Error Bounds on the DC Power Flow Approximation: A Convex Relaxation Approach

An Application of Pascal's and Andy's Talks

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The Power Flow Equations

- Model the relationship between the voltage phasors and the power injections

Polar voltage coordinates: $V_i = |V_i|\angle\theta_i$

$$P_i = |V_i| \sum_{k=1}^{n} |V_k| \left(\mathbf{G}_{ik} \cos\left(\theta_k - \theta_i\right) + \mathbf{B}_{ik} \sin\left(\theta_k - \theta_i\right)\right)$$

$$Q_i = |V_i| \sum_{k=1}^{n} |V_k| \left(\mathbf{G}_{ik} \sin\left(\theta_k - \theta_i\right) - \mathbf{B}_{ik} \cos\left(\theta_k - \theta_i\right)\right)$$

- Central to many power system optimization and control problems
  - Optimal power flow, unit commitment, voltage stability, contingency analysis, transmission switching, etc.

DC Power Flow Approximation

- Linearization of the power flow equations

$$P_i = \overset{1}{|V_i|} \sum_{k=1}^{n} \overset{1}{|V_k|} \left(\overset{0}{\mathbf{G}_{ik}} \cos(\theta_k - \theta_i) + \mathbf{B}_{ik} \overset{(\theta_k - \theta_i)}{\sin(\theta_k - \theta_i)} \right)$$

$$Q_i = |V_i| \sum_{k=1}^{n} |V_k| \left(\mathbf{G}_{ik} \sin(\theta_k - \theta_i) - \mathbf{B}_{ik} \cos(\theta_k - \theta_i) \right)$$

$$\Rightarrow \quad P_i^{DC} = \sum_{k=1}^{n} \mathbf{B}_{ik} (\theta_k - \theta_i)$$

- Advantages:
  - Fast and reliable solution using linear programming
- Disadvantages:
  - No consideration of voltage magnitudes or reactive power
  - Approximation error

DC Power Flow Accuracy

- Many studies of DC power flow accuracy:
  - [Yan & Sekar ' 02], [Liu & Gross ' 02], [Baldick ' 04], [Overbye, Cheng, & Sun ' 04], [Baldick, Dixit & Overbye ' 05], [Purchala, Meeus,
    Van Dommelen & Belmans ' 05], [Van Hertem, Verboomen, Purchala, Belmans & Kling ' 06], [Li & Bo ' 07], [Duthaler, Emery, Andersson, & Kurzidem ' 08], [Stott, Jardim & Alsac '09], [Qi, Shi & Tylavsky ' 12], [Coffrin, Van Hentenryck & Bent ' 12]
- Accuracy depends on the application and test case

> "At no stage in the tests were we able to discern any statistical pattern in the dc-flow error scatters. This defeated all our attempts to find concise, meaningful indices with which to characterize and display dc-model accuracies." [Stott, Jardim & Alsac '09]

Problem Formulation

Assessing DC Power Flow Accuracy

- Goal: bound the worst-case error in the active power injections between the DC and AC power flow models

Today's approach:

Alternate representation:

Power Injections

P_i, Q_i

Voltages

$|V_i|, \theta_i$

DC Power Flow

$$P_i^{DC} = \sum_{k=1}^{n} \mathbf{B}_{ik}\left(\theta_k - \theta_i\right)$$

$$P_{flow,ik}^{DC} = \left(\mathbf{PTDF}\right) P_i$$

Future Work!

