Multihadron systems

Steve Sharpe
University of Washington

Implementing the three-particle quantization condition: a progress report

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Outline

- Motivations for studying 3 (or more) particles
- Status of theoretical formalism for 2 and 3 particles
- Numerical implementation of 3-particle QC
  - Isotropic approximation
  - Including higher partial waves
  - Isotropic approx. v2: including two-particle bound states
- Conclusions & outlook

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ADVERT: 5 slide talk on application to lattice results for $3\pi^+$ spectrum

3-particle papers: RFT approach

Max Hansen & SRS:

"Relativistic, model-independent, three-particle quantization condition,"

arXiv:1408.5933 (PRD) [HS14]

"Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,"

arXiv:1504.04028 (PRD) [HS15]

"Perturbative results for 2- & 3-particle threshold energies in finite volume,"

arXiv:1509.07929 (PRD) [HSPT15]

"Threshold expansion of the 3-particle quantization condition,"

arXiv:1602.00324 (PRD) [HSTH15]

"Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,"

arXiv: 1609.04317 (PRD) [HSBS16]

"Lattice QCD and three-particle decays of Resonances,"

arXiv: 1901.00483 (to appear in Ann. Rev. Nucl. Part. Science) [HSREV19]

Raúl Briceño, Max Hansen & SRS:

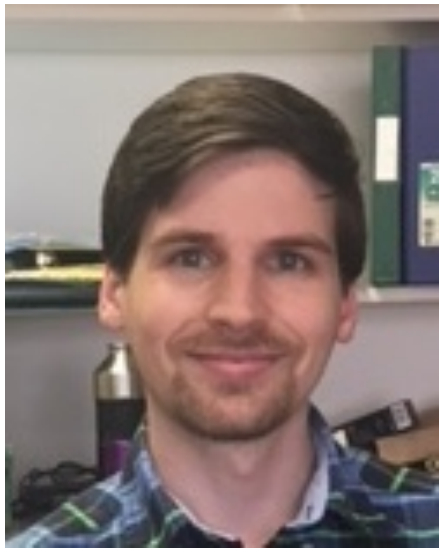
"Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles," arXiv:1701.07465 (PRD) [BHS17]

"Numerical study of the relativistic three-body quantization condition in the isotropic approximation," arXiv:1803.04169 (PRD) [BHS18]

"Three-particle systems with resonant sub-processes in a finite volume," arXiv:1810.01429 (PRD 19) [BHS19]

SRS

"Testing the threshold expansion for three-particle energies at fourth order in φ^4 theory," arXiv:1707.04279 (PRD) [SPT17]



Tyler Blanton, Fernando Romero-López & SRS:

"Implementing the three-particle quantization condition including higher partial waves," arXiv:1901.07095 (JHEP) [BRS19]

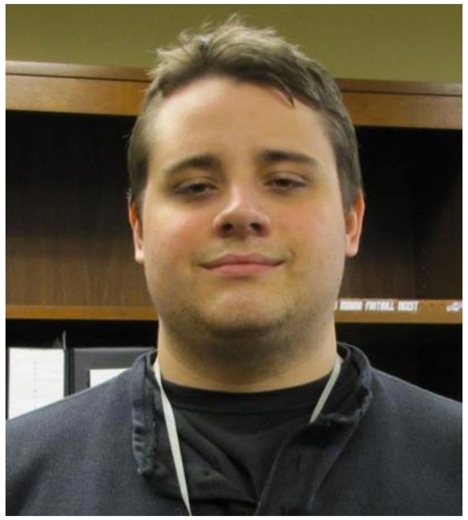
"I=3 three-pion scattering amplitude from lattice QCD," in progress

Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:

"Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states", arXiv:1908.02411 (JHEP to appear)

Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

"Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism," arXiv:1905.11188 (PRD to appear)



Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

"On the Equivalence of Three-Particle Scattering Formalisms," arXiv:1905.12007 (PRD)

Alternate 3-particle approaches

★ **NREFT approach**

- H.-W. Hammer, J.-Y. Pang & A. Rusetsky, 1706.07700, JHEP & 1707.02176, JHEP [Formalism & examples]
- M. Döring et al., 1802.03362, PRD [Numerical implementation]
- J.-Y. Pang et al., 1902.01111, PRD [large volume expansion for excited levels]

★ **Finite-volume unitarity (FVU) approach**

- M. Mai & M. Döring, 1709.08222, EPJA [formalism]
- M. Mai et al., 1706.06118, EPJA [unitary parametrization of $\mathcal{M}_3$ used in FVU approach]
- M. Mai & M. Döring, 1807.04746, PRL [3 pion spectrum at finite-volume from FVU]

★ **HALQCD approach [Sinya Aoki's talk?]**

- T. Doi et al. (HALQCD collab.), 1106.2276, Prog.Theor.Phys. [3 nucleon potentials in NR regime]

Motivations for studying three (or more) particles using LQCD

Studying resonances

Studying resonances

- Most resonances have 3 (or more) particle decay channels
  - $\omega(782, I^G J^{PC} = 0^- 1^{--}) \to 3\pi$ (no subchannel resonances)
  - $a_2(1320, I^G J^{PC} = 1^- 2^{++}) \to \rho\pi \to 3\pi$
  - Roper: $N(1440) \to \Delta\pi \to N\pi\pi$ (branching ratio 25-50%)
  - $X(3872) \to J/\Psi\pi\pi$
  - $Z_c(3900) \to \pi J/\psi, \pi\pi\eta_c, \bar{D}D^*$ (studied by HALQCD)

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  - $Z_c(3900) \to \pi J/\psi, \pi\pi\eta_c, \bar{D}D^*$ (studied by HALQCD)

- N.B. If a resonance has both 2- and 3-particle strong decays, then 2-particle methods fail—channels cannot be separated as they can in experiment

Weak decays

Weak decays

- Calculating weak decay amplitudes/form factors involving 3 particles, e.g. $K \to \pi\pi\pi$

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- N.B. Can study weak $K \to 2\pi$ decays independently of $K \to 3\pi$, since strong interactions do not mix these final states (in isospin-symmetric limit)

A more distant motivation

Observation of CP violation in charm decays

CERN-EP-2019-042
13 March 2019

LHCb collaboration†

Abstract

A search for charge-parity (CP) violation in $D^0 \to K^- K^+$ and $D^0 \to \pi^- \pi^+$ decays is reported, using pp collision data corresponding to an integrated luminosity of $6\,\mathrm{fb}^{-1}$ collected at a center-of-mass energy of $13\,\mathrm{TeV}$ with the LHCb detector. The flavor of the charm meson is inferred from the charge of the pion in $D^*(2010)^+ \to D^0\pi^+$ decays or from the charge of the muon in $\overline{B} \to D^0 \mu^- \bar{\nu}_\mu X$ decays. The difference between the CP asymmetries in $D^0 \to K^- K^+$ and $D^0 \to \pi^- \pi^+$ decays is measured to be $\Delta A_{CP} = [-18.2 \pm 3.2\,(\text{stat.}) \pm 0.9\,(\text{syst.})] \times 10^{-4}$ for π-tagged and $\Delta A_{CP} = [-9 \pm 8\,(\text{stat.}) \pm 5\,(\text{syst.})] \times 10^{-4}$ for μ-tagged D^0 mesons. Combining these with previous LHCb results leads to

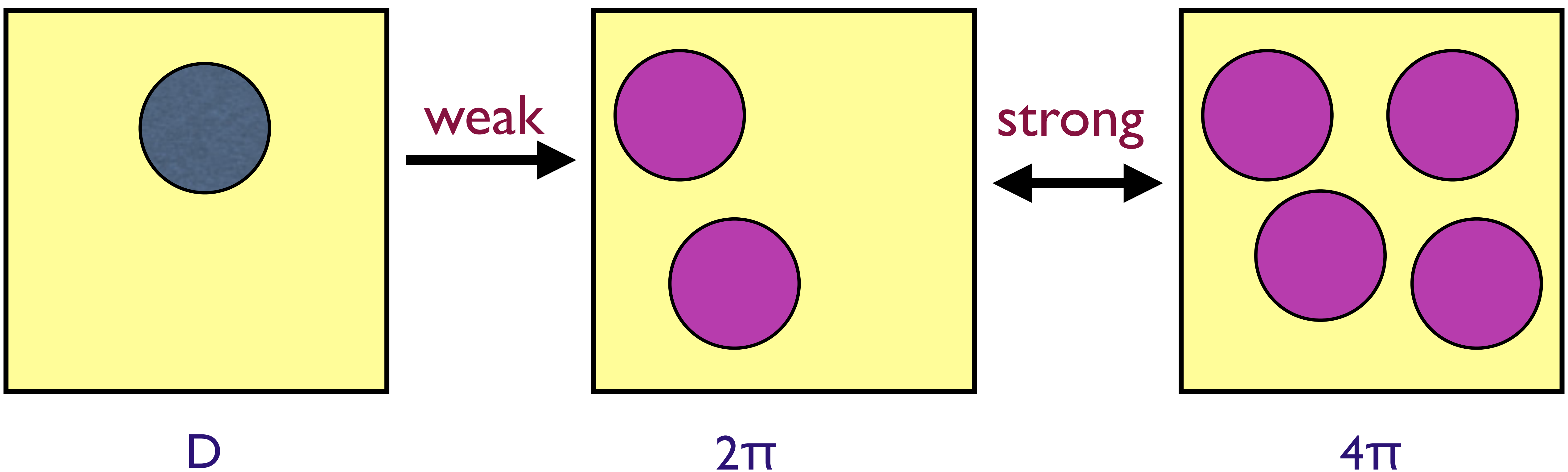
$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4},$$

where the uncertainty includes both statistical and systematic contributions. The measured value differs from zero by more than five standard deviations. This is the first observation of CP violation in the decay of charm hadrons.

5.3σ effect

A more distant motivation

- Calculating CP-violation in $D \to \pi\pi, K\bar{K}$ in the Standard Model
- Finite-volume state is a mix of $2\pi, K\bar{K}, \eta\eta, 4\pi, 6\pi, \ldots$
- Need 4 (or more) particles in the box!



3-body interactions

3-body interactions

- Determining NN & NNN interactions
  - Input for effective field theory treatments of larger nuclei & nuclear matter
  - NNN interaction important for determining properties of neutron stars
- Similarly, $\pi\pi\pi$, $\pi K\bar{K}$, … interactions needed for study of pion/kaon condensation

LQCD spectrum already includes 3+ particle states

Two- and three-pion finite-volume spectra at maximal isospin from lattice QCD

[arXiv:1905.04277]

Ben Hörz*
Nuclear Science Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA

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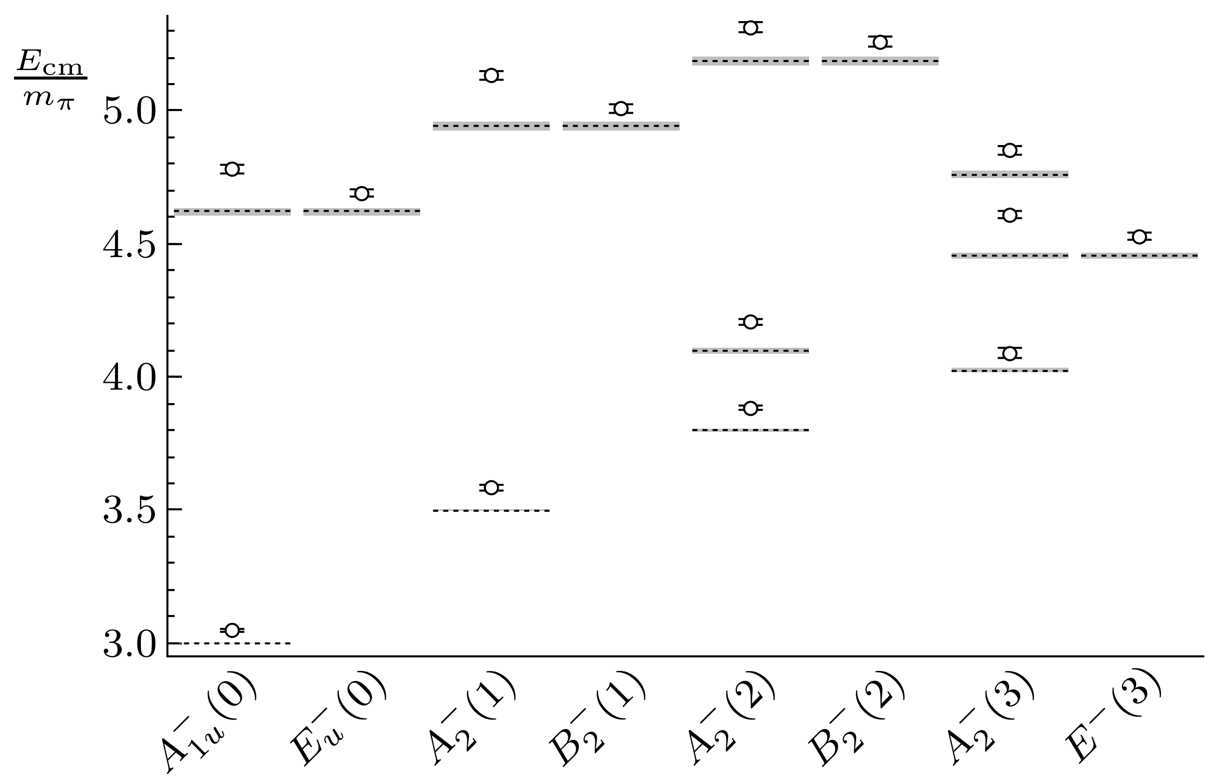
Helmholtz-Institut Mainz, Johannes Gutenberg-Universität, 55099 Mainz, Germany

(Dated: May 13, 2019)

We present the three-pion spectrum with maximum isospin in a finite volume determined from lattice QCD, including, for the first time, excited states across various irreducible representations at zero and nonzero total momentum, in addition to the ground states in these channels. The required correlation functions, from which the spectrum is extracted, are computed using a newly implemented algorithm which reduces the number of operations, and hence speeds up the computation by more than an order of magnitude. The results for the $I = 3$ three-pion and the $I = 2$ two-pion spectrum are publicly available, including all correlations, and can be used to test the available three-particle finite-volume approaches to extracting three-pion interactions.



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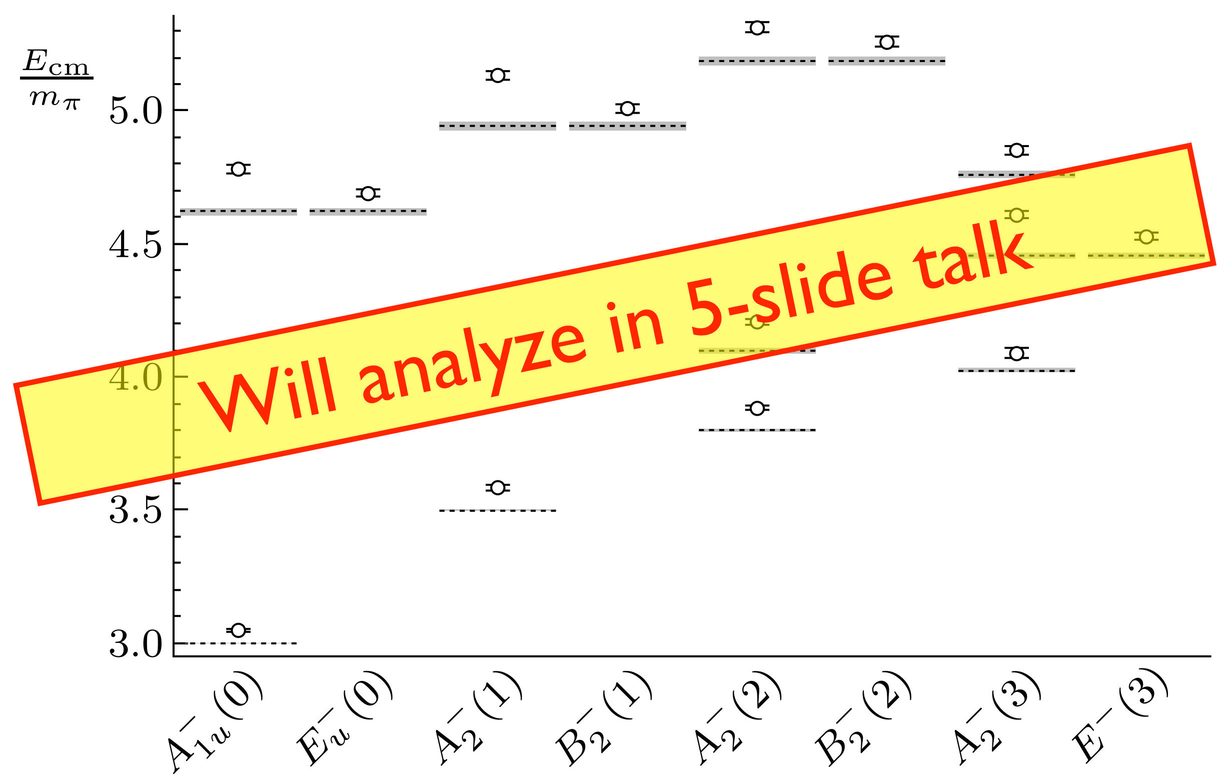
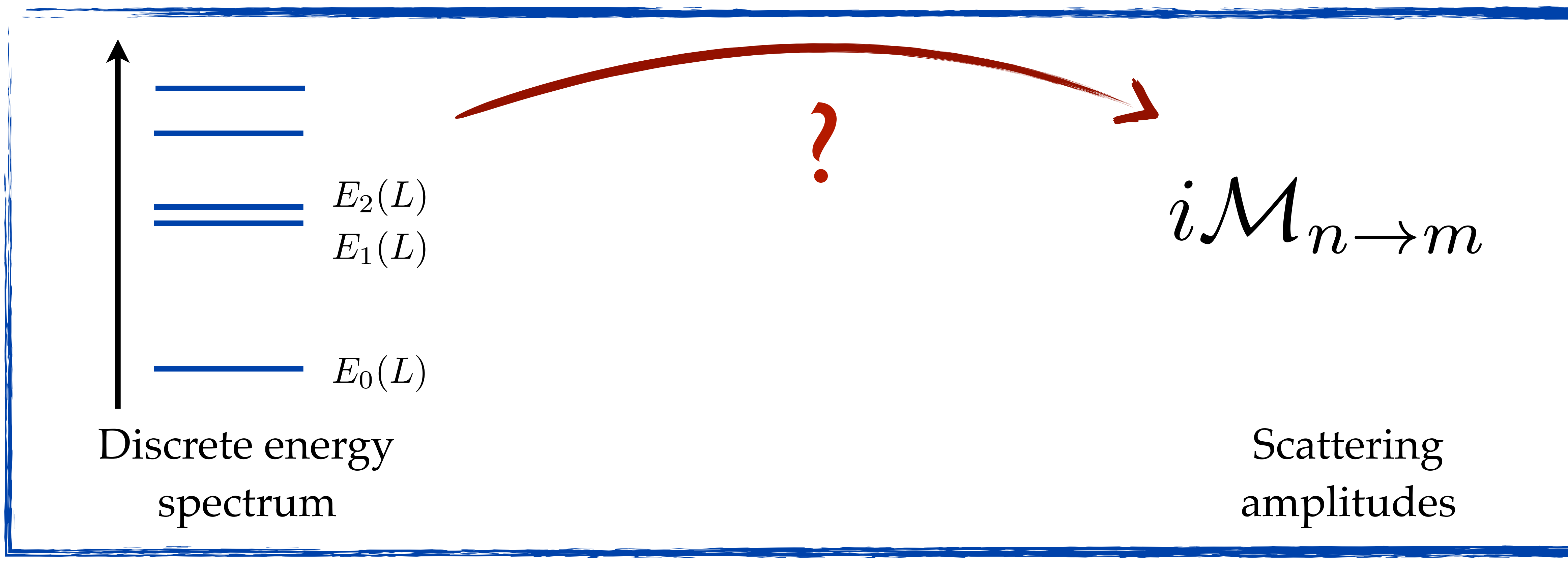


Status of theoretical formalism for 2 & 3 particles

The fundamental issue

- Lattice QCD can calculate energy levels of multiparticle systems in a box
- How are these related to infinite-volume scattering amplitudes (which determine resonance properties)?



