</td>
<td>0.00010</td>
</tr>
</tbody>
</table>

39 Bus
| Gen. Reactance | Max. % | Max. MW | Avg. % | Avg. MW | Max. % | Max. MW | Avg. % | Avg. MW |
| 0.02   | 0.00011 | 0.00069 | 0.00001 | 0.00009 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 0.06   | 0.00011 | 0.00068 | 0.00001 | 0.00009 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 0.10   | 0.00010 | 0.00061 | 0.00001 | 0.00008 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 0.14   | 0.00009 | 0.00057 | 0.00001 | 0.00007 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 0.20   | 0.00013 | 0.00081 | 0.00002 | 0.00010 | 9.49169 | 5.17440 | 0.25634 | 0.25016 |
Minimizing the rotor angle while maintaining a certain dispatching distance

**Power Restoration with Transient Stability**

**TABLE II**

<table>
<thead>
<tr>
<th></th>
<th>6 Bus</th>
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<td><strong>Maximum dispatch limit (L2)</strong></td>
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<td><strong>Maximum dispatch limit (L2)</strong></td>
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</tr>
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Beyond Steady States

- Using the dynamics to correct the steady state
  - hierarchical approach
  - feedback loop

- The power flow equations
  - need to be solved exactly
  - the relaxations are “cheating”

- Scalability is the next frontier
  - more realistic dynamic models
  - thermal limits
  - scaling performance
Open Issues

- **Power Flows**
  - Can we get scalable approximations with performance guarantees on these problems?
  - Can we get scalable relaxations?
  - Can we define what normal operating conditions means?

- **Dynamics**
  - Can we get scalable optimization approaches?

- **Stochastic and robust optimisation**
  - Generalizing the deterministic case