AC Power Flow

$$P_i^{AC} = |V_i| \sum_{k=1}^{n} |V_k| \left(\mathbf{G}_{ik}\cos\left(\theta_k - \theta_i\right) + \mathbf{B}_{ik}\sin\left(\theta_k - \theta_i\right)\right)$$

$$Q_i = |V_i| \sum_{k=1}^{n} |V_k| \left(\mathbf{G}_{ik}\sin\left(\theta_k - \theta_i\right) - \mathbf{B}_{ik}\cos\left(\theta_k - \theta_i\right)\right)$$

$$P_{flow,ik}^{AC} = |V_i||V_k|\left(\mathbf{G}_{ik}\cos\left(\theta_i - \theta_k\right) + \mathbf{B}_{ik}\sin\left(\theta_i - \theta_k\right)\right) - |V_k|^2\,\mathbf{G}_{ik}$$

Error

$$\left\| P^{DC} - P^{AC} \right\|$$

$$\left\| P_{flow}^{DC} - P_{flow}^{AC} \right\|$$

Worst-Case Error Formulation

$$\max_{|V|,\theta} \; \left|\left| P^{DC} - P^{AC} \right|\right|_{\infty}$$

$$\text{s.t.} \quad \left|V_i^{min}\right| \leq |V_i| \leq \left|V_i^{max}\right|$$

$$\theta_{ik}^{min} \leq \theta_i - \theta_k \leq \theta_{ik}^{max}$$

$$P_i^{min} \leq P_i^{AC} \leq P_i^{max}$$

$$Q_i^{min} \leq Q_i \leq Q_i^{max}$$

$$P_i^{DC} = \sum_{k=1}^{n} \mathbf{B}_{ik} \left(\theta_k - \theta_i\right)$$

$$P_i^{AC} = |V_i| \sum_{k=1}^{n} |V_k| \; \left(\mathbf{G}_{ik} \cos\left(\theta_k - \theta_i\right) + \mathbf{B}_{ik} \sin\left(\theta_k - \theta_i\right)\right)$$

$$Q_i = |V_i| \sum_{k=1}^{n} |V_k| \; \left(\mathbf{G}_{ik} \sin\left(\theta_k - \theta_i\right) - \mathbf{B}_{ik} \cos\left(\theta_k - \theta_i\right)\right)$$

Maximize Error

Operational Constraints

Non-Convex!

DC Power Flow

AC Power Flow

Handling the Objective Function

- Maximize the infinity norm by solving $2n$ optimization problems:

$$\max_{|V|,\theta} \quad ||P^{DC} - P^{AC}||_{\infty}$$

$$\max_{\ell\in\{1,\ldots,n\}} \left\{ \max_{|V|,\theta} \left| P_\ell^{DC} - P_\ell^{AC} \right| \right\}$$

For each $\ell = 1, \ldots, n$ and $\sigma = \{-1, 1\}$, solve (in parallel):

$$\max_{|V|,\theta} \quad \sigma \cdot \left(P_\ell^{DC} - P_\ell^{AC}\right)$$

Select the largest absolute value among all the solutions

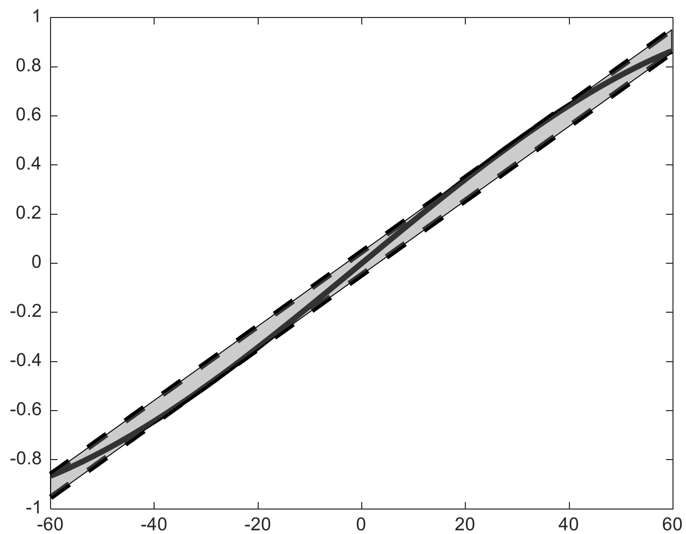
Formulation

Handling the Power Flow Equations via Convex Relaxations

- Formulating the DC power flow requires a representation of the voltage angles: $P_i^{DC} = \sum_{k=1}^{n} \mathbf{B}_{ik} \left(\theta_k - \theta_i\right)$