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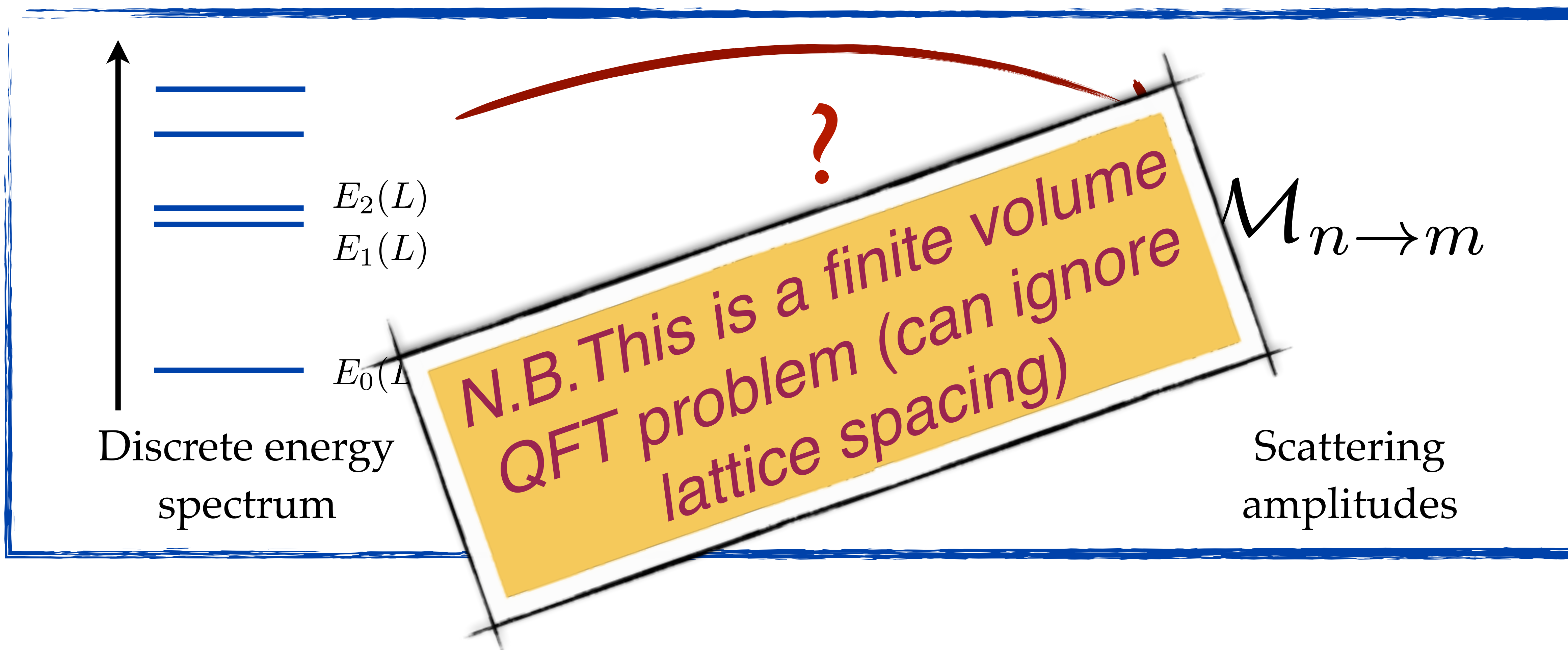
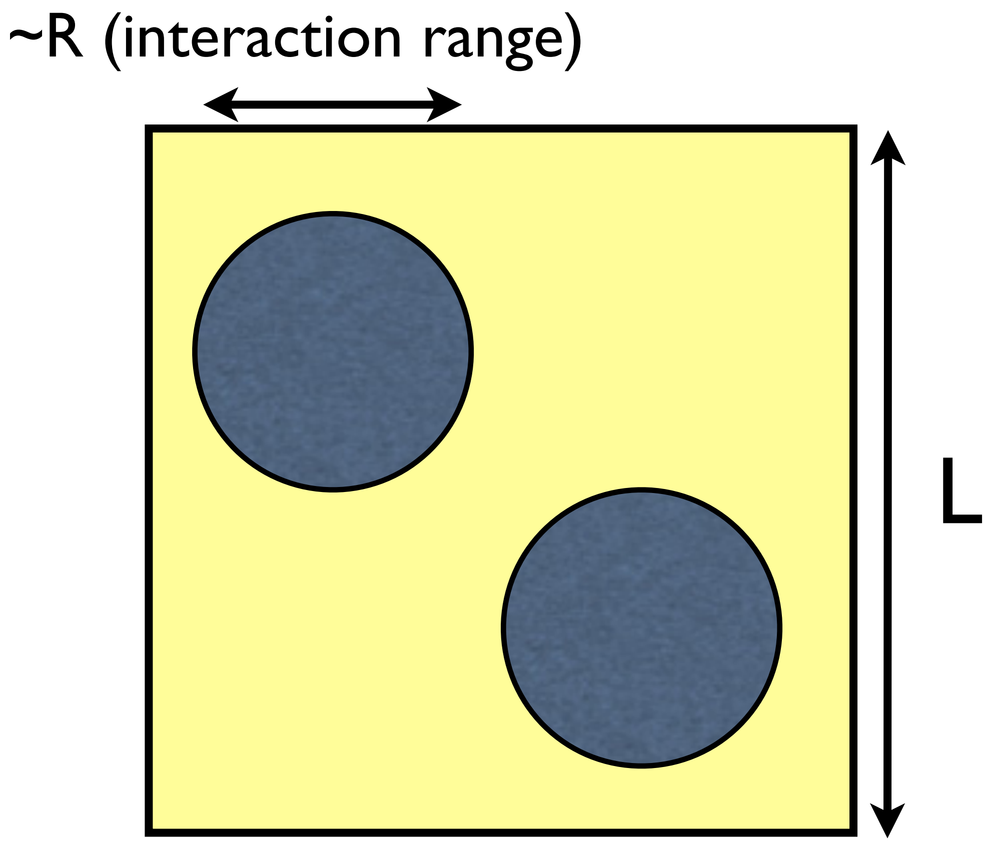
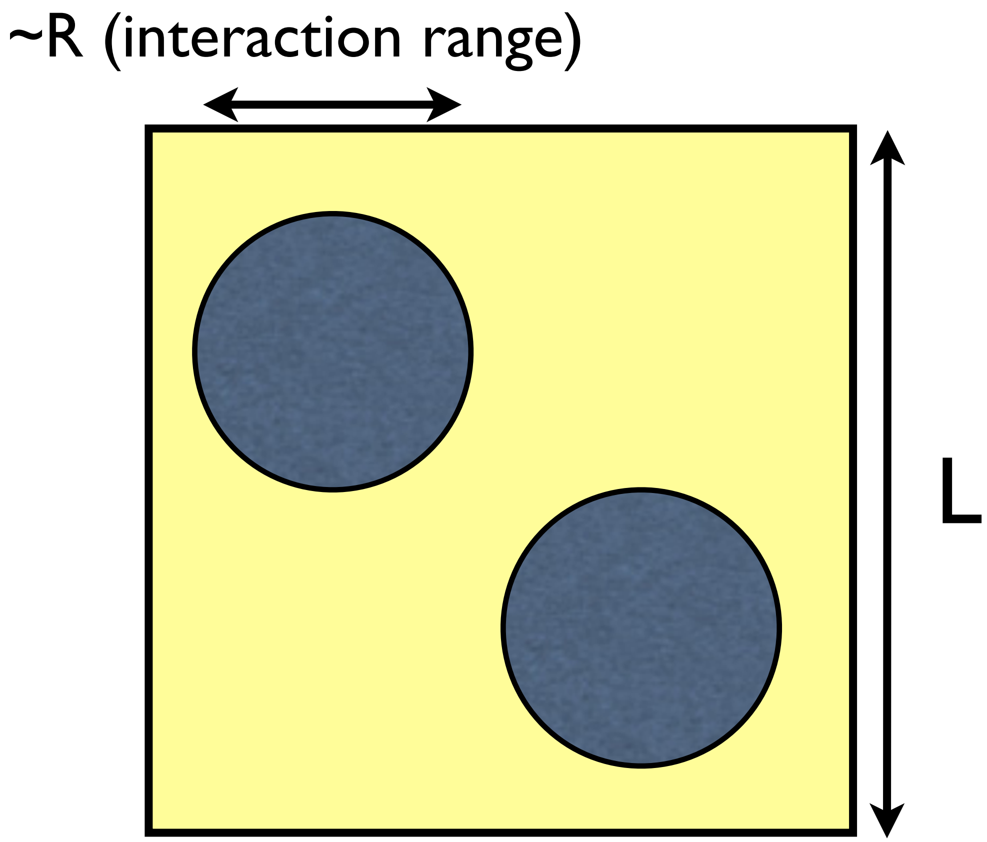


Foundational results



- If $L > 2R$, spectrum related to $2\rightarrow 2$ scattering amplitude up to corrections $\sim e^{-M_\pi L}$ (arising from tail of interaction) [Lüscher, 86,91]

Foundational results



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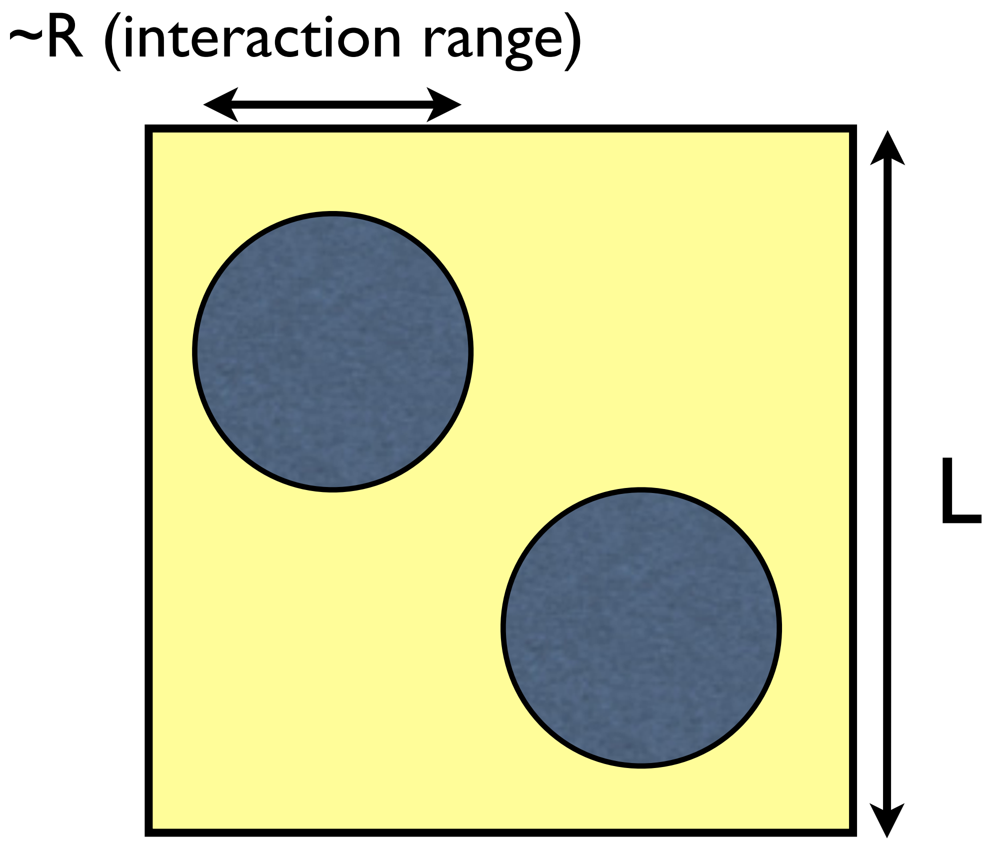
We ignore such exponentially-suppressed corrections throughout:
If $M_\pi L$=4 / 5 / 6, $\exp(-M_\pi L)$~2 / 0.7 / 0.2%

Foundational results



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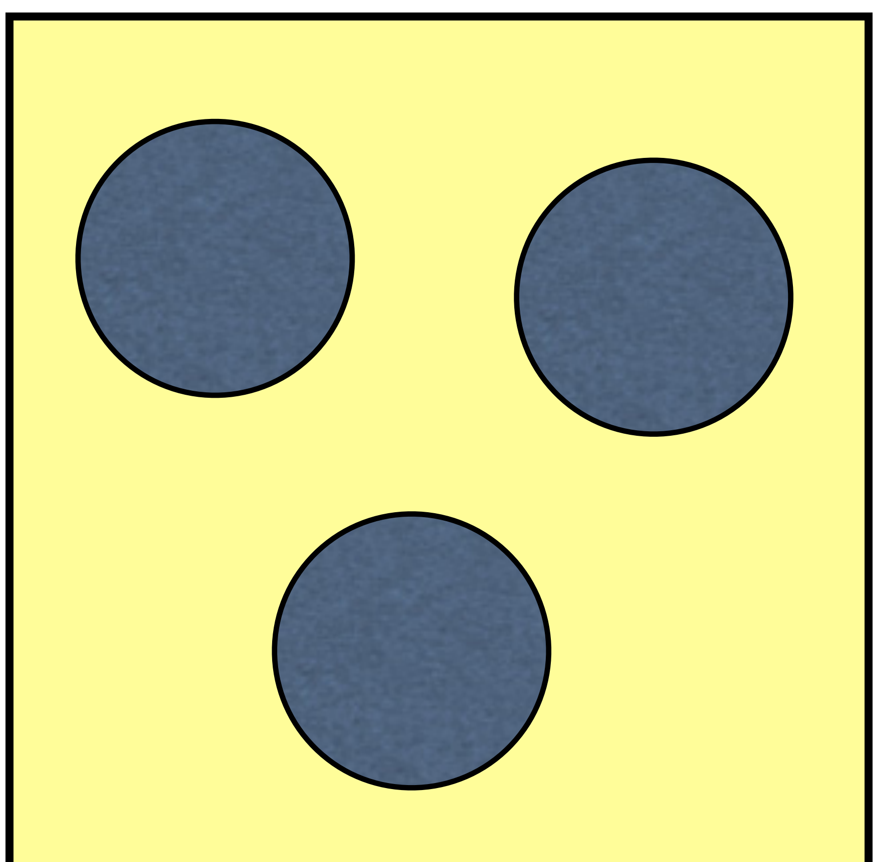

- Spectrum is related to 2→2, 2→3 & 3→3 scattering amplitudes up to corrections $\sim e^{-M_\pi L}$ [Polejaeva & Rusetsky, 12]

Single-channel 2-particle quantization condition

[Lüscher 86 & 91; Rummukainen & Gottlieb 85; Kim, Sachrajda & SRS 05; …]

- Two particles (say pions) in cubic box of size L with PBC and total momentum **P**
- Below inelastic threshold (4 pions if have Z_2 symmetry), the finite-volume spectrum $E_1, E_2, \ldots$ is given by solutions to a equation in partial-wave (*l,m)* space (up to exponentially suppressed corrections)

$$\boxed{\det\left[F_{PV}(E,\overrightarrow{P},L)^{-1} + \mathscr{K}_2(E^*)\right] = 0}$$

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$$\boxed{\det\left[F_{PV}(E,\vec{P},L)^{-1}+\mathcal{K}_2(E^*)\right]=0}$$

- $\mathcal{K}_2 \sim \tan\delta/q$ is the K-matrix, which is diagonal in *l,m*
- F_{PV} is a known kinematical "zeta-function" depending on the box parameters; it is off-diagonal in l,m, since the box violates rotation symmetry
- Beware when reading the literature, as each collaboration uses different notation for what I call F: sometimes B (box function), sometimes M

Single-channel 2-particle quantization condition

$$\boxed{\det\left[F_{PV}(E,\overrightarrow{P},L)^{-1}+\mathscr{K}_2(E^*)\right]=0}$$

- Infinite-dimensional determinant must be truncated to be practical; truncate by assuming that $\mathcal{K}_2$ vanishes above l_{max}
- If $l_{max}=0$, obtain one-to-one relation between energy levels and $\mathcal{K}_2$

$$\boxed{\mathscr{K}_2^{(\ell=0)}(E_n^*)=-\frac{1}{F_{PV;00;00}(E_n,\overrightarrow{P},L)}}$$

$E_n^*=\sqrt{E_n^2-\vec{P}^2}$

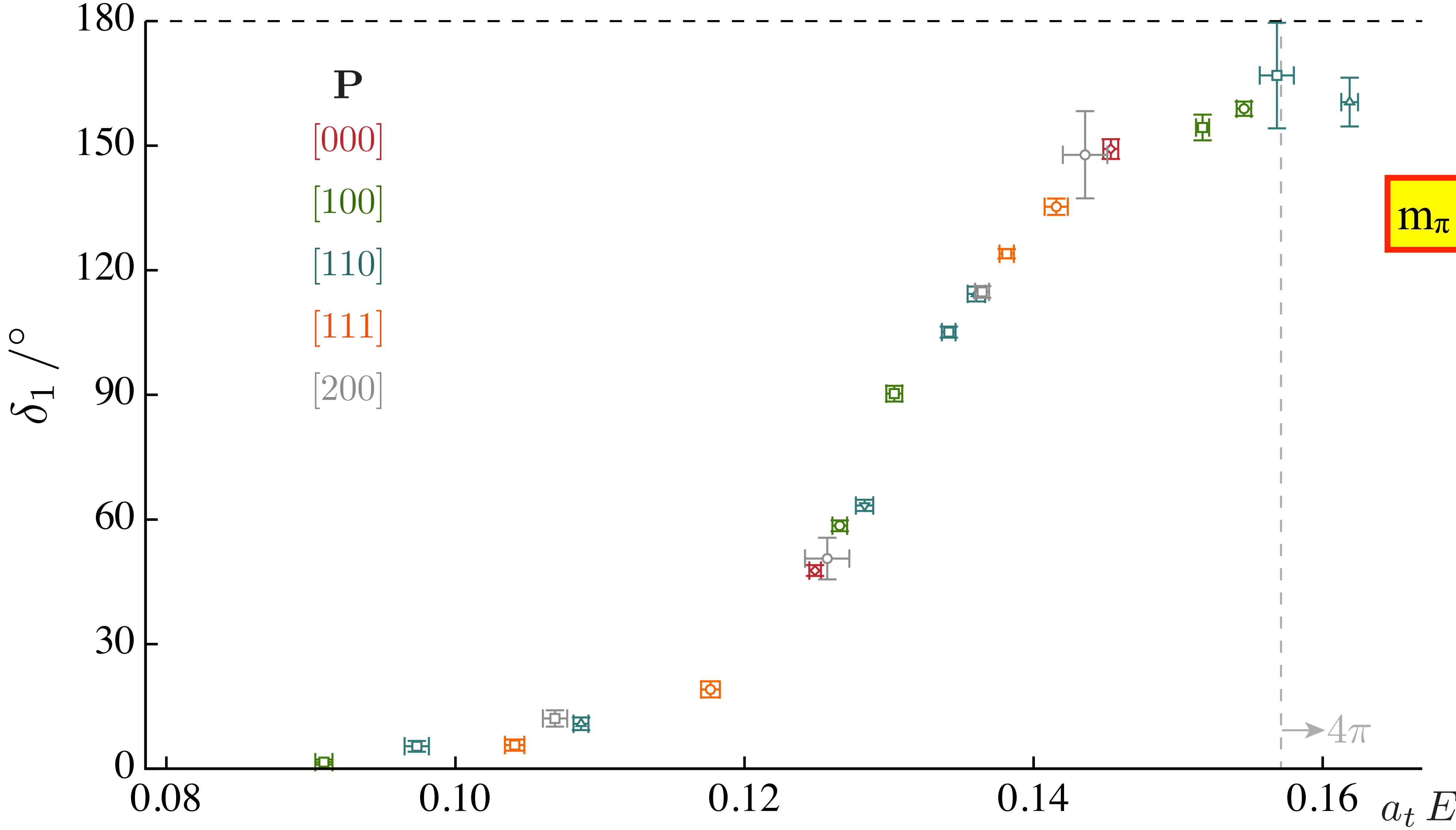
CM energy

"measured" energy-level

ρ resonance from LQCD

- Most results to date assume $l_{max}=1$ and work with unphysical quark masses

[Wilson, Briceño, Dudek, Edwards & Thomas, 1507.02599]



