Error Bounds on the DC Power Flow Approximation: A Convex Relaxation Approach

Krishnamurthy Dvijotham  
Pacific Northwest National Laboratory

Daniel Molzahn  
Argonne National Laboratory

Santa Fe Winter School and Conference  
January 13, 2017
The Power Flow Equations

- Model the relationship between the voltage phasors and the power injections

Polar voltage coordinates: \( V_i = |V_i| \angle \theta_i \)

\[
P_i = |V_i| \sum_{k=1}^{n} |V_k| \left( G_{ik} \cos(\theta_k - \theta_i) + B_{ik} \sin(\theta_k - \theta_i) \right)
\]

\[
Q_i = |V_i| \sum_{k=1}^{n} |V_k| \left( G_{ik} \sin(\theta_k - \theta_i) - B_{ik} \cos(\theta_k - \theta_i) \right)
\]

- Central to many power system optimization and control problems
  - Optimal power flow, unit commitment, voltage stability, contingency analysis, transmission switching, etc.
DC Power Flow Approximation

• Linearization of the power flow equations

\[
P_i = |V_i| \sum_{k=1}^{n} |V_k| \left( G_{ik} \cos (\theta_k - \theta_i) + B_{ik} \sin (\theta_k - \theta_i) \right)
\]

\[
Q_i = |V_i| \sum_{k=1}^{n} |V_k| \left( G_{ik} \sin (\theta_k - \theta_i) - B_{ik} \cos (\theta_k - \theta_i) \right)
\]

\[
P_{i}^{DC} = \sum_{k=1}^{n} B_{ik} (\theta_k - \theta_i)
\]

• Advantages:
  – Fast and reliable solution using linear programming

• Disadvantages:
  – No consideration of voltage magnitudes or reactive power
  – Approximation error
**DC Power Flow Accuracy**

- Many studies of DC power flow accuracy:
  - [Yan & Sekar ’02], [Liu & Gross ’02], [Baldick ’04], [Overbye, Cheng, & Sun ’04], [Baldick, Dixit & Overbye ’05], [Purchala, Meeus, Van Dommelen & Belmans ’05], [Van Hertem, Verboomen, Purchala, Belmans & Kling ’06], [Li & Bo ’07], [Duthaler, Emery, Andersson, & Kurzidem ’08], [Stott, Jardim & Alsac ‘09], [Qi, Shi & Tylavsky ’12], [Coffrin, Van Hentenryck & Bent ’12]

- Accuracy depends on the application and test case

  “At no stage in the tests were we able to discern any statistical pattern in the dc-flow error scatters. This defeated all our attempts to find concise, meaningful indices with which to characterize and display dc-model accuracies.”  
  [Stott, Jardim & Alsac ‘09]
Problem Formulation
Assessing DC Power Flow Accuracy

- Goal: bound the worst-case error in the active power injections between the DC and AC power flow models

**Formulation**

**DC Power Flow**

\[ P_{\text{flow}_{i,k}}^{\text{DC}} = \sum_{k=1}^{n} (\mathbf{P}_i \cdot \mathbf{DF}) - P_i \]

**AC Power Flow**

\[
\begin{align*}
P_{\text{flow}_{i,k}}^{\text{AC}} & = |V_i||V_k| (G_{ik} \cos(\theta_i - \theta_k) \\
& \quad + B_{ik} \sin(\theta_i - \theta_k) - |V_k|^2 G_{ik} \\
& \quad - B_{ik} \cos(\theta_i - \theta_k)) \\
Q_{i,k} & = |V_i||V_k| (G_{ik} \sin(\theta_i - \theta_k) \\
& \quad - B_{ik} \cos(\theta_i - \theta_k))
\end{align*}
\]

**Future Work!**

**Today's presentation:** Alternate approach:

**Error**

\[ |P_{\text{flow}_{i,k}}^{\text{DC}} - P_{\text{flow}_{i,k}}^{\text{AC}}| \]
Worst-Case Error Formulation

\[
\max_{|V|, \theta} \left| P_{DC} - P_{AC} \right|_{\infty}
\]

s.t. \[
\begin{align*}
|V_{min}^i| & \leq |V_i| \leq |V_{max}^i| \\
\theta_{min}^{ik} & \leq \theta_i - \theta_k \leq \theta_{max}^{ik} \\
P_{min}^i & \leq P_{AC}^i \leq P_{max}^i \\
Q_{min}^i & \leq Q_i \leq Q_{max}^i \\
P_{DC}^i & = \sum_{k=1}^{n} B_{ik} (\theta_k - \theta_i) \\
P_{AC}^i & = |V_i| \sum_{k=1}^{n} |V_k| \left( G_{ik} \cos(\theta_k - \theta_i) + B_{ik} \sin(\theta_k - \theta_i) \right) \\
Q_i & = |V_i| \sum_{k=1}^{n} |V_k| \left( G_{ik} \sin(\theta_k - \theta_i) - B_{ik} \cos(\theta_k - \theta_i) \right)
\end{align*}
\]
Handling the Objective Function

• Maximize the infinity norm by solving \(2n\) optimization problems:

\[
\max_{|V|, \theta} \left\| P^{DC} - P^{AC} \right\|_\infty
\]

\[
\max_{\ell \in \{1, \ldots, n\}} \left\{ \max_{|V|, \theta} \left| P^{DC}_\ell - P^{AC}_\ell \right| \right\}
\]