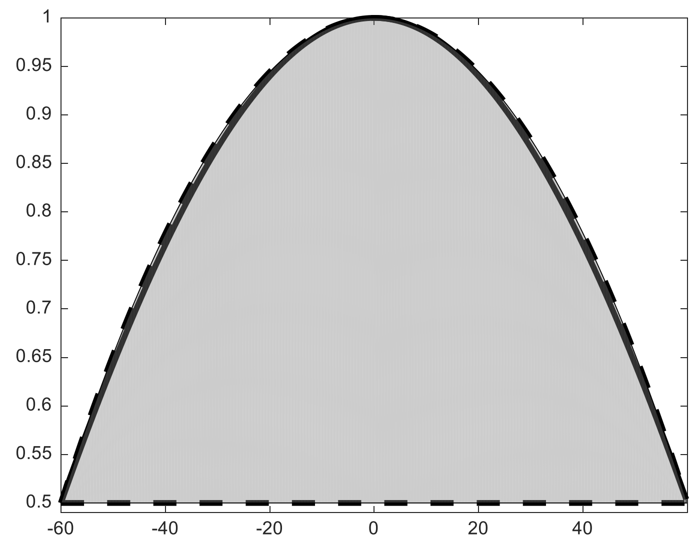
- QC Relaxation to the rescue! [Coffrin, Hijazi & Van Hentenryck '15]

$\sin\left(\theta_k - \theta_i\right)$



$\theta_k - \theta_i$

$\cos\left(\theta_k - \theta_i\right)$



$\theta_k - \theta_i$

Formulation

Further Tightening the Relaxation

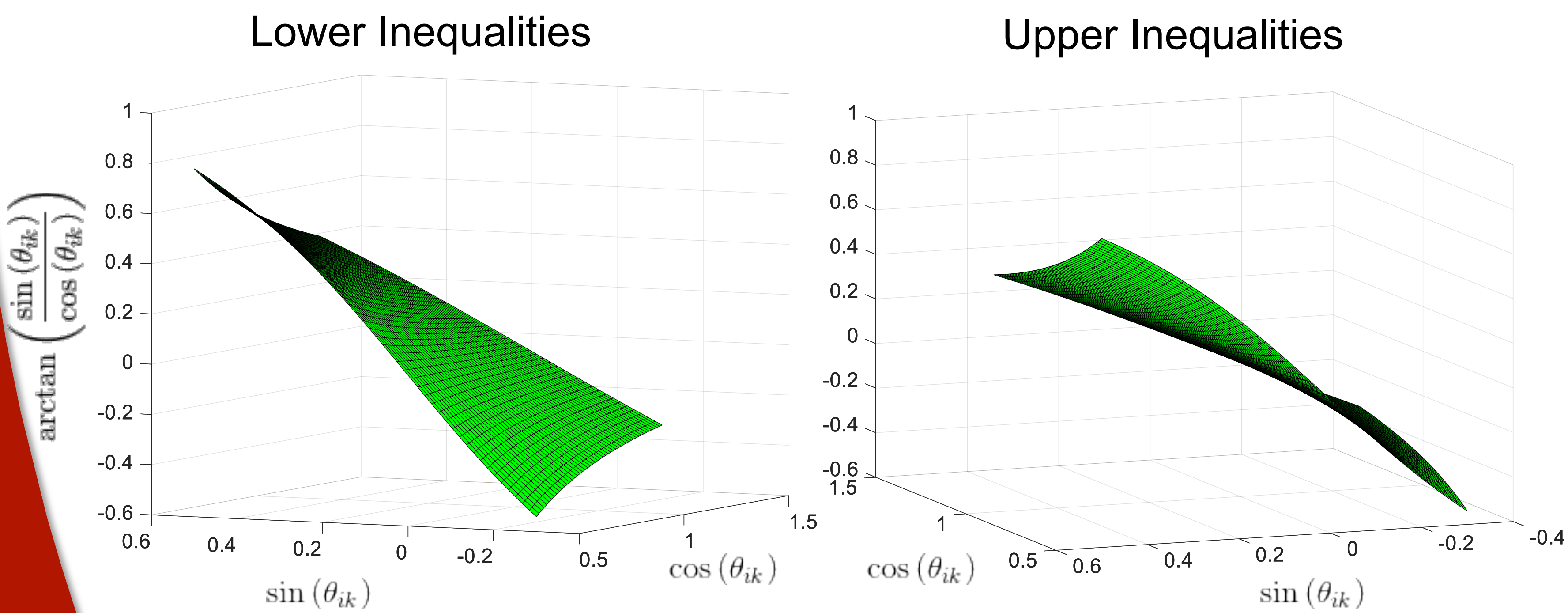
- Augment the QC relaxation with
  - A Semidefinite Programming Relaxation of the power flow equations in rectangular coordinates [Lavaei & Low '12]
  - Lifted Nonlinear Cuts implied by the angle difference and voltage magnitude limits [Coffrin, Hijazi & Van Hentenryck '15], [Chen, Atamturk & Oren '15]
  - Arctangent Envelopes [Kocuk, Dey & Sun '16]
- Apply a bound tightening algorithm to improve upon the specified operational limits [Kocuk, Dey & Sun '15], [Chen, Atamturk & Oren '15], [Coffrin, Hijazi & Van Hentenryck '16]