Generalizations

- Multiple two-particle channels [Hu, Feng & Liu, hep-lat/0504019; Lage, Meissner & Rusetsky, 0905.0069; Hansen & SS, 1204.0826; Briceño & Davoudi, 1204.1110]
    - e.g. $J^{PC} = 0^{++} \quad \pi\pi + K\bar{K}\,(+\eta\eta)$

$$\boxed{\det\left[\begin{pmatrix} F_{PV}^{\pi\pi}(E,\vec{P},L)^{-1} & 0 \\ 0 & F_{PV}^{K\bar{K}}(E,\vec{P},L)^{-1} \end{pmatrix} + \begin{pmatrix} \mathscr{K}_2^{\pi\pi}(E^*) & \mathscr{K}_2^{\pi K}(E^*) \\ \mathscr{K}_2^{\pi K}(E^*) & \mathscr{K}_2^{KK}(E^*) \end{pmatrix}\right] = 0}$$

Generalizations

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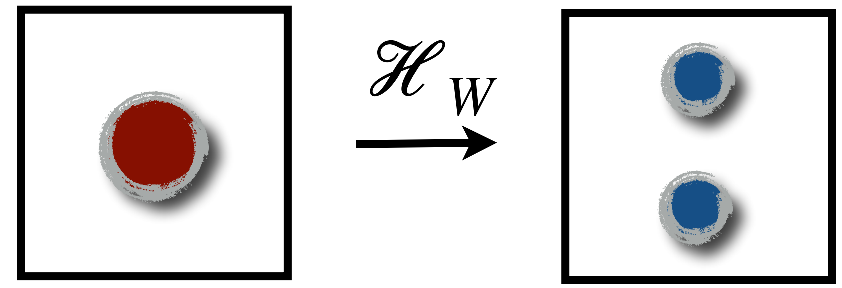
- Even if truncate to $l_{\max}=0$, there is no longer a one-to-one relation between energy levels and K-matrix elements
- Must parametrize the (enlarged) K matrix in some way and fit parameters to multiple spectral levels
- Using these parametrizations can study pole structure of scattering amplitude
- Approach is very similar to that used analyzing scattering data

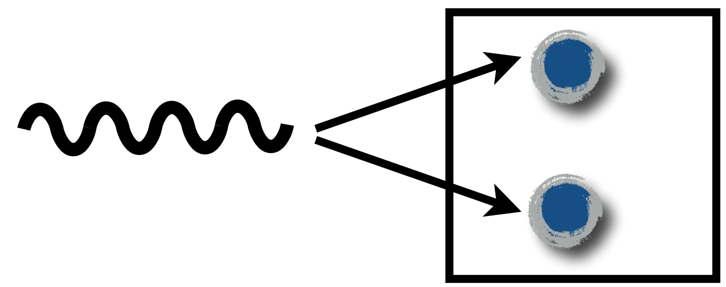
Generalizations

$\langle \pi\pi, \mathrm{out}|\mathcal{H}|K\rangle$ from



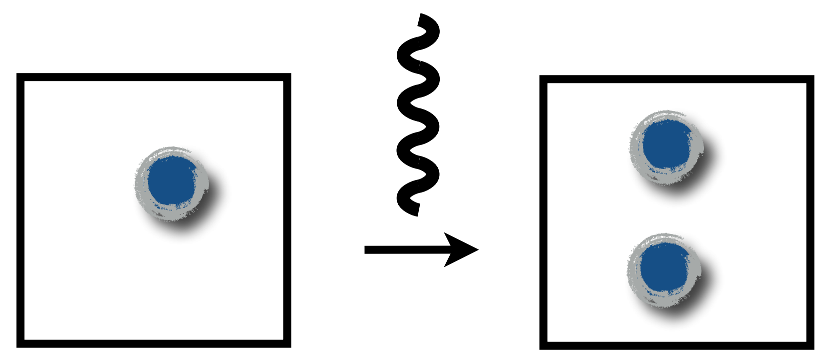
[Lellouch & Lüscher]

$\langle \pi\pi,\ \mathrm{out}|\mathcal{J}_\mu|0\rangle$ from



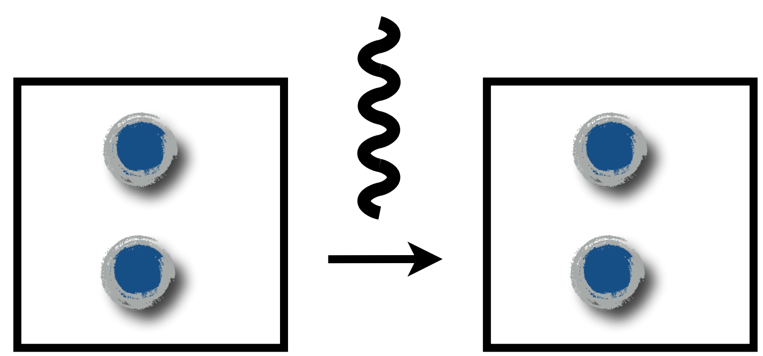
[Meyer, 2011]

$\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle$ from



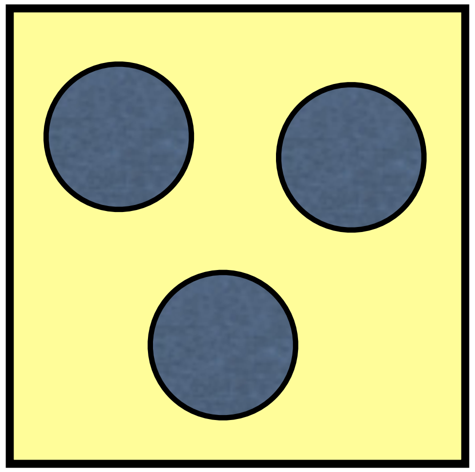
[Agadjanov *et al.*, 2014; Briceño, Hansen & Walker-Loud, 2015; Briceño & Hansen, 2016]

$\langle \pi\pi, \mathrm{out}\,|\,\mathscr{J}_\mu\,|\,\pi\pi, \mathrm{in}\rangle$ from

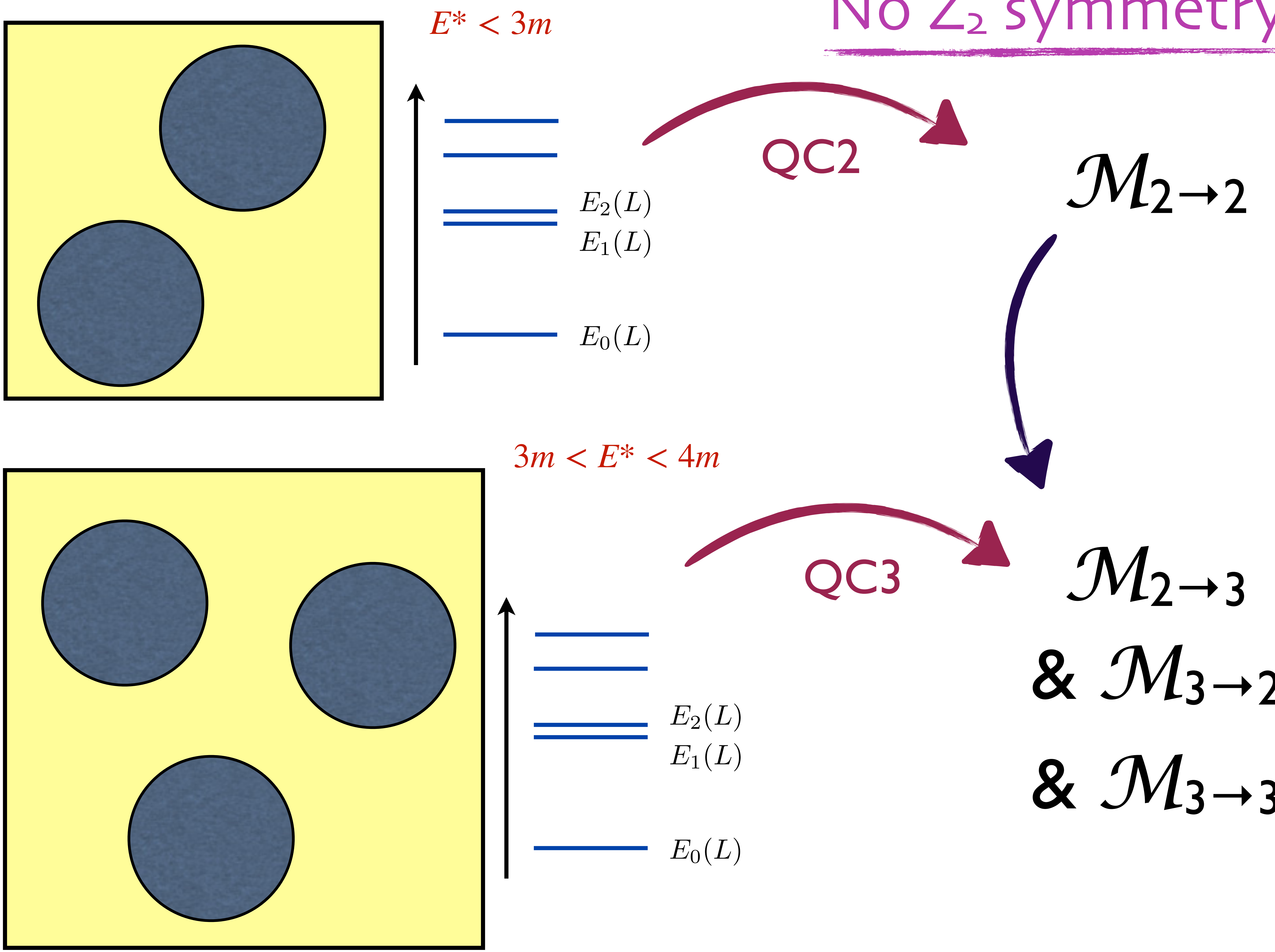


[Briceño, Hansen, 2015; Baroni, Briceño, Hansen & Ortega-Gama, 2018; talk by Alessandro Baroni]

3-particle quantization condition (QC3)



No Z_2 symmetry



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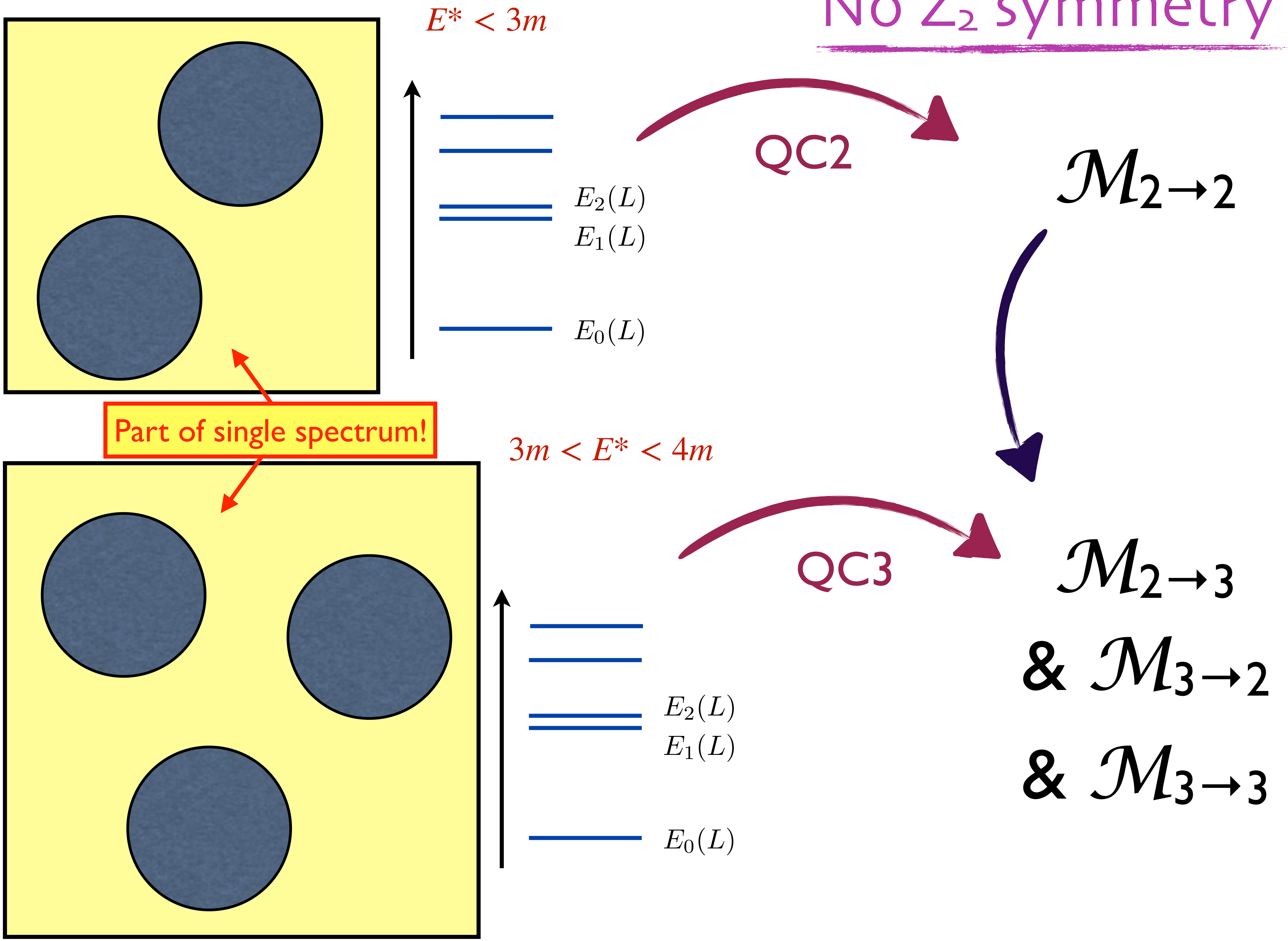
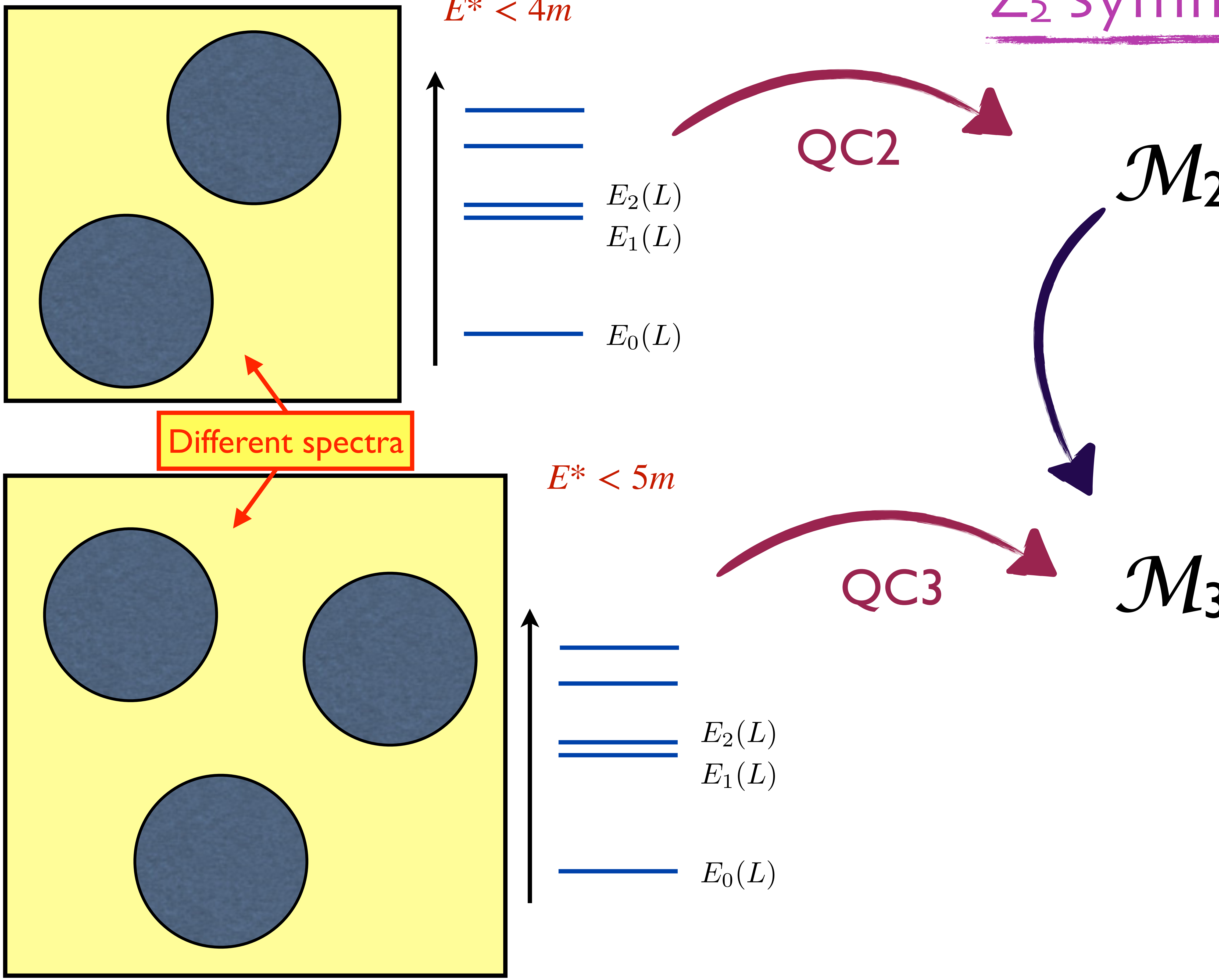
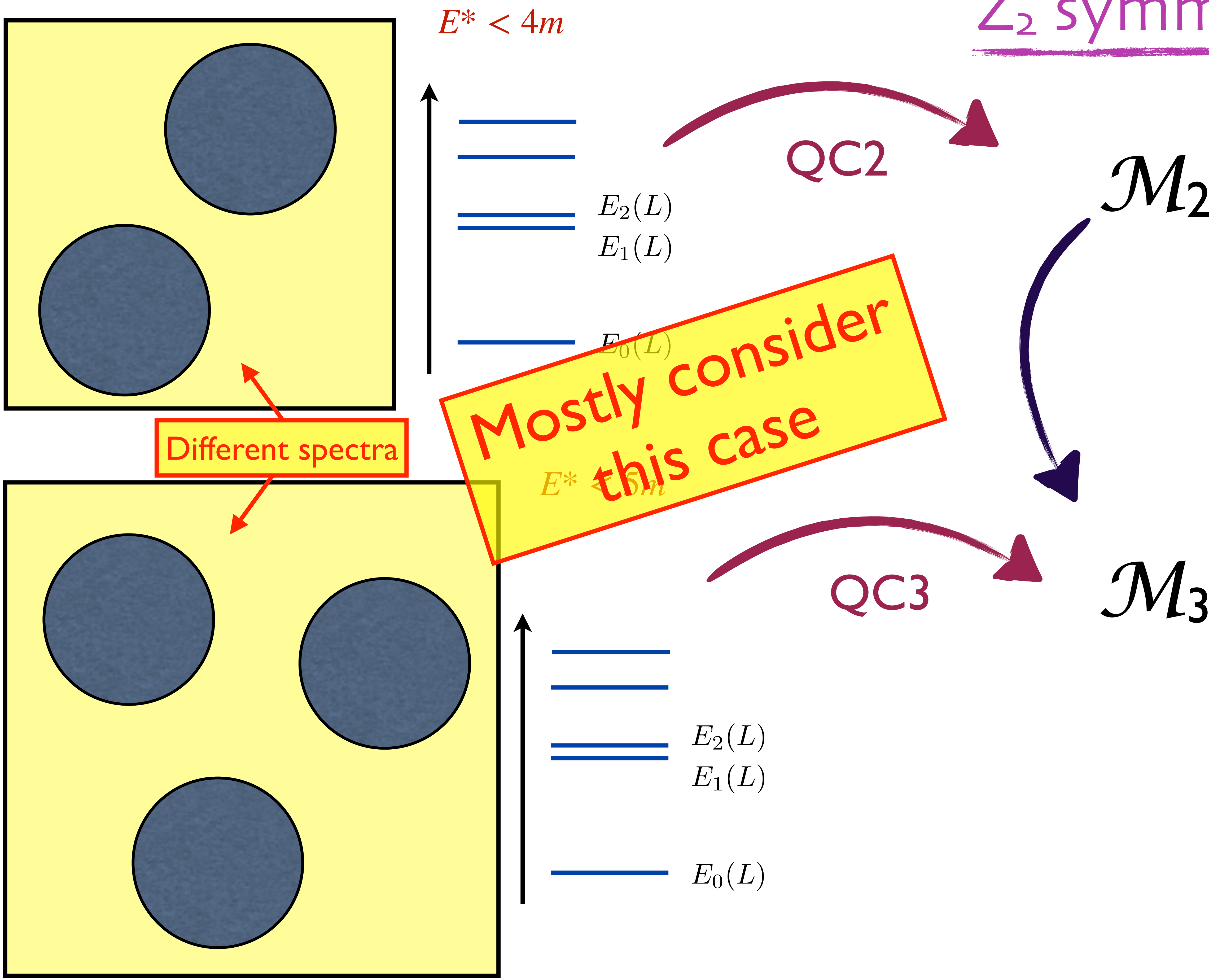