For each \(\ell = 1, \ldots, n\) and \(\sigma = \{-1, 1\}\), solve (in parallel):

\[
\max_{|V|, \theta} \sigma \cdot \left( P^{DC}_\ell - P^{AC}_\ell \right)
\]

Select the largest absolute value among all the solutions.
Handling the Power Flow Equations via Convex Relaxations

• Formulating the DC power flow requires a representation of the voltage angles: \( P_{i}^{DC} = \sum_{k=1}^{n} B_{ik} (\theta_k - \theta_i) \)

• QC Relaxation to the rescue!  
  \[ \sin (\theta_k - \theta_i) \]  
  \[ \cos (\theta_k - \theta_i) \]

[Coiffin, Hijazi & Van Hentenryck ‘15]
Further Tightening the Relaxation

- Augment the QC relaxation with
  - A Semidefinite Programming Relaxation of the power flow equations in rectangular coordinates [Lavaei & Low ‘12]
  - Lifted Nonlinear Cuts implied by the angle difference and voltage magnitude limits [Coffrin, Hijazi & Van Hentenryck ‘15], [Chen, Atamturk & Oren ‘15]
  - Arctangent Envelopes [Kocuk, Dey & Sun ‘16]

- Apply a bound tightening algorithm to improve upon the specified operational limits [Kocuk, Dey & Sun ‘15], [Chen, Atamturk & Oren ‘15], [Coffrin, Hijazi & Van Hentenryck ‘16]
Semidefinite Relaxation of the Power Flow Equations

- Write power flow equations as $z^H A_i z = c_i$
  where $z = [V_1 \ldots V_n]^T$ with voltage phasors $V \in \mathbb{C}^n$

- Define matrix $W = zz^H$

- Rewrite as $\text{rank} (W) = 1$ and

  
  $\begin{cases} 
  \text{trace} (A_i W) = c_i \\
  W \succeq 0 
  \end{cases}$

- Relaxation:
  - Do not enforce $\text{rank} (W) = 1$ 
  - A solution with $\text{rank} (W) = 1$ implies zero relaxation gap and recovery of the globally optimal voltage profile. This is not necessary for our problem: we only require a lower bound.

Formulation [Lavaei & Low ’12]
**Arctangent Envelopes**

- Enclose the arctangent function using linear inequalities  
  [Kocuk, Dey, & Sun ‘16]
Formulation Summary

For each $\ell = 1, \ldots, n$ and $\sigma = \{-1, 1\}$, solve (in parallel):

$$\max_{|V|, \theta} \sigma \cdot \left( P^{DC}_\ell - P^{AC}_\ell \right)$$

s.t. $|V^\text{min}_i| \leq |V_i| \leq |V^\text{max}_i|$

$$\theta^\text{min}_{ik} \leq \theta_i - \theta_k \leq \theta^\text{max}_{ik}$$

$$P^\text{min}_i \leq P^{AC}_i \leq P^\text{max}_i$$

$$Q^\text{min}_i \leq Q_i \leq Q^\text{max}_i$$

$$P^{DC}_i = \sum_{k=1}^{n} B_{ik} (\theta_k - \theta_i)$$

$$P^{AC}_i = |V| \sum_{k=1}^{n} |V_k| \left( G_{ik} \cos (\theta_k - \theta_i) + B_{ik} \sin (\theta_k - \theta_i) \right)$$

$$Q_i = |V| \sum_{k=1}^{n} |V_k| \left( G_{ik} \sin (\theta_k - \theta_i) - B_{ik} \cos (\theta_k - \theta_i) \right)$$

Maximize the absolute value of the error at each bus

Bound-tightened operational constraints

DC Power Flow

QC Relaxation
+ SDP Relaxation
+ Lifted Nonlinear Cuts
+ Arctangent Envelopes
Results for the IEEE Test Cases
**Results**

Worst-case power injection error (MW) for the IEEE 14-bus system

- Upper bound from the relaxation
- Lower bound on the worst-case error from a local solution
- Range from random injections
- Average from random injections
Results

Worst-case power injection error (MW) for the IEEE 14-bus system, by bus

5% Allowed Variation in Injections

50% Allowed Variation in Injections
Results

IEEE 30-Bus System

- Upper bound from the relaxation
- Lower bound on the worst-case error from a local solution
- Range from random injections
- Average from random injections

IEEE 57-Bus System

- Upper bound from the relaxation
- Lower bound on the worst-case error from a local solution
- Range from random injections
- Average from random injections

IEEE 118-Bus System

- Upper bound from the relaxation
- Lower bound on the worst-case error from a local solution
- Range from random injections
- Average from random injections
Conclusion
Conclusions

• We proposed an algorithm that uses convex relaxations to **bound the worst-case error of the DC power flow**

• Results for several IEEE test cases show:
  – The bound is **reasonably tight**
  – The DC power flow **can have large errors** for some operating conditions

• Next steps:
  – Application to **other linear approximations** and test cases
  – Comparison with **other error bounds**
  – Determination of **physical explanations** for large errors
  – Design of **new linearizations** informed by the worst-case error
Questions?