Semidefinite Relaxation of the Power Flow Equations

- Write power flow equations as $z^H \mathbf{A}_i z = c_i$
  where $z = \begin{bmatrix} V_1 & \ldots & V_n \end{bmatrix}^\intercal$ with voltage phasors $V \in \mathbb{C}^n$
- Define matrix $\mathbf{W} = zz^H$
- Rewrite as $\operatorname{rank}(\mathbf{W}) = 1$ and $\begin{cases} \operatorname{trace}(\mathbf{A}_i \mathbf{W}) = c_i \\ \mathbf{W} \succeq 0 \end{cases}$
- Relaxation:
  Do not enforce $\operatorname{rank}(\mathbf{W}) = 1$ [Lavaei & Low '12]
  - A solution with $\operatorname{rank}(\mathbf{W}) = 1$ implies zero relaxation gap and recovery of the globally optimal voltage profile. This is not necessary for our problem: we only require a lower bound.

Arctangent Envelopes

- Enclose the arctangent function using linear inequalities [Kocuk, Dey, & Sun '16]



Formulation

Formulation Summary

For each $\ell = 1, \ldots, n$ and $\sigma = \{-1, 1\}$, solve (in parallel):

$$\max_{|V|,\theta} \quad \sigma \cdot \left(P_\ell^{DC} - P_\ell^{AC}\right)$$

Maximize the absolute value of the error at each bus

$$\text{s.t.} \quad \left|V_i^{min}\right| \leq |V_i| \leq \left|V_i^{max}\right|$$