Z_2 symmetry



$E^* < 4m$
$E_2(L)$
$E_1(L)$
$E_0(L)$
QC2
$\mathcal{M}_2$
Different spectra
$E^* < 5m$
QC3
$\mathcal{M}_3$
$E_2(L)$
$E_1(L)$
$E_0(L)$

Z_2 symmetry



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Different spectra
Mostly consider this case
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QC3
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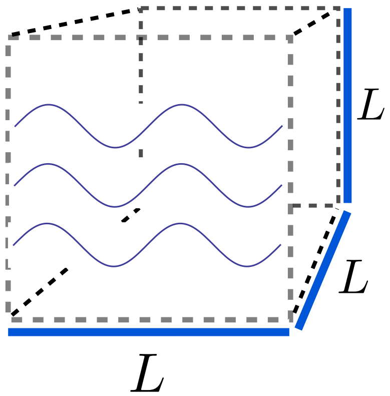
Two-step method

2 & 3 particle
spectrum from LQCD



Quantization conditions

$$\det\left[F_2^{-1} + \mathscr{K}_2\right] = 0$$

$$\det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$$

Intermediate, unphysical scattering quantity

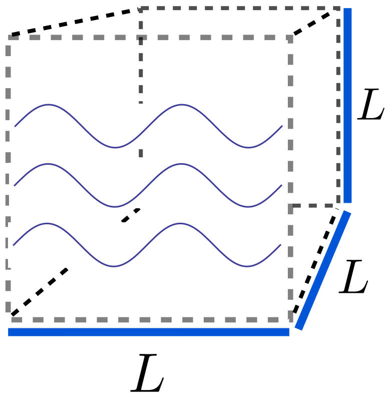
Integral equations in infinite volume

Scattering amplitudes

$$\mathscr{M}_{22}\,,\quad \mathscr{M}_{23}\,,\quad \mathscr{M}_{32}\,,\quad \mathscr{M}_{23}$$

Two-step method

2 & 3 particle spectrum from LQCD



Quantization conditions

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...ediate, unphysical
...ering quantity

Integral equ...
infinite vo...

Need for two steps common to all approaches (though intermediate quantities differ)

Scattering amplitudes

$$\mathcal{M}_{22}\,,\quad \mathcal{M}_{23}\,,\quad \mathcal{M}_{32}\,,\quad \mathcal{M}_{23}$$

QC2

$$\boxed{\det\left[F_{\mathrm{PV}}(E,\vec{P},L)^{-1}+\mathscr{K}_2(E^*)\right]=0}$$

- Total momentum (E, **P**)
- Matrix indices are *l, m*
- F_{PV} is a finite-volume geometric function
- $\mathcal{K}_2$ is a physical infinite-volume amplitude, which is real and has no threshold cusps
- $\mathcal{K}_2$ is algebraically related to $\mathcal{M}_2$

$$\frac{1}{\mathscr{M}_2^{(\ell)}}\equiv\frac{1}{\mathscr{K}_2^{(\ell)}}-i\rho$$

QC2 $\longrightarrow$ QC3 [HS14]

$$\det\left[F_{\rm PV}(E,\vec{P},L)^{-1}+\mathcal{K}_2(E^*)\right]=0 \quad \longrightarrow \quad \det\left[F_3(E,\vec{P},L)^{-1}+\mathcal{K}_{\rm df,3}(E^*)\right]=0$$

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- Matrix indices are *l, m*
- $F_{\rm PV}$ is a finite-volume geometric function
- $\mathcal{K}_2$ is a physical infinite-volume amplitude, which is real and has no threshold cusps
- $\mathcal{K}_2$ is algebraically related to $\mathcal{M}_2$

$$\frac{1}{\mathcal{M}_2^{(\ell)}} \equiv \frac{1}{\mathcal{K}_2^{(\ell)}} - i\rho$$

- Total momentum (E, **P**)
- Matrix indices are ***k***, *l, m*
- F_3 depends on geometric functions ($F_{\rm PV}$ and G) and also on $\mathcal{K}_2$
  - F_3 is known if first solve QC2
- $\mathcal{K}_{\rm df,3}$ is an infinite-volume 3-particle amplitude, which is real and has no threshold cusps
- It is cutoff dependent and thus unphysical
- It is related to $\mathcal{M}_3$ via integral equations [HS15]

Matrix indices

- All quantities are infinite-dimensional matrices with indices describing 3 on-shell particles

[finite volume "spectator" momentum: **k**=2π**n**/L] x [2-particle CM angular momentum: *l,m*]

Describes three on-shell particles with total energy-momentum (E, **P**)

- For large spectator-momentum **k**, the other two particles are below threshold; must include such configurations, by analytic continuation, up to a cut-off at k~m [Polejaeva & Rusetsky, `12]

F_3 collects 2-particle interactions

$$F_3 = \frac{1}{2\omega L^3}\left[\frac{F}{3} - F\frac{1}{\mathscr{K}_2^{-1} + F + G}F\right]$$

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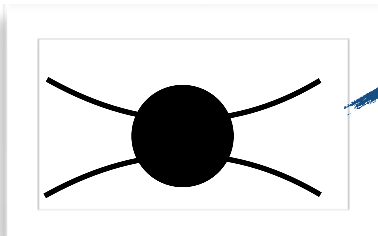

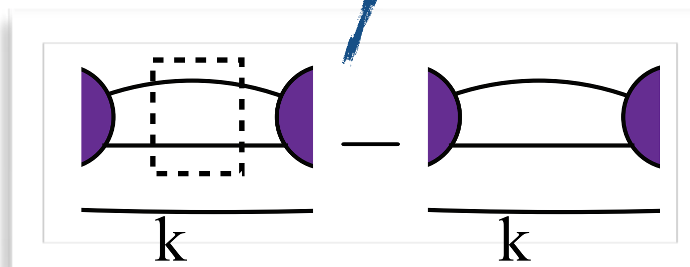

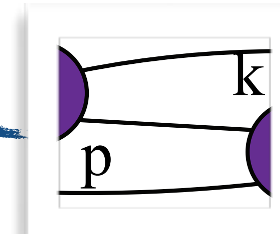

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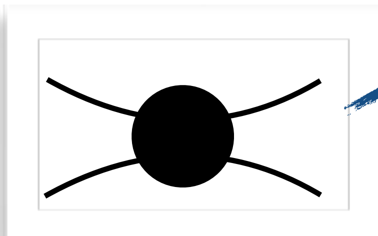
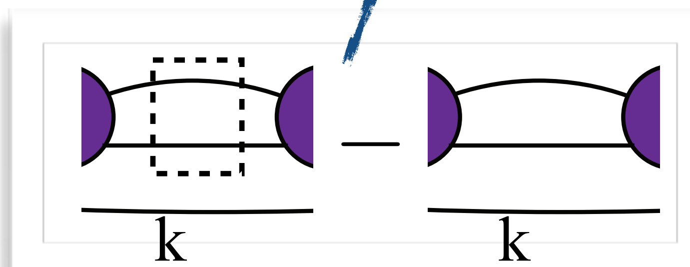
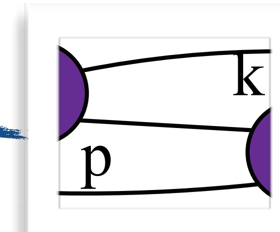
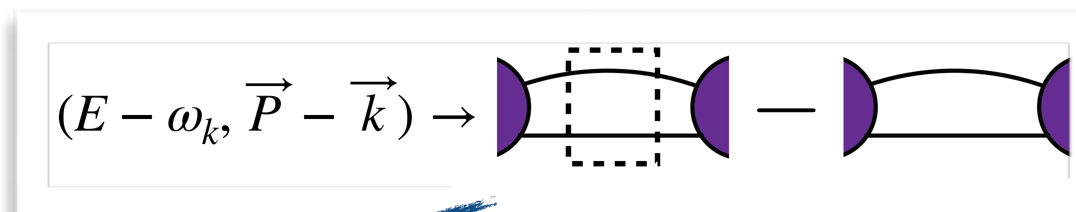






- F & G are known geometrical functions, containing cutoff function H



$$F_{p\ell'm';k\ell m} = \delta_{pk}\, H(\vec{k})\, F_{\mathrm{PV},\ell'm';\ell m}(E-\omega_k, \vec{P}-\vec{k}, L)$$

$$G_{p\ell'm';k\ell m} = \left(\frac{k^*}{q_p^*}\right)^{\ell'} \frac{4\pi Y_{\ell'm'}(\hat{k}^*)H(\vec{p})H(\vec{k})Y^*_{\ell m}(\hat{p}^*)}{(P-k-p)^2 - m^2}\left(\frac{p^*}{q_k^*}\right)^{\ell}\frac{1}{2\omega_k L^3}$$

Relativistic form introduced in [BHS17]

Divergence-free K matrix

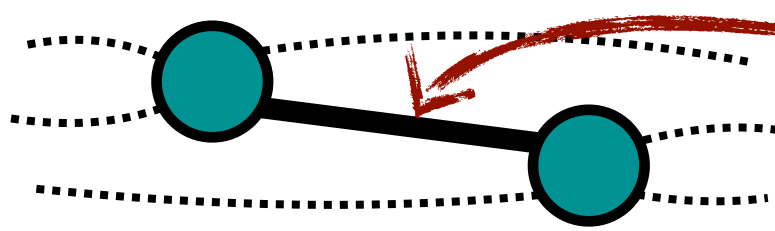
$$\det\left[F_3(E,\overrightarrow{P},L)^{-1}+\mathscr{K}_{\mathrm{df},3}(E^*)\right]=0$$

Divergence-free K matrix

$$\det\left[F_3(E,\vec{P},L)^{-1} + \mathcal{K}_{\mathrm{df},3}(E^*)\right] = 0$$

Three-to-three amplitude has kinematic singularities

$i\mathcal{M}_{3\to3} \equiv$ **fully connected correlator with six external legs amputated and projected on shell**

$=$  $+\cdots$

Certain external momenta put this on-shell!

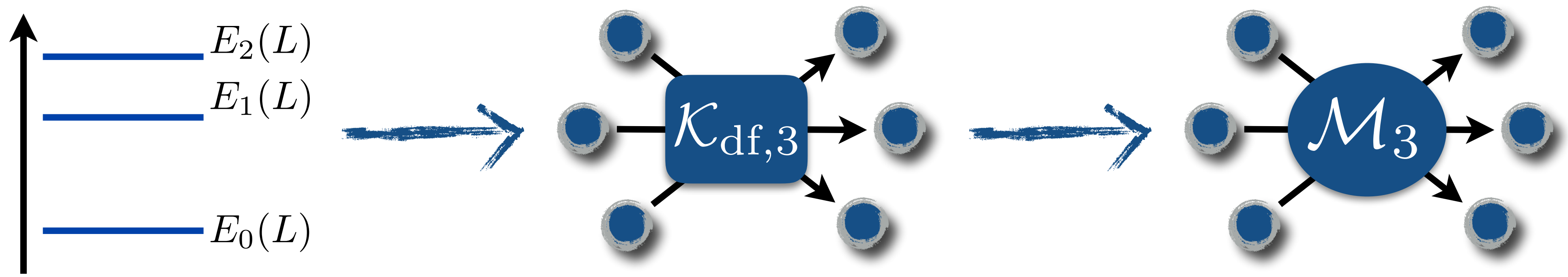
[Artwork from Hansen, HMI lectures]

- To have a nonsingular (divergence-free) quantity, need to subtract pole & higher order singularities, leading to $\mathcal{M}_{\mathrm{df},3}$, which is finite but cutoff dependent
- Replace iε with PV prescription obtain K matrix, $\mathcal{K}_{\mathrm{df},3}$, that is real, has no unitary cusps, and is like a quasilocal interaction; it is also cutoff dependent
- $\mathcal{K}_{\mathrm{df},3}$ has the same symmetries as $\mathcal{M}_3$: relativistic invariance, particle interchange, T-reversal

Status of relativistic approach

- Original work applied to scalars with G-parity & no subchannel resonances or dimers [Hansen & SRS, arXiv:1408.5933 & 1504.04248]

$$\det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$$



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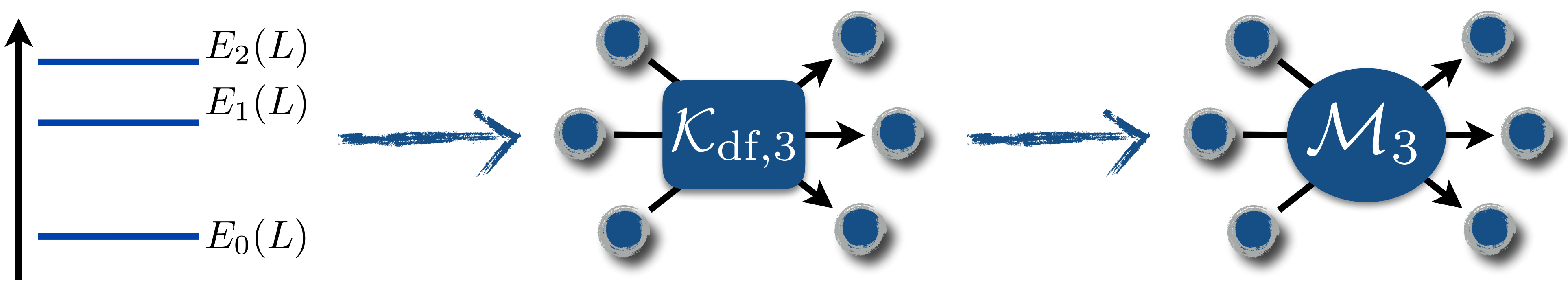


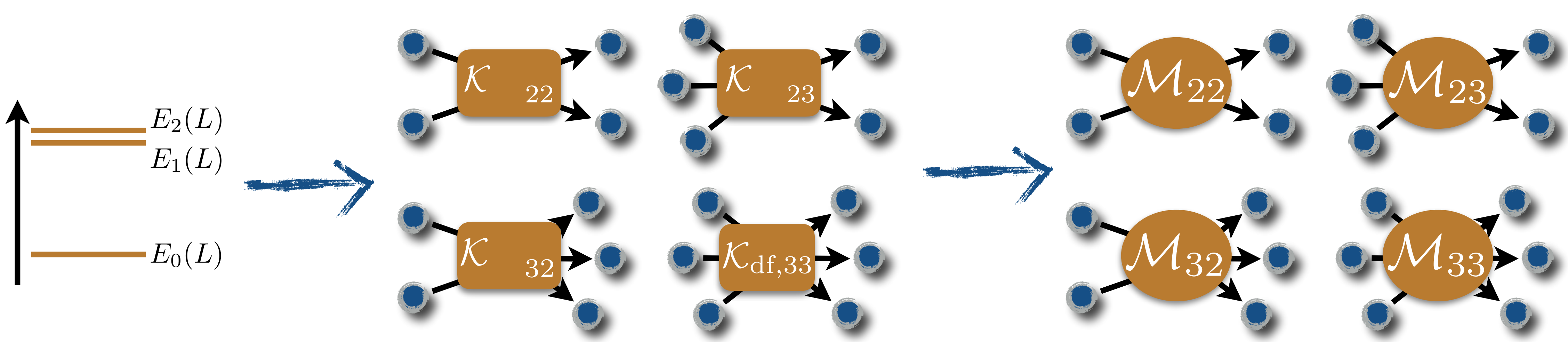
UPDATE: subchannel resonances and dimers are allowed by using a modified PV pole-prescription [BBHRS19]

Status of relativistic approach

- Second major step: removing G-parity constraint, allowing 2↔3 processes [Briceño, Hansen & SRS, arXiv:1701.07465]

F_2 appears in 2-particle quantization condition

$$\det\left[\begin{pmatrix} F_2 & 0 \\ 0 & F_3 \end{pmatrix}^{-1} + \begin{pmatrix} \mathcal{K}_{22} & \mathcal{K}_{23} \\ \mathcal{K}_{32} & \mathcal{K}_{\mathrm{df},33} \end{pmatrix}\right] = 0$$



Status of relativistic approach

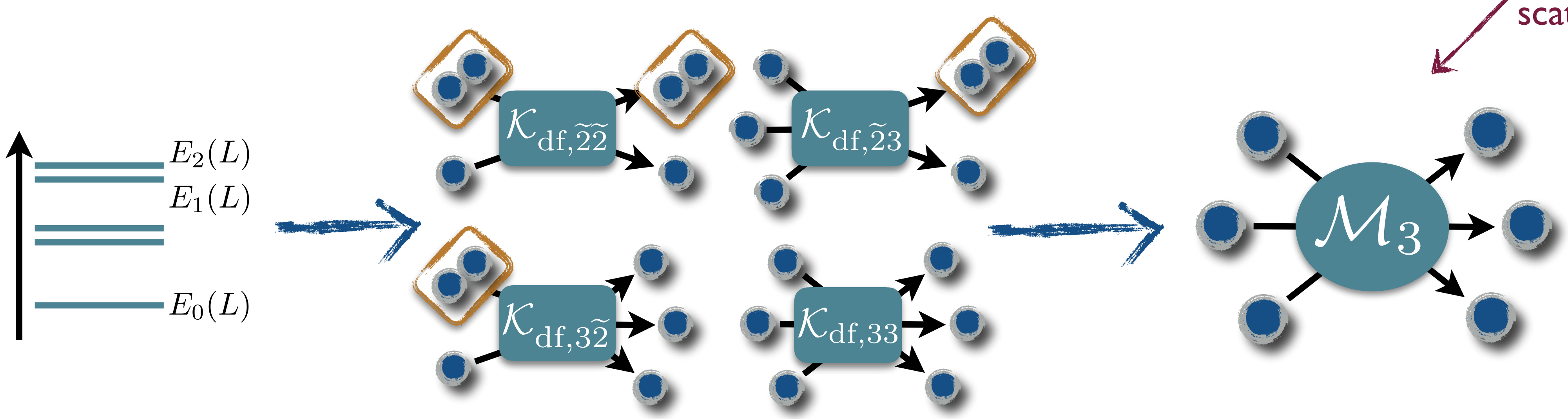
- Final major step: allowing subchannel resonance (i.e. pole in $\mathcal{K}_2$)
  [Briceño, Hansen & SRS, arXiv:1810.01429]

Determined by K2 & Lüscher finite-volume zeta functions

resonance + particle channel (not physical, but forced on us by derivation)

$$\det\left[\begin{pmatrix} F_{\tilde{2}\tilde{2}} & F_{\tilde{2}3} \\ F_{3\tilde{2}} & F_{33} \end{pmatrix}^{-1} + \begin{pmatrix} \mathcal{K}_{\mathrm{df},\tilde{2}\tilde{2}} & \mathcal{K}_{\mathrm{df},\tilde{2}3} \\ \mathcal{K}_{\mathrm{df},3\tilde{2}} & \mathcal{K}_{\mathrm{df},33} \end{pmatrix}\right] = 0$$

No unphysical channel in final scattering amplitude



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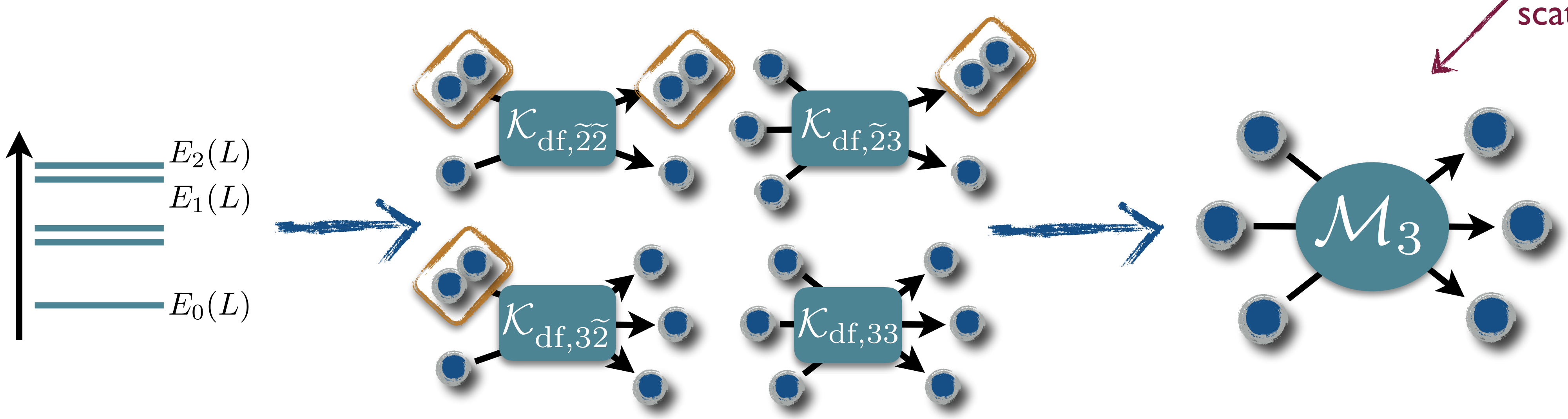

UPDATE: this elaboration is avoidable

Implementation of QC3

Status

- Formalism of [HS14, HS15] (Z_2 symmetry) has been implemented numerically in three approximations:

1. Isotropic, s-wave low-energy approximation, with no dimers [BHS18]

2. Including d waves in $\mathcal{K}_2$ and $\mathcal{K}_{\mathrm{df},3}$, with no dimers [BRS19]

3. Both 1 & 2 with dimers and two-particle resonances (using modified PV prescription) [BBHRS19]

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I will show results from these calculations

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  - Corresponds to first approximation above

  - Ease of implementation comparable in the three approaches

Truncation

$$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$$

matrices with indices:

[finite volume "spectator" momentum: **k**=2π**n**/L] x [2-particle CM angular momentum: *l,m*]

- To use quantization condition, one must truncate matrix space, as for the two-particle case
- Spectator-momentum space is truncated by cut-off function H(**k**)
- Need to truncate sums over l,m in $\mathcal{K}_2$ & $\mathcal{K}_{\mathrm{df},3}$

Truncating sum over l

- In 2-particle case, we know that s-wave scattering dominates at low energies; can then systematically add in higher waves (suppressed by q^{2l})
- Implement using the effective-range expansion (ERE) for partial waves of $\mathcal{K}_2$

s wave: $$\frac{1}{\mathcal{K}_2^{(0)}} = \frac{1}{16\pi E_2}\left[-\frac{1}{a_0} + r_0\frac{q^2}{2} + P_0 r_0^3 q^4 + \ldots\right],$$

$E_2 = \sqrt{s}$

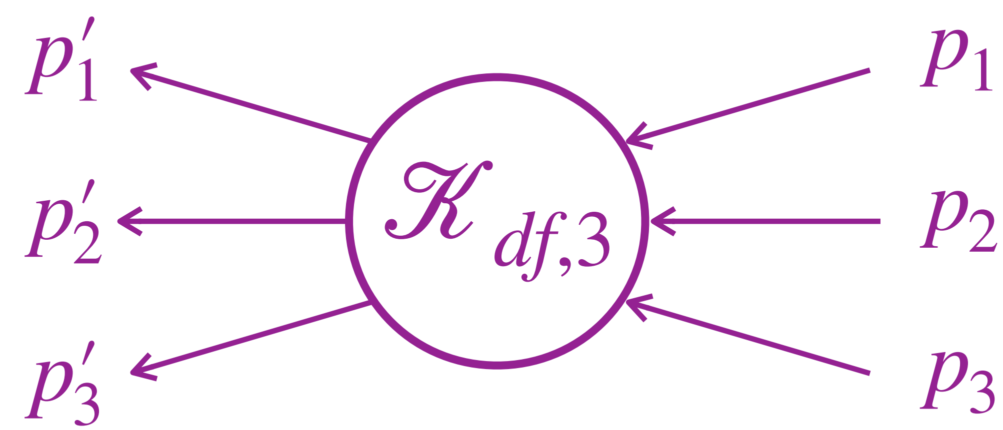
CM energy of two particles

q is momentum in CM frame

d wave: $$\frac{1}{\mathcal{K}_2^{(2)}} = -\frac{1}{16\pi E_2}\frac{1}{q^4}\frac{1}{a_2^5} + \ldots$$

Truncating sum over l

- Implement the same approach for $\mathcal{K}_{\mathrm{df},3}$, making use of the facts that it is relativistically invariant and completely symmetric under initial- & final-state permutations, and T invariant, and expanding about threshold [BHS18, BRS19]



3 $\quad s_{ij} \equiv (p_i + p_j)^2$

+

3 $\quad s'_{ij} \equiv (p'_i + p'_j)^2$

+

9 $\quad t_{ij} \equiv (p_i - p'_j)^2$

=15
building blocks
(but only 8 are independent)

$$\Delta \equiv \frac{s - 9m^2}{9m^2}$$

$$\Delta_i \equiv \frac{s_{jk} - 4m^2}{9m^2}$$

$$\Delta'_i \equiv \frac{s'_{jk} - 4m^2}{9m^2}$$

$$\widetilde{t}_{ij} \equiv \frac{t_{ij}}{9m^2}$$

Expand in these dimensionless quantities

Truncating sum over l

- Enforcing the symmetries, one finds [BRS19]

$$m^2\mathcal{K}_{\rm df,3} = \mathcal{K}^{\rm iso} + \mathcal{K}_{\rm df,3}^{(2,A)}\Delta_A^{(2)} + \mathcal{K}_{\rm df,3}^{(2,B)}\Delta_B^{(2)} + \mathcal{O}(\Delta^3)$$

$$\mathcal{K}^{\rm iso} = \mathcal{K}_{\rm df,3}^{\rm iso} + \mathcal{K}_{\rm df,3}^{\rm iso,1}\Delta + \mathcal{K}_{\rm df,3}^{\rm iso,2}\Delta^2$$

$$\Delta_A^{(2)} = \sum_{i=1}^{3}(\Delta_i^2 + \Delta_i'^2) - \Delta^2$$

$$\Delta_B^{(2)} = \sum_{i,j=1}^{3}\widetilde{t}_{ij}^{\,2} - \Delta^2$$

Convenient linear combinations

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"isotropic" since independent of momenta $\Rightarrow \ \ell' = \ell = 0$

leading order term

only term at linear order

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Convenient linear combinations

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These terms contain $\ell' = 0,2$ & $\ell = 0,2$

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Summary of approximations

$$\frac{1}{\mathscr{K}_2^{(0)}} = -\frac{1}{16\pi E_2}\left[\frac{1}{a_0} + \cancel{r_0\frac{q^2}{2}} + \cancel{P_0 r_0^3 q^4}\right], \qquad \frac{1}{\mathscr{K}_2^{(2)}} = -\frac{1}{16\pi E_2}\cancel{\frac{1}{q^4}\frac{1}{a_2^5}}$$

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1. Isotropic: $\ell_{\mathrm{max}} = 0$
   - Parameters: $a_0 \equiv a$ & $\mathscr{K}_{\mathrm{df},3}^{\mathrm{iso}}$
   - Corresponds to approximations used in NREFT & FVU approaches

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   - Corresponds to approximations used in NREFT & FVU approaches

2. "d wave": $\ell_{\max} = 2$
   - Parameters: $a_0,\ r_0,\ P_0,\ a_2,\ \mathscr{K}_{\mathrm{df},3}^{\mathrm{iso}},\ \mathscr{K}_{\mathrm{df},3}^{\mathrm{iso},1},\ \mathscr{K}_{\mathrm{df},3}^{\mathrm{iso},2},\ \mathscr{K}_{\mathrm{df},3}^{2,A},$ & $\mathscr{K}_{\mathrm{df},3}^{2,B}$

Numerical implementation: isotropic approximation

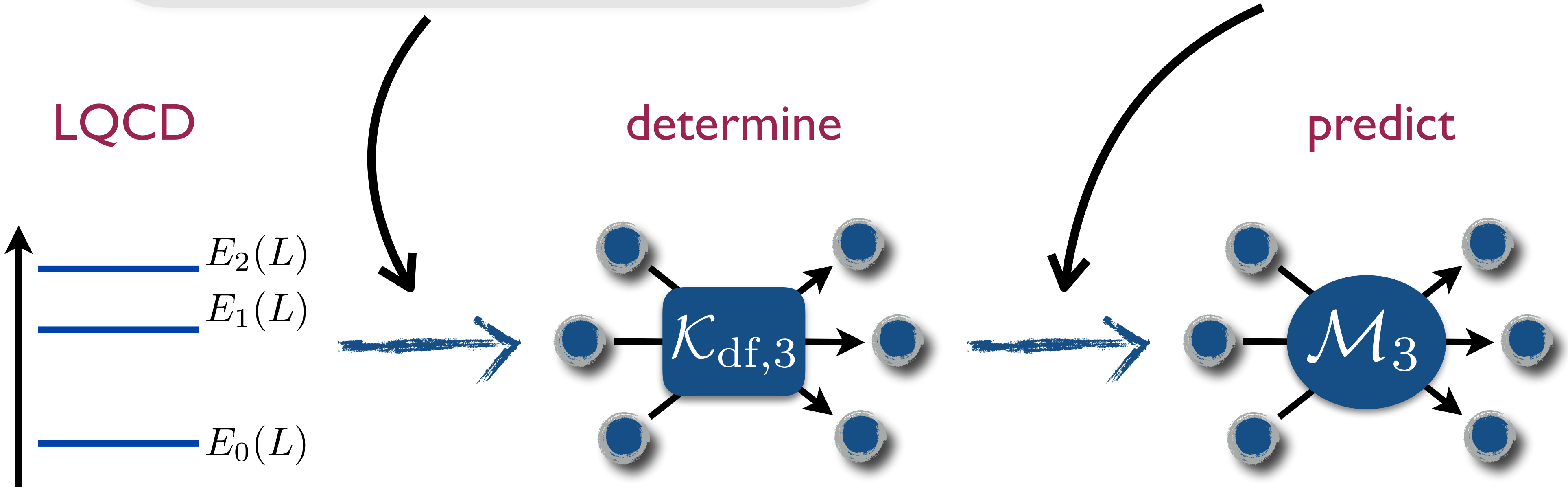
[BHS18]

Overview

$$\det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$$

Integral equations

DREAM:



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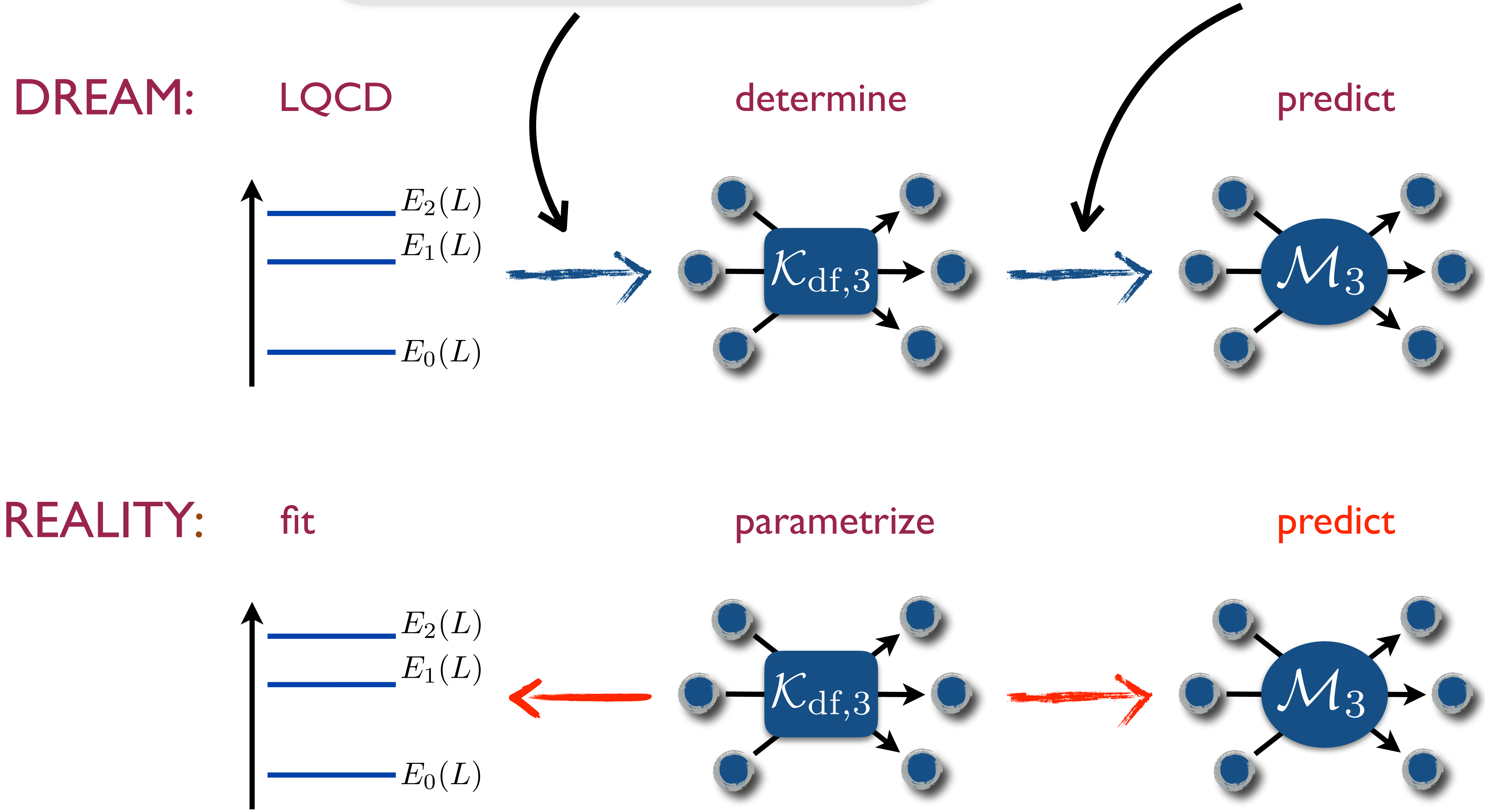

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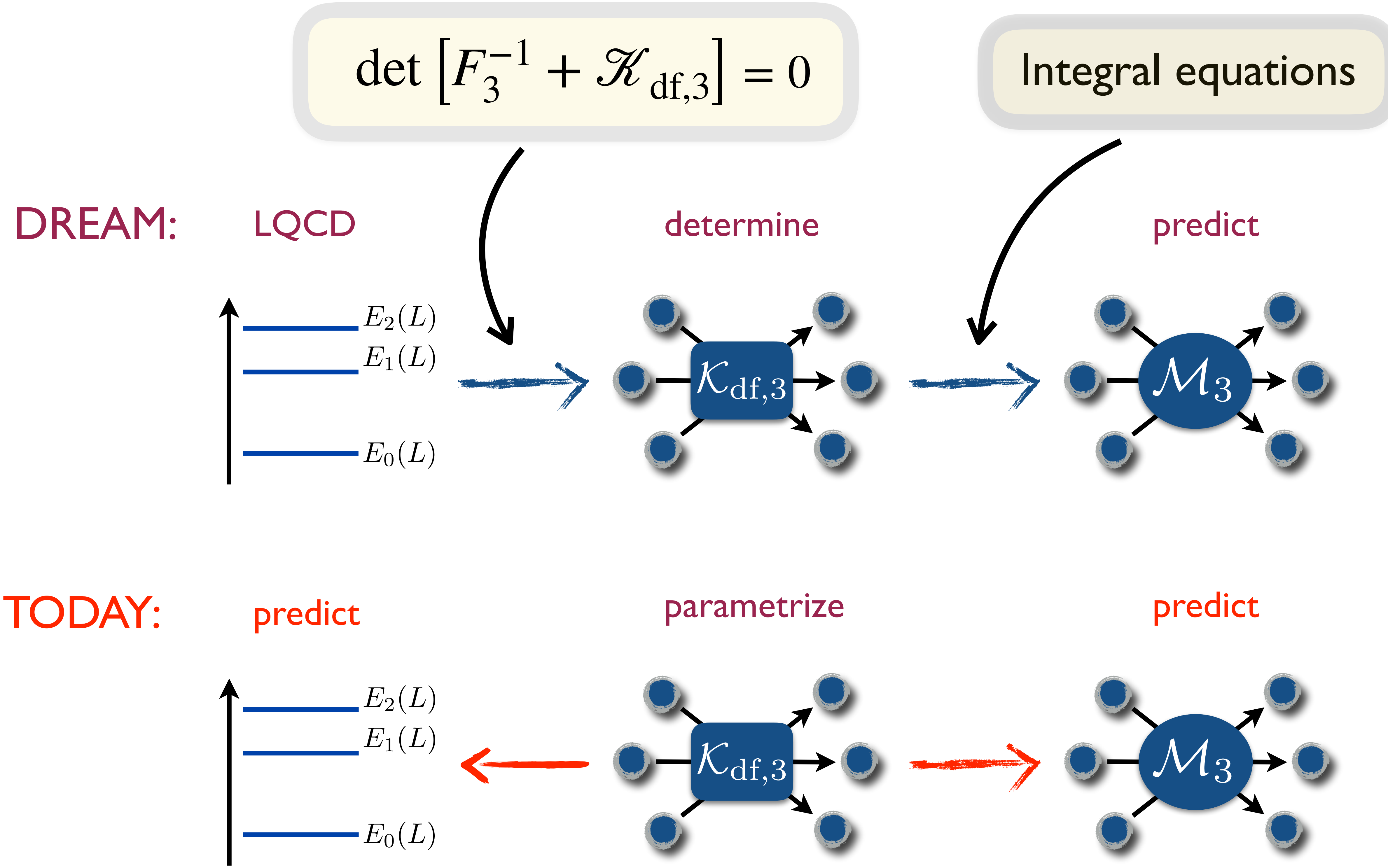
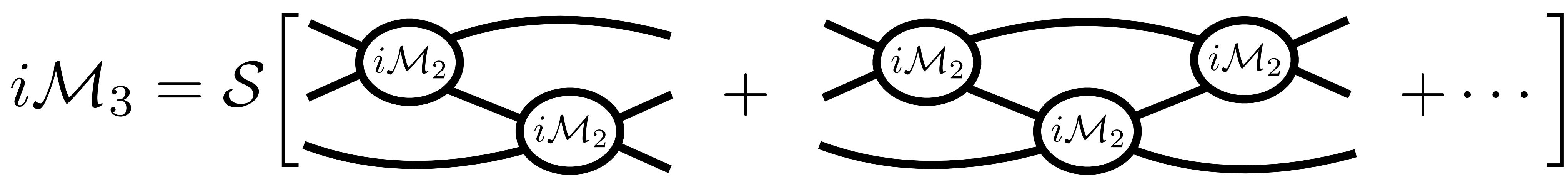

$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$
Integral equations
DREAM:
LQCD
determine
predict
$E_2(L)$
$E_1(L)$
$E_0(L)$
$\mathcal{K}_{\mathrm{df},3}$
$\mathcal{M}_3$
TODAY:
predict
parametrize
predict
$E_2(L)$
$E_1(L)$
$E_0(L)$
$\mathcal{K}_{\mathrm{df},3}$
$\mathcal{M}_3$

Implementing the isotropic QC3

- Two parameters: $a = a_0$ & $\mathscr{K}_{\mathrm{df},3}^{\mathrm{iso}}$
  - Consider only rest frame, **P=0**
  - Consider solutions only in the A_1^+ irrep (only irrep sensitive to $\mathscr{K}_{\mathrm{df},3}^{\mathrm{iso}}$)