$$\theta_{ik}^{min} \leq \theta_i - \theta_k \leq \theta_{ik}^{max}$$

$$P_i^{min} \leq P_i^{AC} \leq P_i^{max}$$

$$Q_i^{min} \leq Q_i \leq Q_i^{max}$$

Bound-tightened operational constraints

$$P_i^{DC} = \sum_{k=1}^{n} \mathbf{B}_{ik} \left(\theta_k - \theta_i\right)$$

DC Power Flow

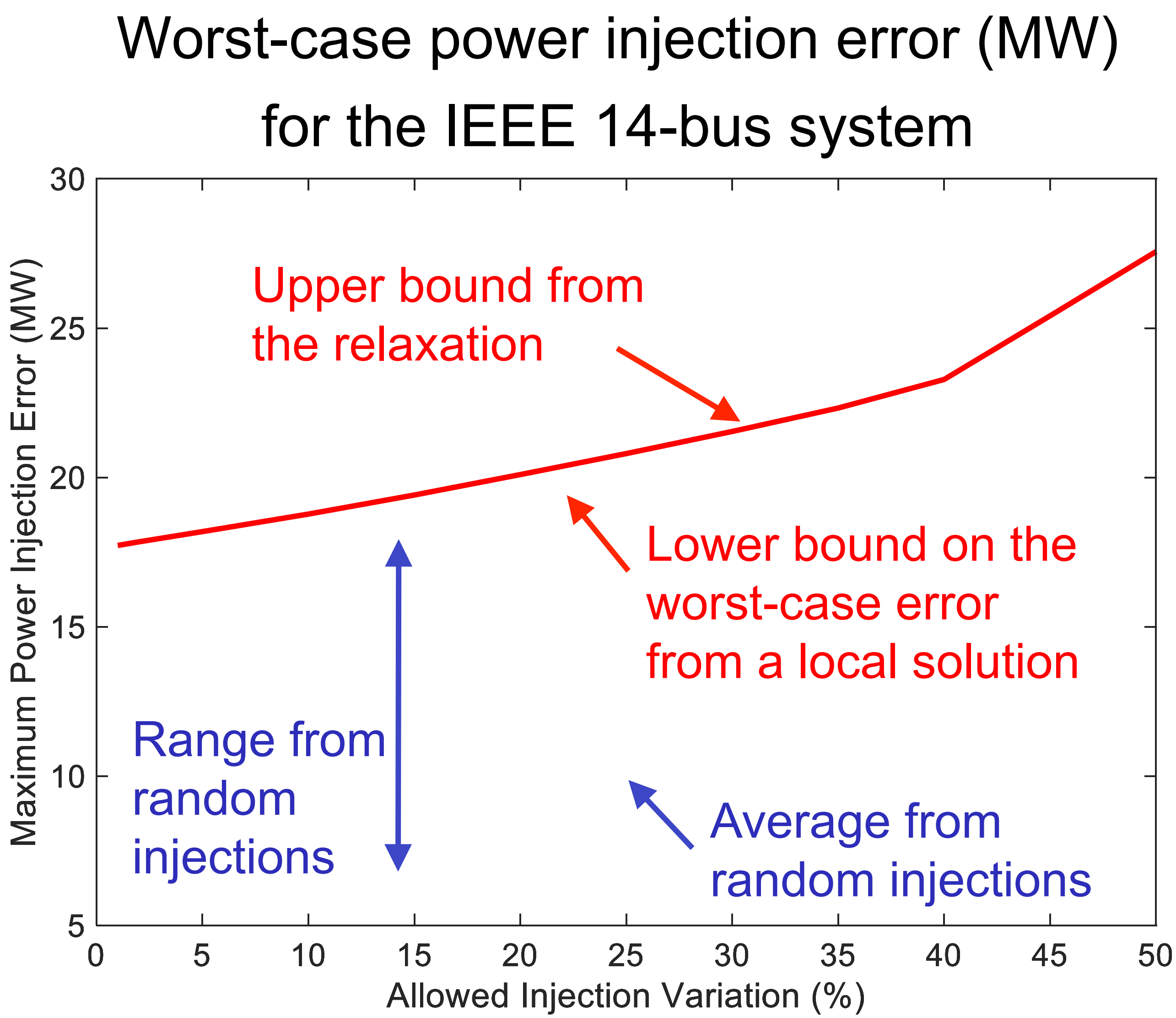
$$P_i^{AC} = |V_i| \sum_{k=1}^{n} |V_k| \left(\mathbf{G}_{ik} \cos\left(\theta_k - \theta_i\right) + \mathbf{B}_{ik} \sin\left(\theta_k - \theta_i\right)\right)$$

$$Q_i = |V_i| \sum_{k=1}^{n} |V_k| \left(\mathbf{G}_{ik} \sin\left(\theta_k - \theta_i\right) - \mathbf{B}_{ik} \cos\left(\theta_k - \theta_i\right)\right)$$

QC Relaxation
+ SDP Relaxation
+ Lifted Nonlinear Cuts
+ Arctangent Envelopes

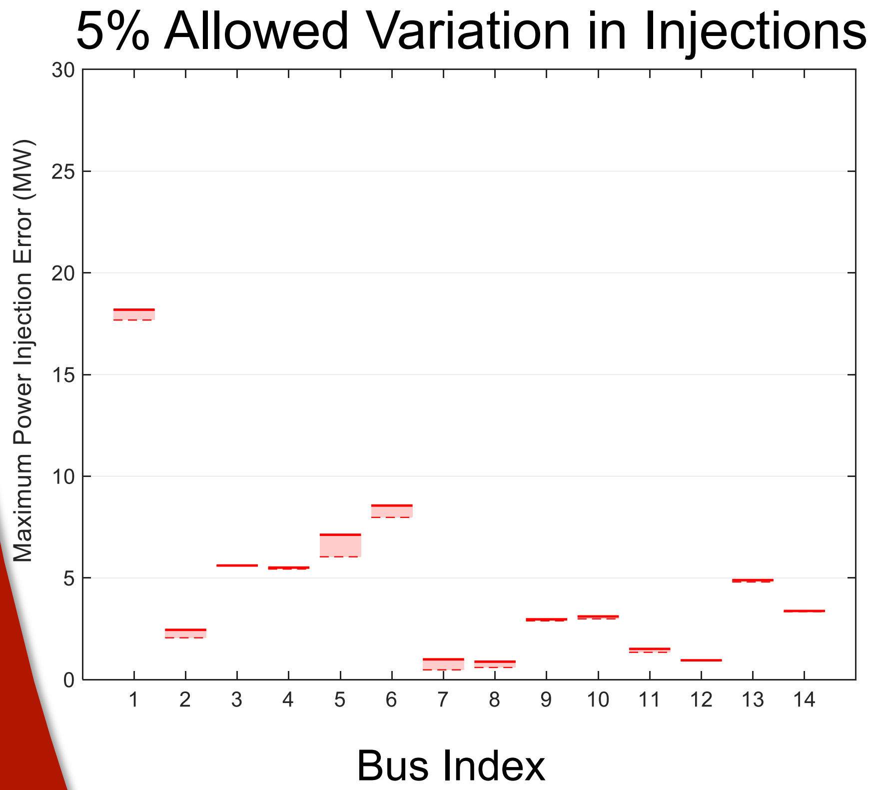
Results for the IEEE Test Cases

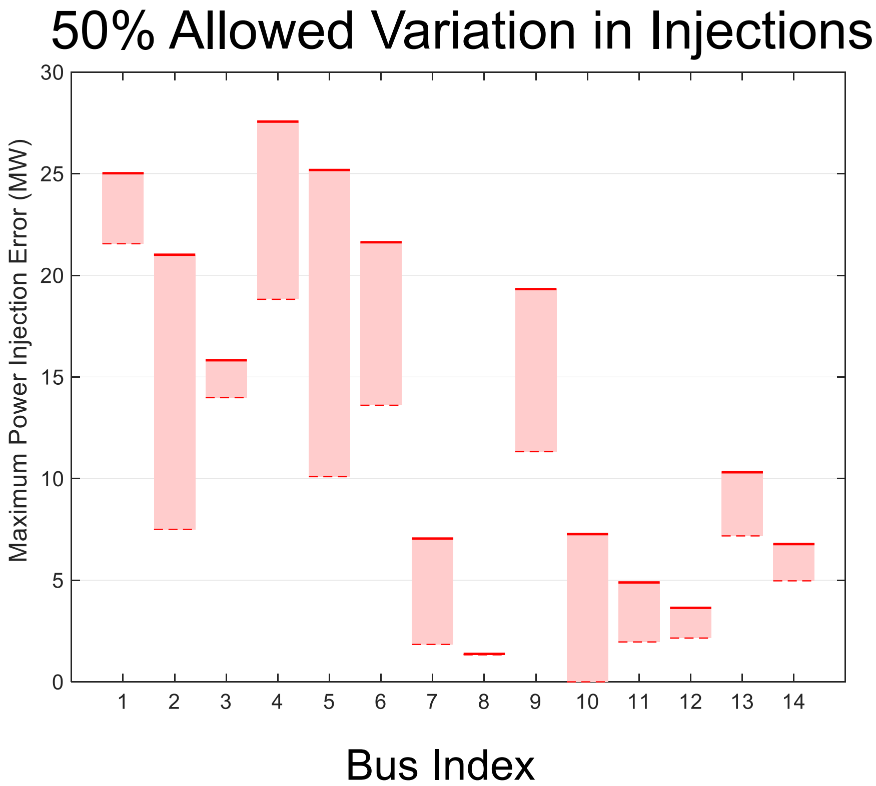
Results


Worst-case power injection error (MW)
for the IEEE 14-bus system
Maximum Power Injection Error (MW)
30
25
20
15
10
5
0
5
10
15
20
25
30
35
40
45