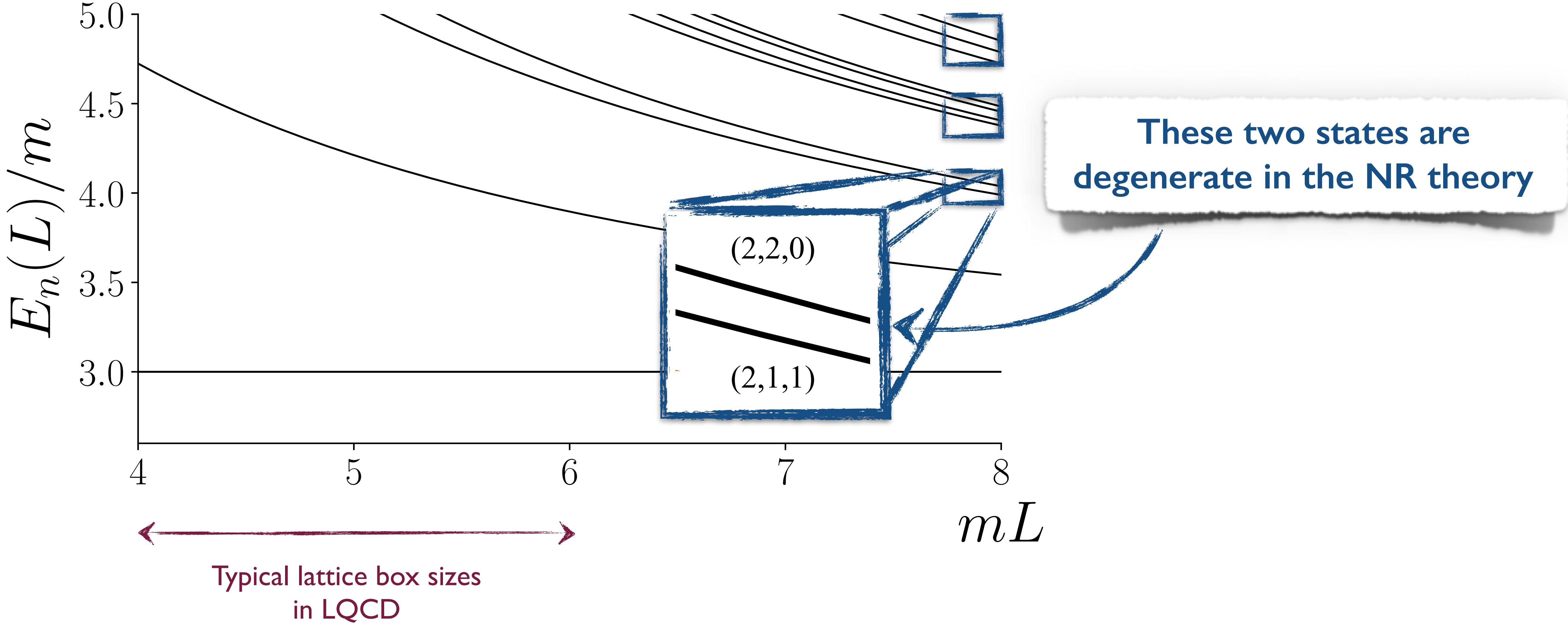
Solutions with $\mathcal{K}_{\mathrm{df},3}=0$

- Useful benchmark: deviations measure impact of 3-particle interaction
  - Caveat: scheme-dependent since $\mathcal{K}_{\mathrm{df},3}$ depends on cut-off function H
- Qualitative meaning of this limit for $\mathcal{M}_3$:



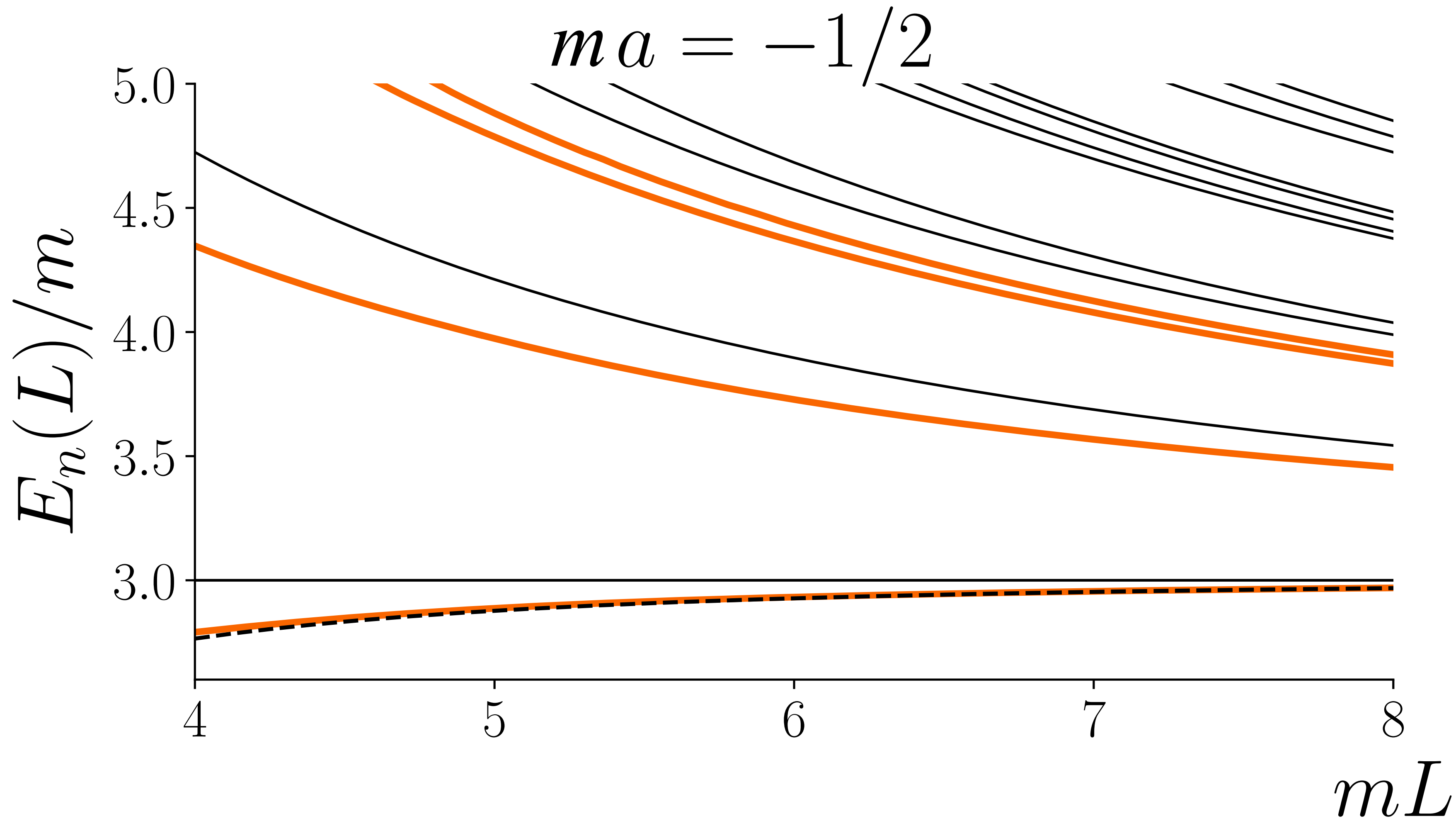
Solutions with $\mathcal{K}_{\mathrm{df},3}=0$

- Noninteracting three-particle states for **P**=0



Solutions with $\mathcal{K}_{df,3}=0$

- Weakly attractive two-particle interaction



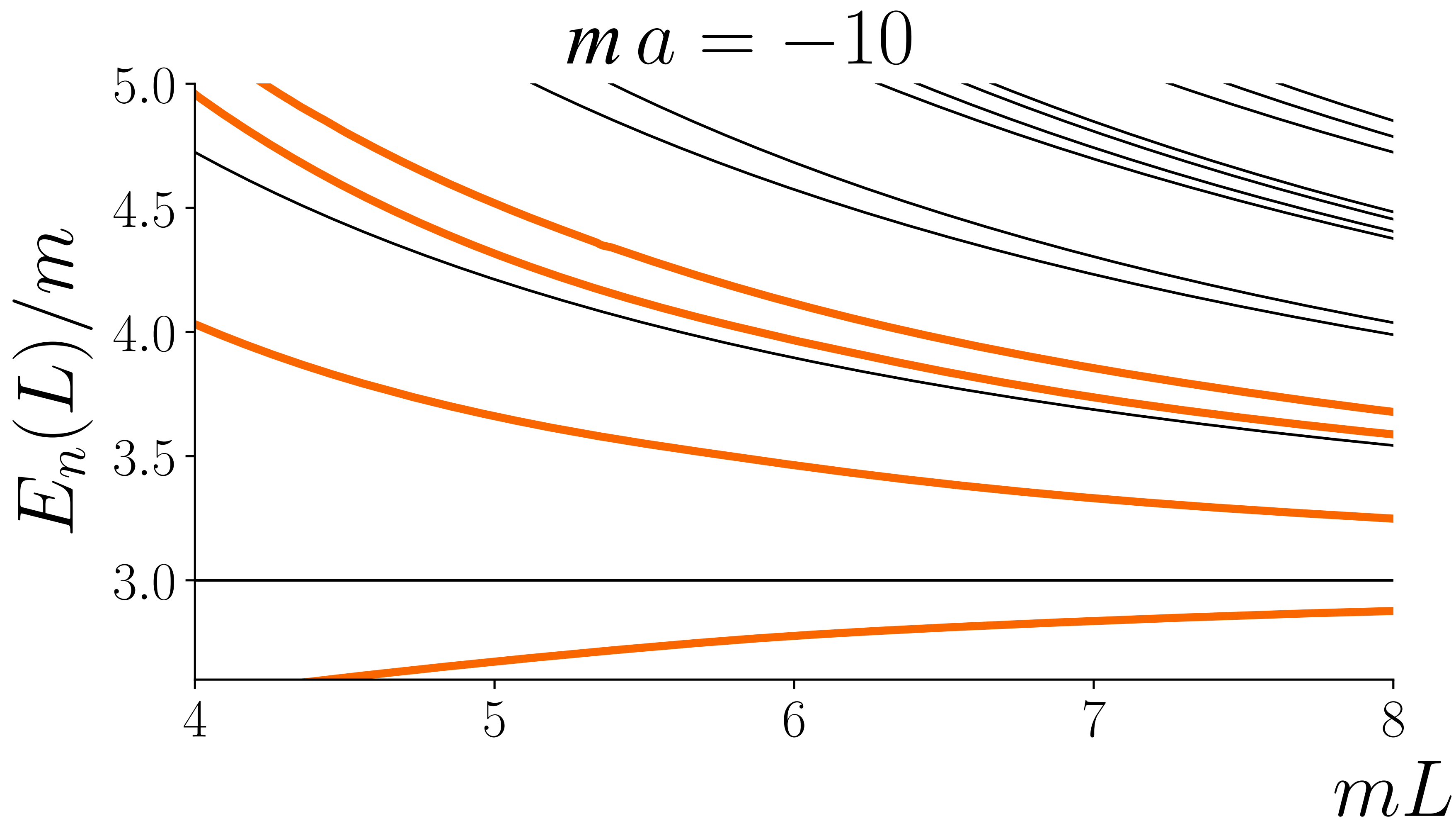
$1/L$ **expansion**

[Beane, Detmold, Savage; Tan; Hansen & SRS]

2-particle interaction enters at $1/L^3$, 3-particle interaction (and relativistic effects) enter at $1/L^6$

Solutions with $\mathcal{K}_{\mathrm{df},3}=0$

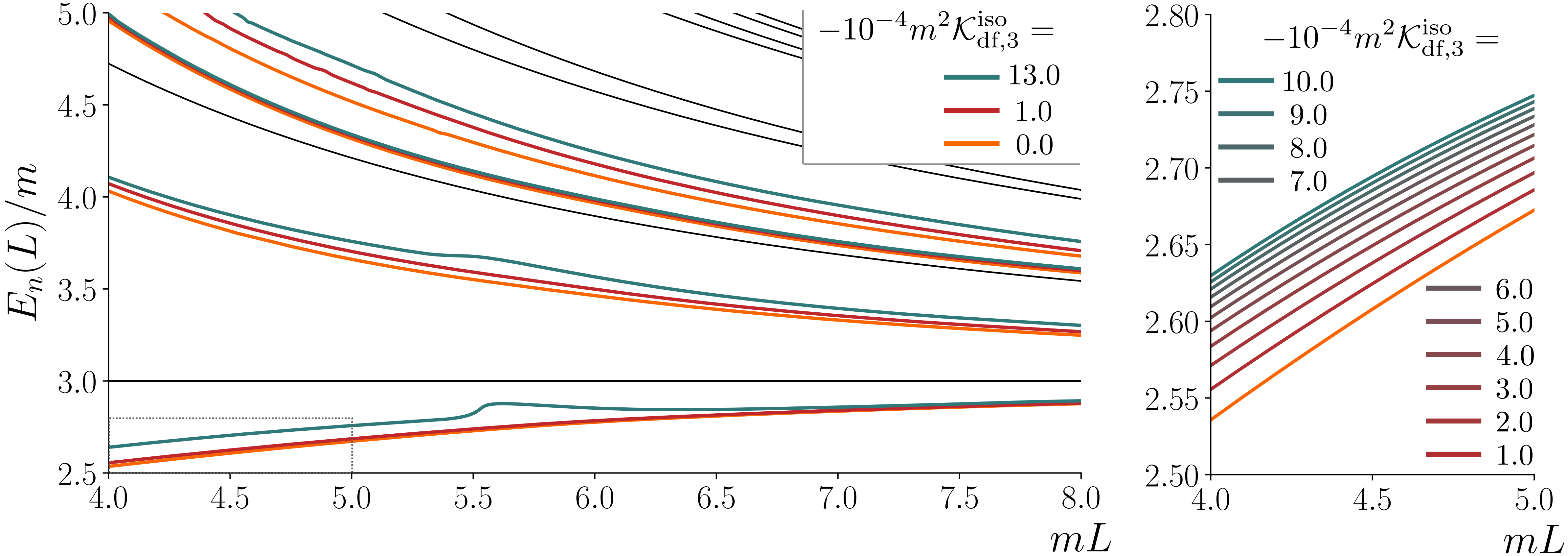
- Strongly attractive two-particle interaction



Threshold expansion not useful since need |a/L| << 1

Impact of $\mathcal{K}_{\mathrm{df},3}$

$ma = -10$ (strongly attractive interaction)



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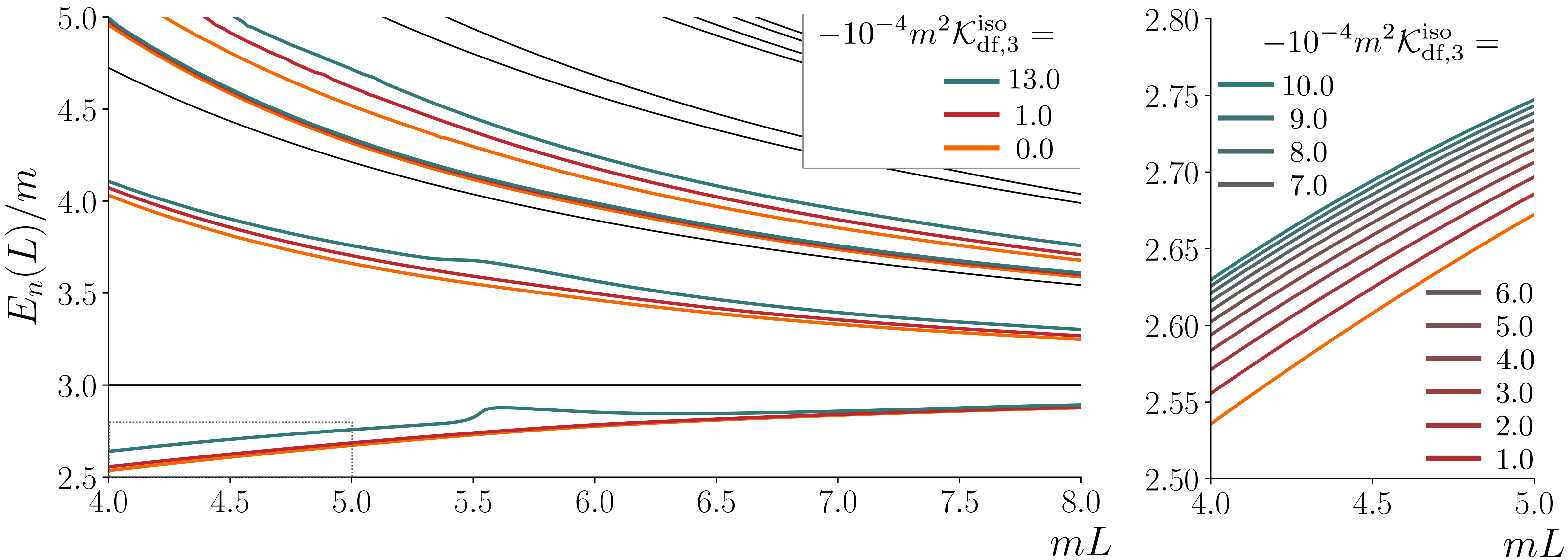
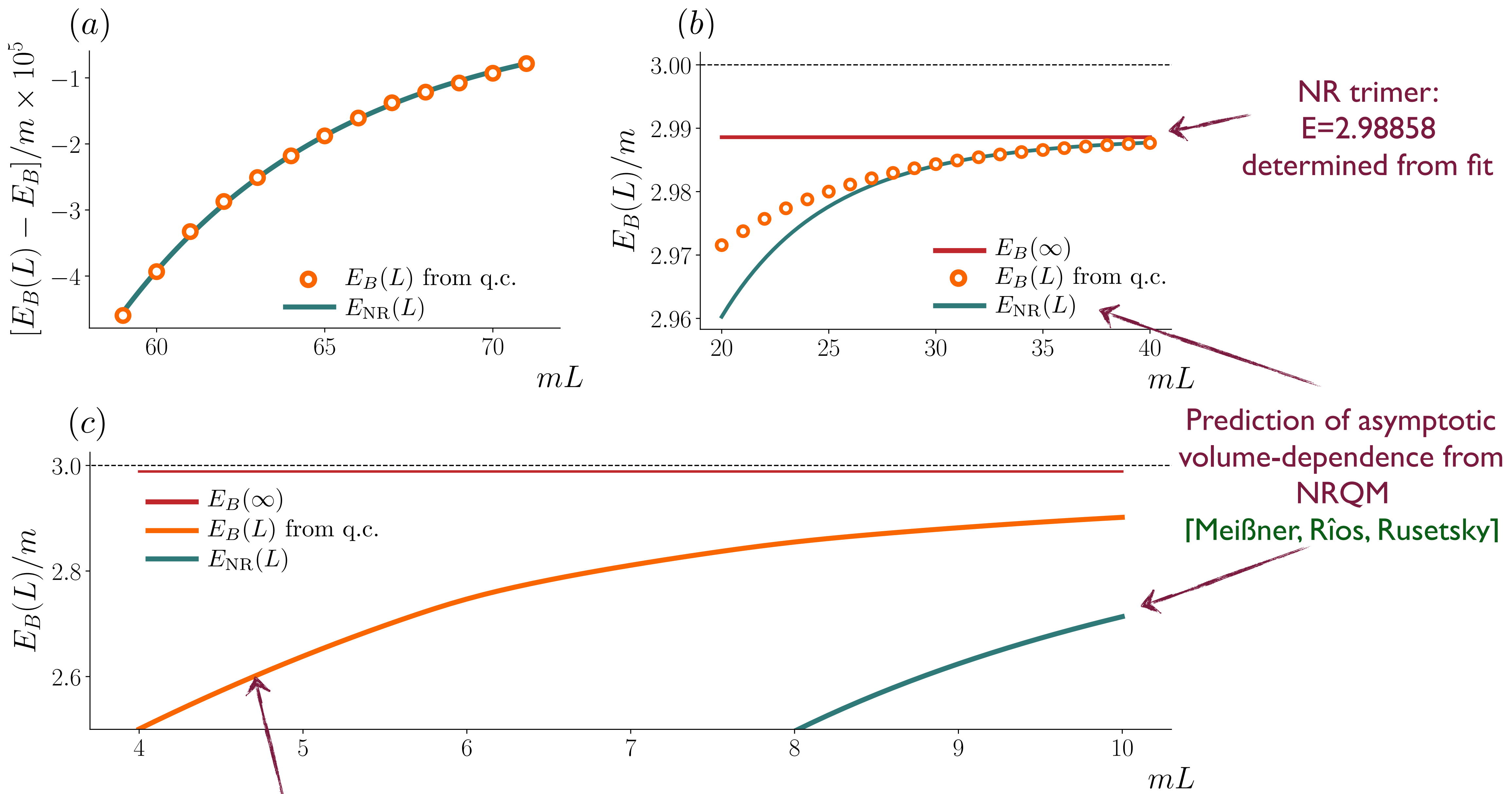


Local 3-particle interaction has significant effect on energies, especially in region of simulations (mL<5), and thus can be determined

Volume-dependence of unitary trimer

$am = -10^4$ & $m^2\mathscr{K}^{\rm iso}_{\rm df,3} = 2500$ (unitary regime, with no dimer)



Need quantization condition to determine finite-volume effects for realistic values of mL

Trimer "wavefunction"

- Solve integral equations numerically to determine $\mathcal{M}_{\mathrm{df},3}$ from $\mathcal{K}_{\mathrm{df},3}$
- Determine wavefunction from residue at bound-state pole

$$\mathcal{M}_{\mathrm{df},3}^{(u,u)}(k,p) \sim -\frac{\Gamma^{(u)}(k)\Gamma^{(u)}(p)^*}{E^2 - E_B^2}$$

- Compare to analytic prediction from NRQM in unitary limit [HSBS16]

$$|\Gamma^{(u)}(k)_{\mathrm{NR}}|^2 = |c||A|^2 \frac{256\pi^{5/2}}{3^{1/4}} \frac{m^2\kappa^2}{k^2(\kappa^2+3k^2/4)} \frac{\sin^2\left(s_0 \sinh^{-1}\frac{\sqrt{3}k}{2\kappa}\right)}{\sinh^2\frac{\pi s_0}{2}}$$

Known constant

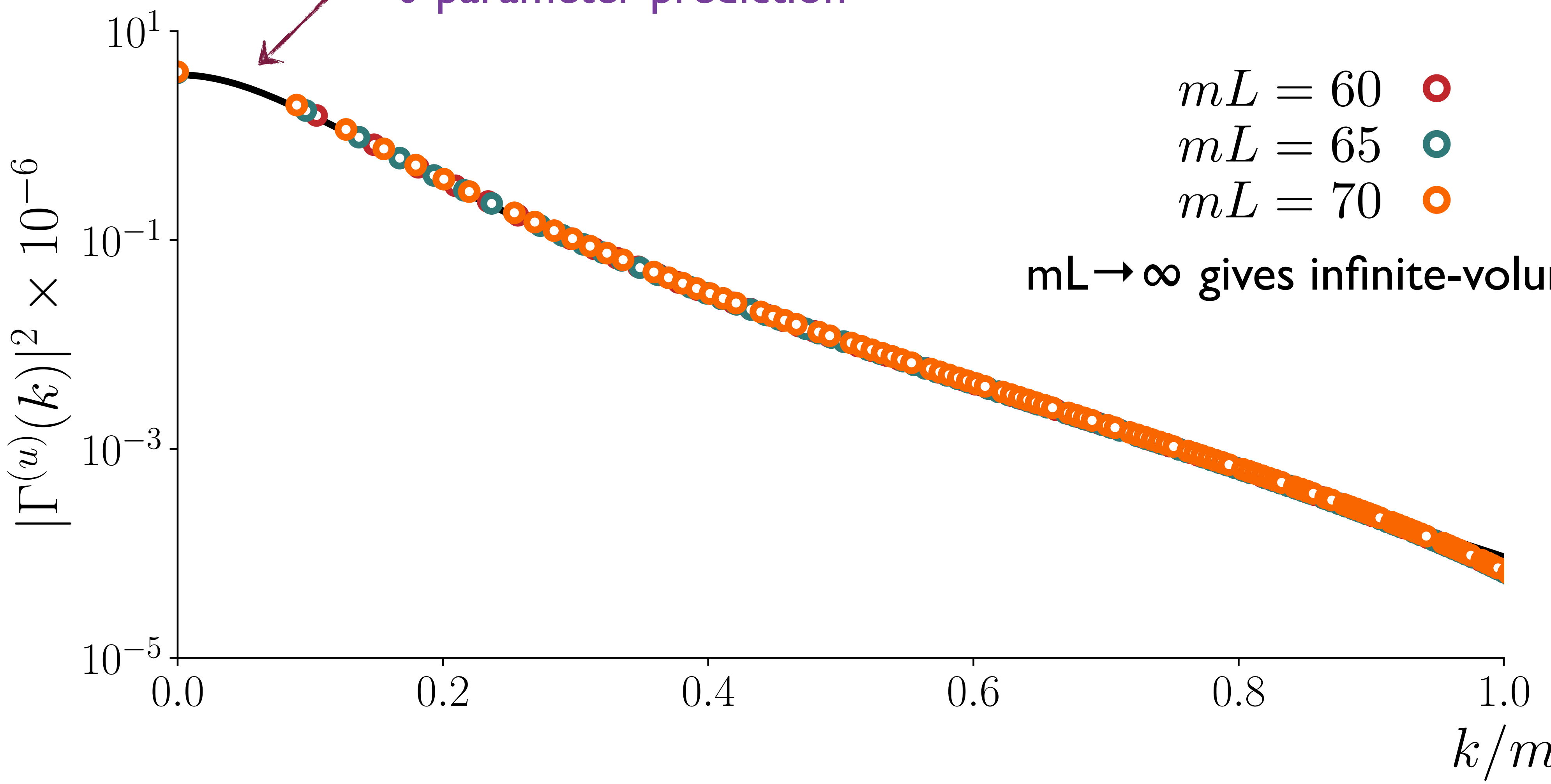
Determined by fit to volume-dependence of bound-state energy

Known constant

Trimer wavefunction

$$|\Gamma^{(u)}(k)_{\rm NR}|^2 = |c||A|^2 \frac{256\pi^{5/2}}{3^{1/4}} \frac{m^2\kappa^2}{k^2(\kappa^2+3k^2/4)} \frac{\sin^2\left(s_0 \sinh^{-1}\frac{\sqrt{3}k}{2\kappa}\right)}{\sinh^2\frac{\pi s_0}{2}}$$

0-parameter prediction



mL→∞ gives infinite-volume result

Trimer wavefunction

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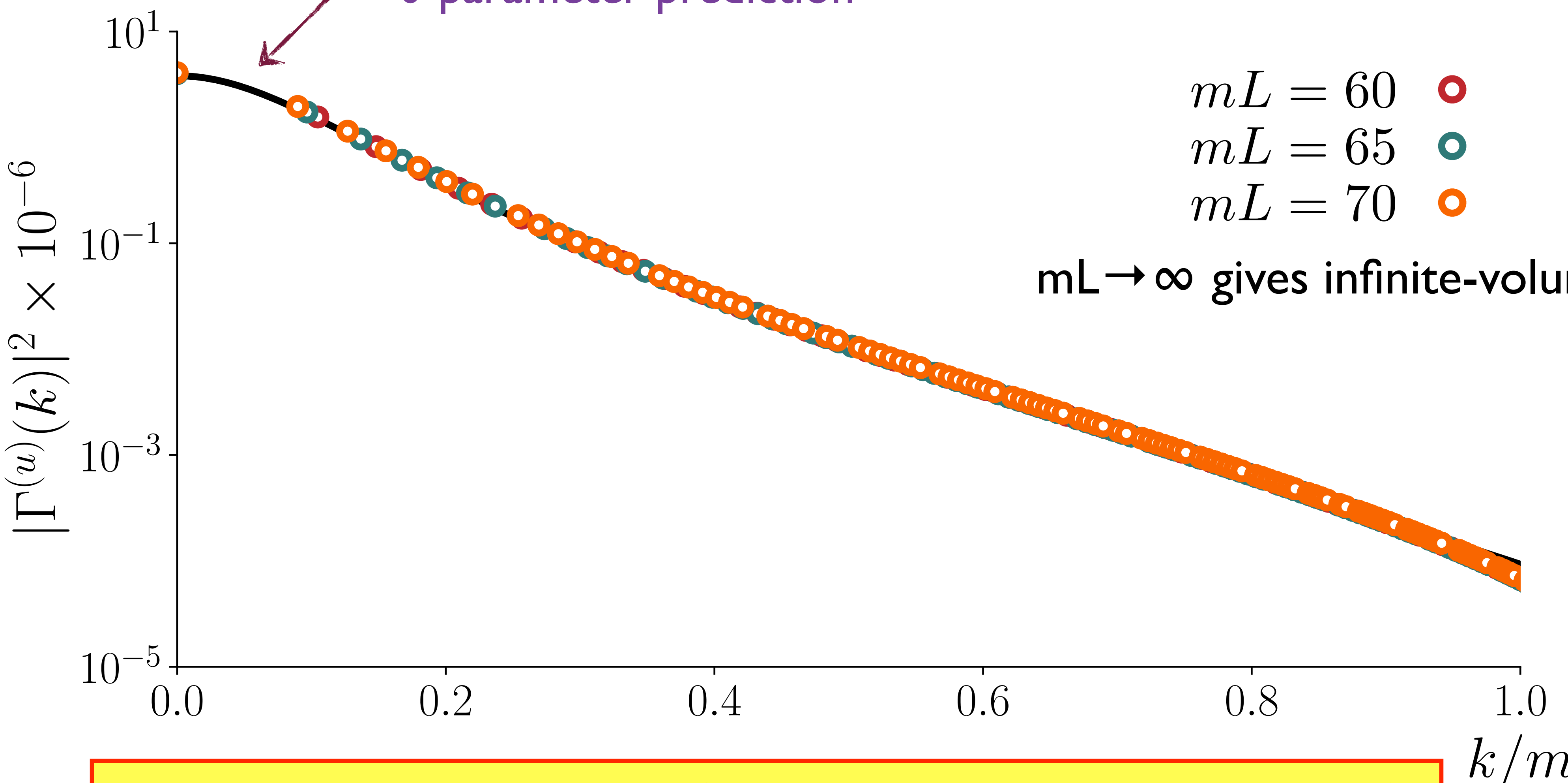


mL→∞ gives infinite-volume result

Works over many orders of magnitude to expected accuracy!
Example of QC3/KtoM giving infinite-volume results

Beyond isotropic: including higher partial waves

[BRS19]

d-wave approximation: $l_{max} = 2$

$$\frac{1}{\mathcal{K}_2^{(0)}} = \frac{1}{16\pi E_2}\left[\frac{1}{a_0} + r_0\frac{q^2}{2} + P_0 r_0^3 q^4\right], \qquad \frac{1}{\mathcal{K}_2^{(2)}} = \frac{1}{16\pi E_2}\frac{1}{q^4}\frac{1}{a_2^5}$$

$$m^2 \mathcal{K}_{\mathrm{df},3} = \mathcal{K}^{\mathrm{iso}} + \mathcal{K}_{\mathrm{df},3}^{(2,A)}\Delta_A^{(2)} + \mathcal{K}_{\mathrm{df},3}^{(2,B)}\Delta_B^{(2)}$$

$$\mathcal{K}^{\mathrm{iso}} = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}\Delta + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},2}\Delta^2$$

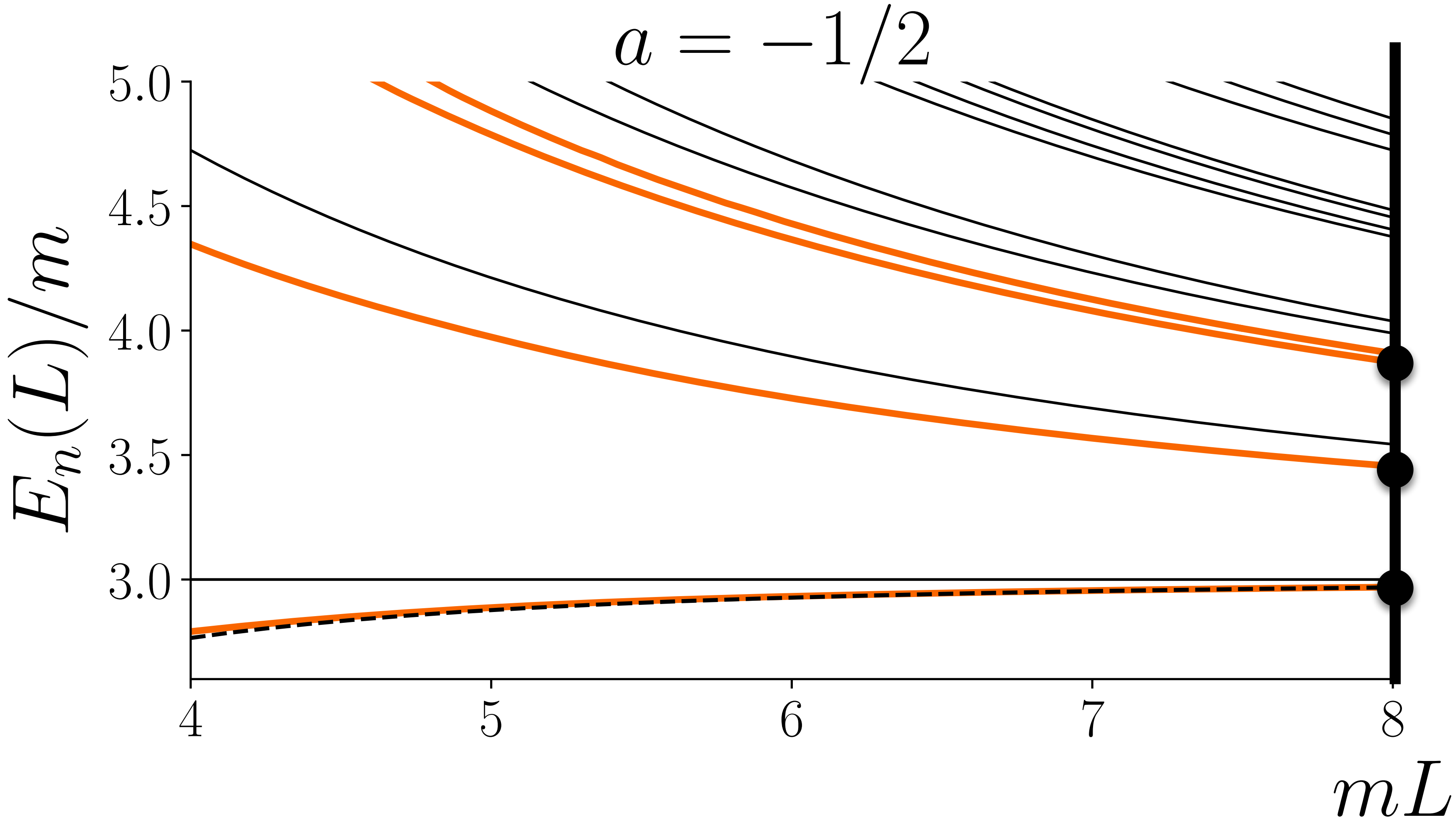
- Parameters: a_0, r_0, P_0, a_2, $\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}$, $\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}$, $\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},2}$, $\mathcal{K}_{\mathrm{df},3}^{2,A}$, & $\mathcal{K}_{\mathrm{df},3}^{2,B}$

$$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$$

- QC3 now involves the determinant of a (finite) matrix
- Consider only **P**=0
- Project onto irreps, determine vanishing of eigenvalues of $1/F_3 + K_{\mathrm{df},3}$

Impact of strong d-wave attraction

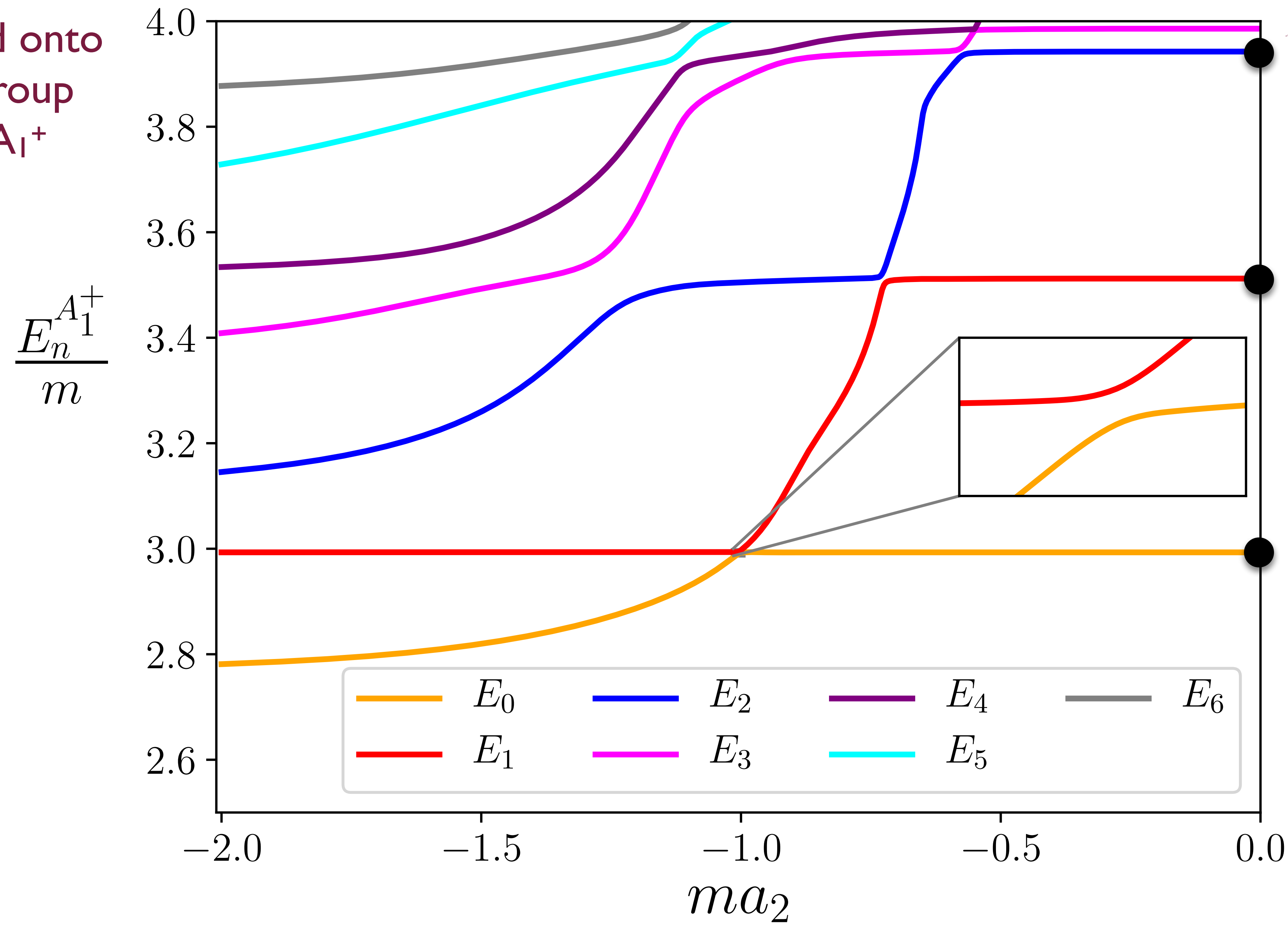
Results from isotropic approximation with $\mathscr{K}_{df,3} = 0$



What happens to these levels as a_2 is turned on?