50
Allowed Injection Variation (%)
Upper bound from
the relaxation
Lower bound on the
worst-case error
from a local solution
Range from
random
injections
Average from
random injections

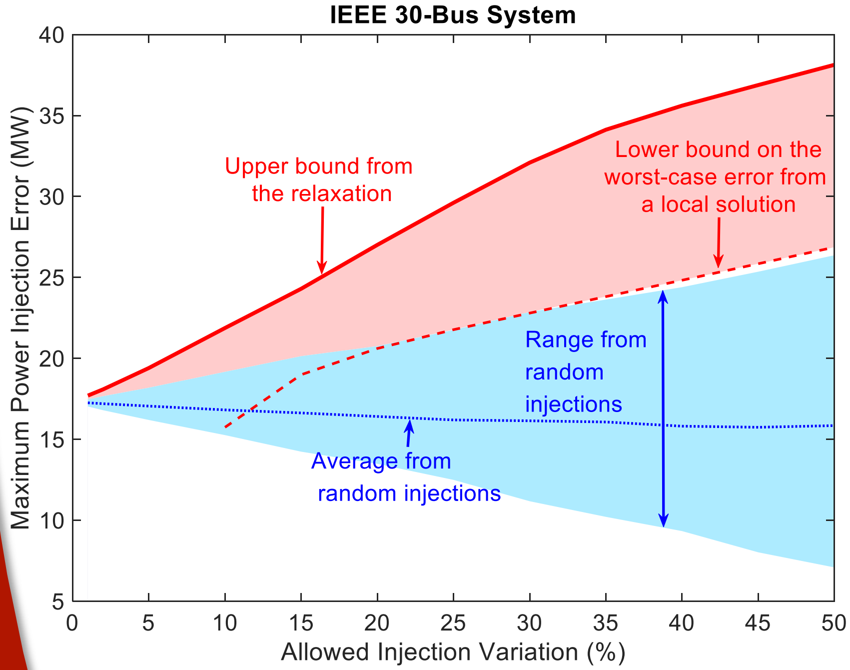
Results

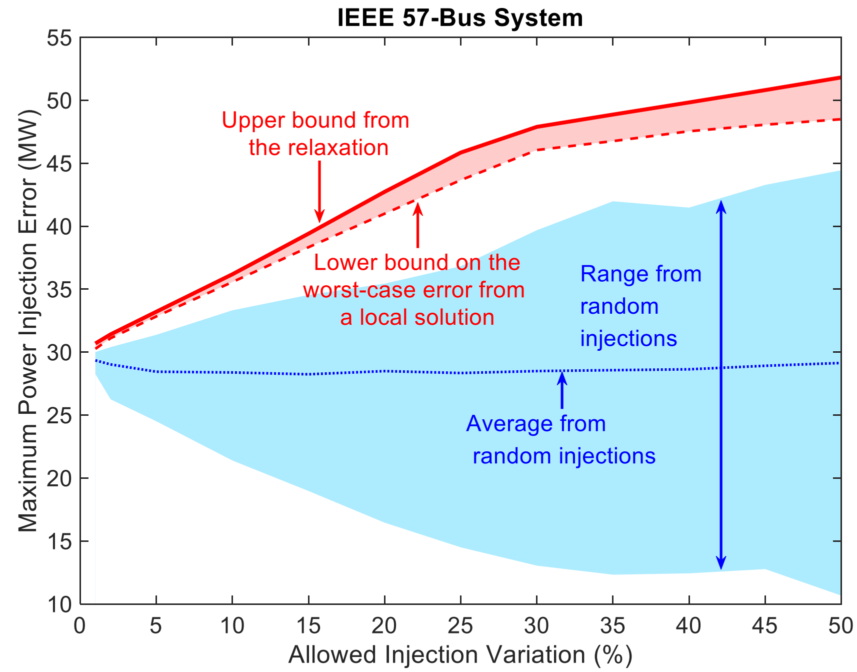
Worst-case power injection error (MW)
for the IEEE 14-bus system, by bus

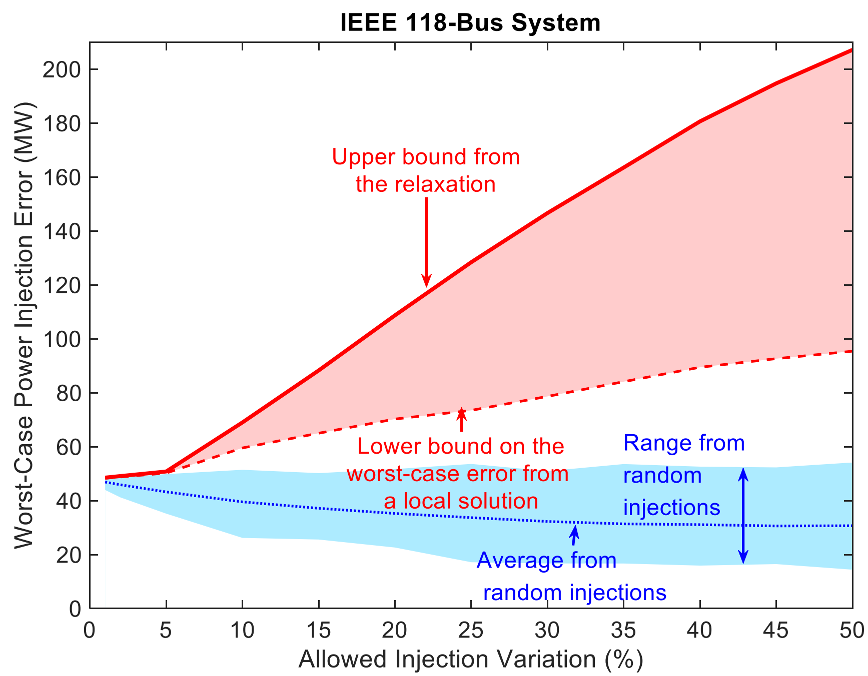




Results


IEEE 30-Bus System
Maximum Power Injection Error (MW)
Allowed Injection Variation (%)
Upper bound from the relaxation
Lower bound on the worst-case error from a local solution
Range from random injections
Average from random injections


IEEE 57-Bus System
Maximum Power Injection Error (MW)
Allowed Injection Variation (%)
Upper bound from the relaxation
Lower bound on the worst-case error from a local solution
Range from random injections
Average from random injections


IEEE 118-Bus System
Worst-Case Power Injection Error (MW)
Allowed Injection Variation (%)
Upper bound from the relaxation
Lower bound on the worst-case error from a local solution
Range from random injections
Average from random injections

Conclusion

Conclusions

- We proposed an algorithm that uses convex relaxations to bound the worst-case error of the DC power flow
- Results for several IEEE test cases show:
  - The bound is reasonably tight
  - The DC power flow can have large errors for some operating conditions
- Next steps:
  - Application to other linear approximations and test cases
  - Comparison with other error bounds
  - Determination of physical explanations for large errors
  - Design of new linearizations informed by the worst-case error

Questions?

K. Dvijotham and D.K. Molzahn, "Error Bounds on the DC Power Flow Approximation: A Convex Relaxation Approach," *IEEE 55th Annual Conference on Decision and Control (CDC)*, December 12-14, 2016.