Impact of strong d-wave attraction

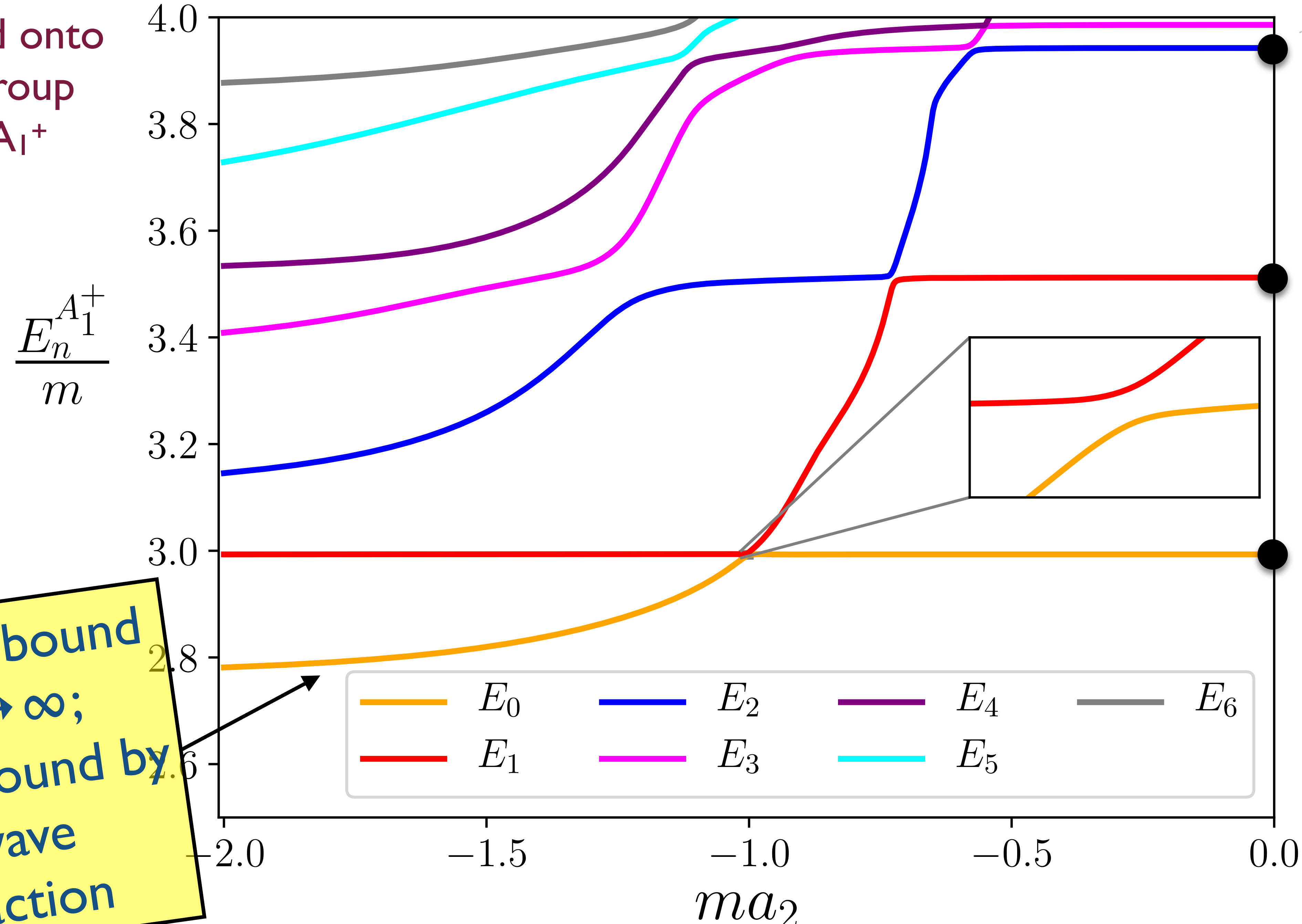
Projected onto cubic-group irrep A_1^+



$$mL = 8.1,\ ma_0 = -0.1,\ r_0 = P_0 = \mathcal{K}_{df,3} = 0$$

Impact of strong d-wave attraction

Projected onto cubic-group irrep A_1^+



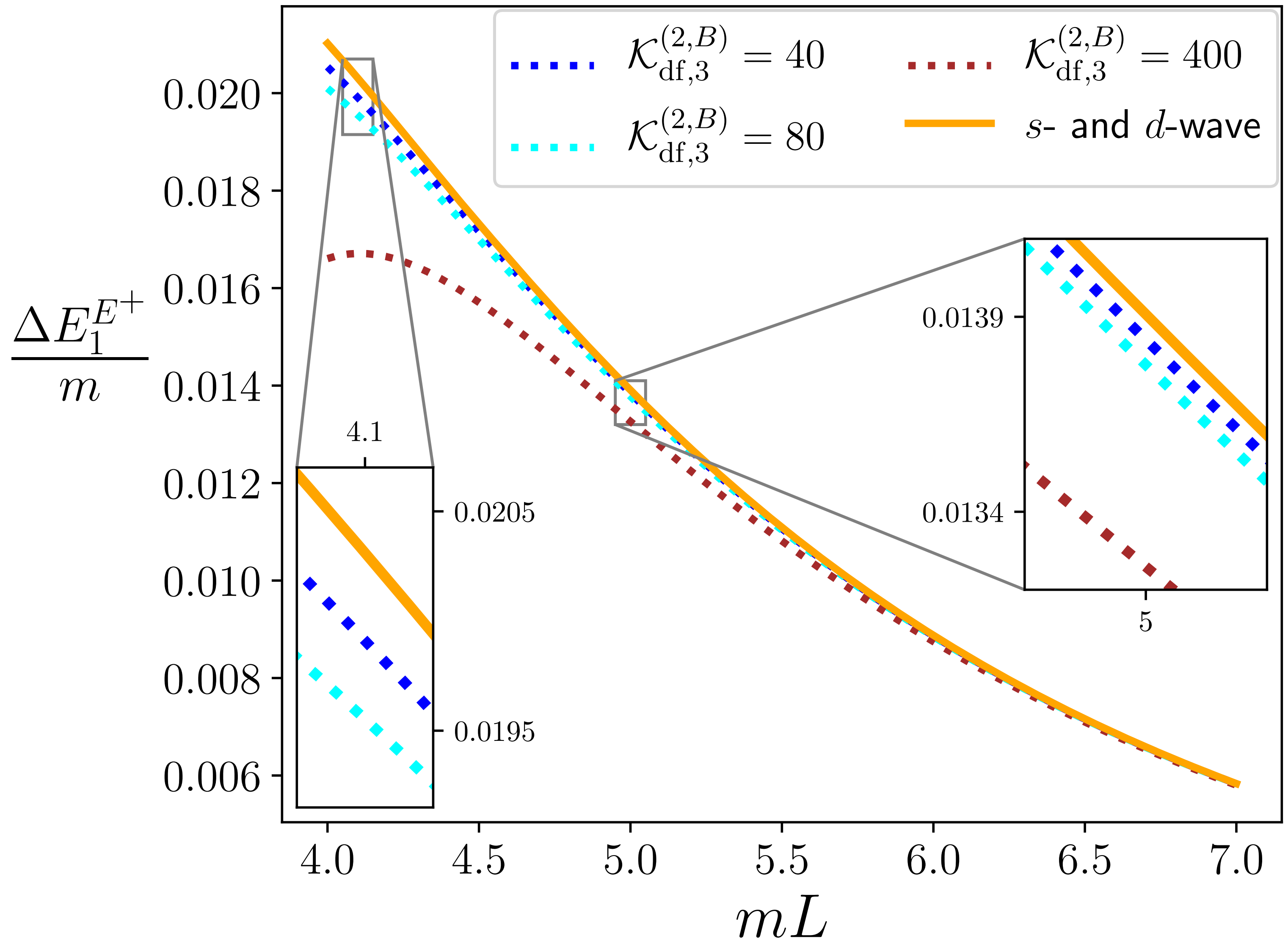
Remains bound as L→∞; trimer bound by d-wave attraction

$$mL = 8.1,\ ma_0 = -0.1,\ r_0 = P_0 = \mathscr{K}_{df,3} = 0$$

Impact of quadratic terms in $\mathcal{K}_{\mathrm{df},3}$

a_0, r_0, P_0, & a_2 set to physical values for $3\pi^+$

Energy shift relative to noninteracting energy for first excited state. Projected into E^+ irrep.



Energies of $3\pi^+$ states need to be determined very accurately to be sensitive to $\mathcal{K}_{\mathrm{df},3}^{(2,B)}$, but this is achievable in ongoing simulations

Numerical implementation: isotropic approximation including dimers

[BBHRS19]

Isotropic approximation: v2

- Same set-up as in [BHS18], except that by modifying the PV pole prescription, the formalism works for am > 1
  - Allows us to study cases where, in infinite-volume, there is a two-particle bound state ("dimer"), which can have relativistic binding energy

$$E_B/m = 2\sqrt{1 - 1/(am)^2} \xrightarrow{am=2} \sqrt{3}$$

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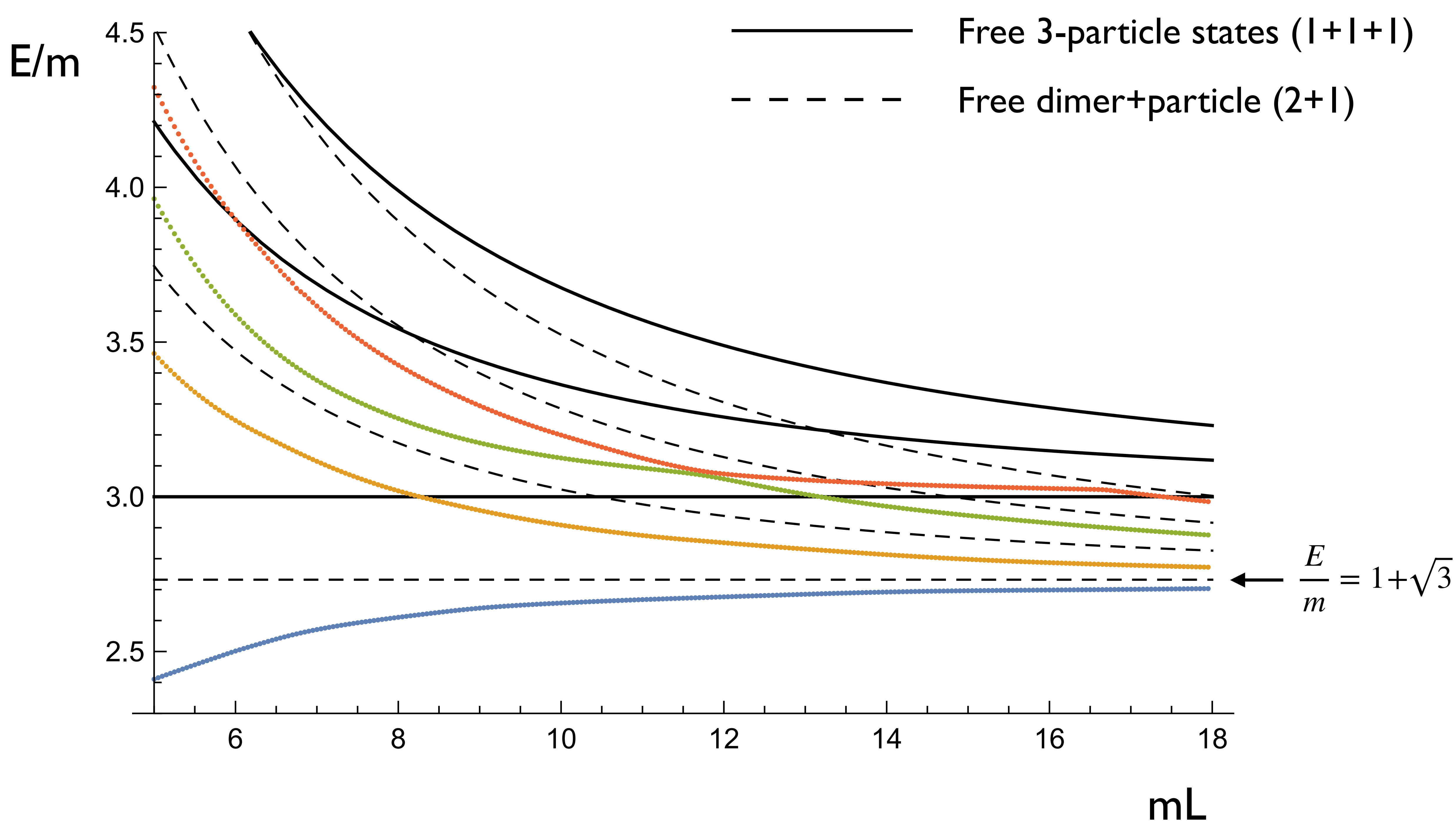
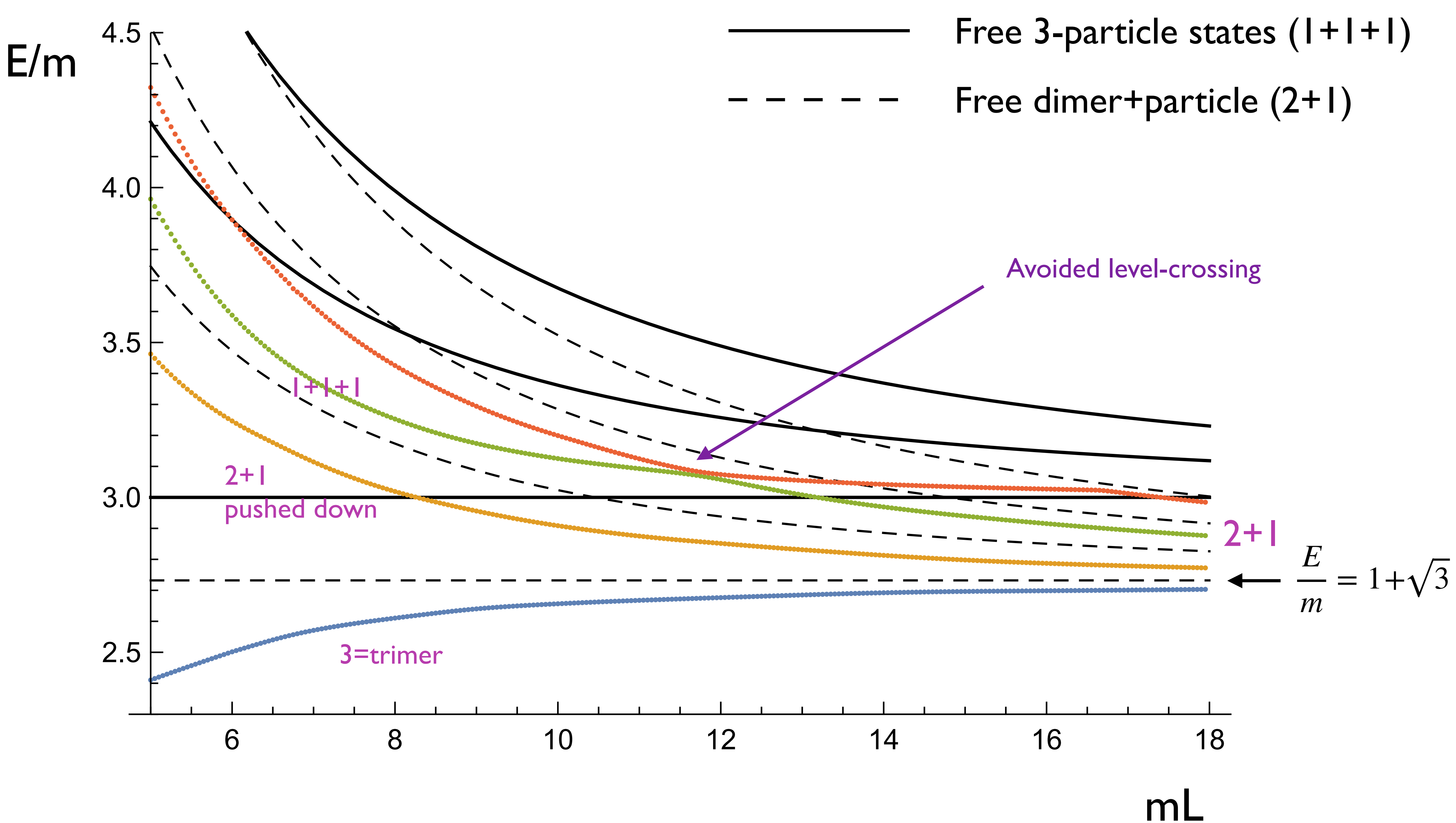
$$E_B/m = 2\sqrt{1 - 1/(am)^2} \xrightarrow{am=2} \sqrt{3}$$

- Interesting case: choose parameters so that there is both a dimer and a trimer
  - This is the analog (without spin) of studying the n+n+p system in which there are neutron + deuteron and tritium states
  - Finite-volume states will have components of all three types

Isotropic approximation: am=2, $\mathcal{K}_{df,3}=0$



Isotropic approximation: am=2, $\mathcal{K}_{df,3}$=0



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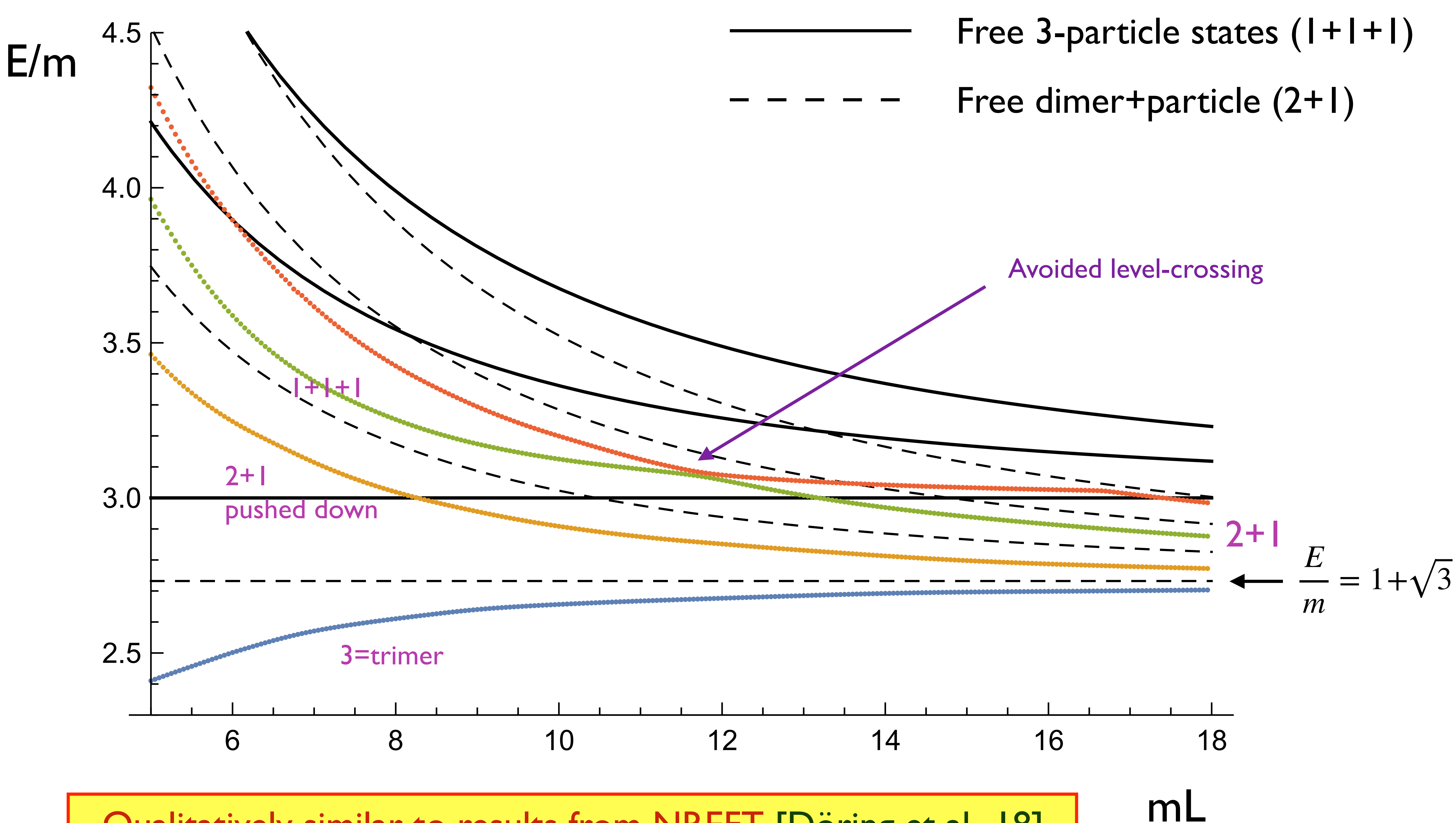
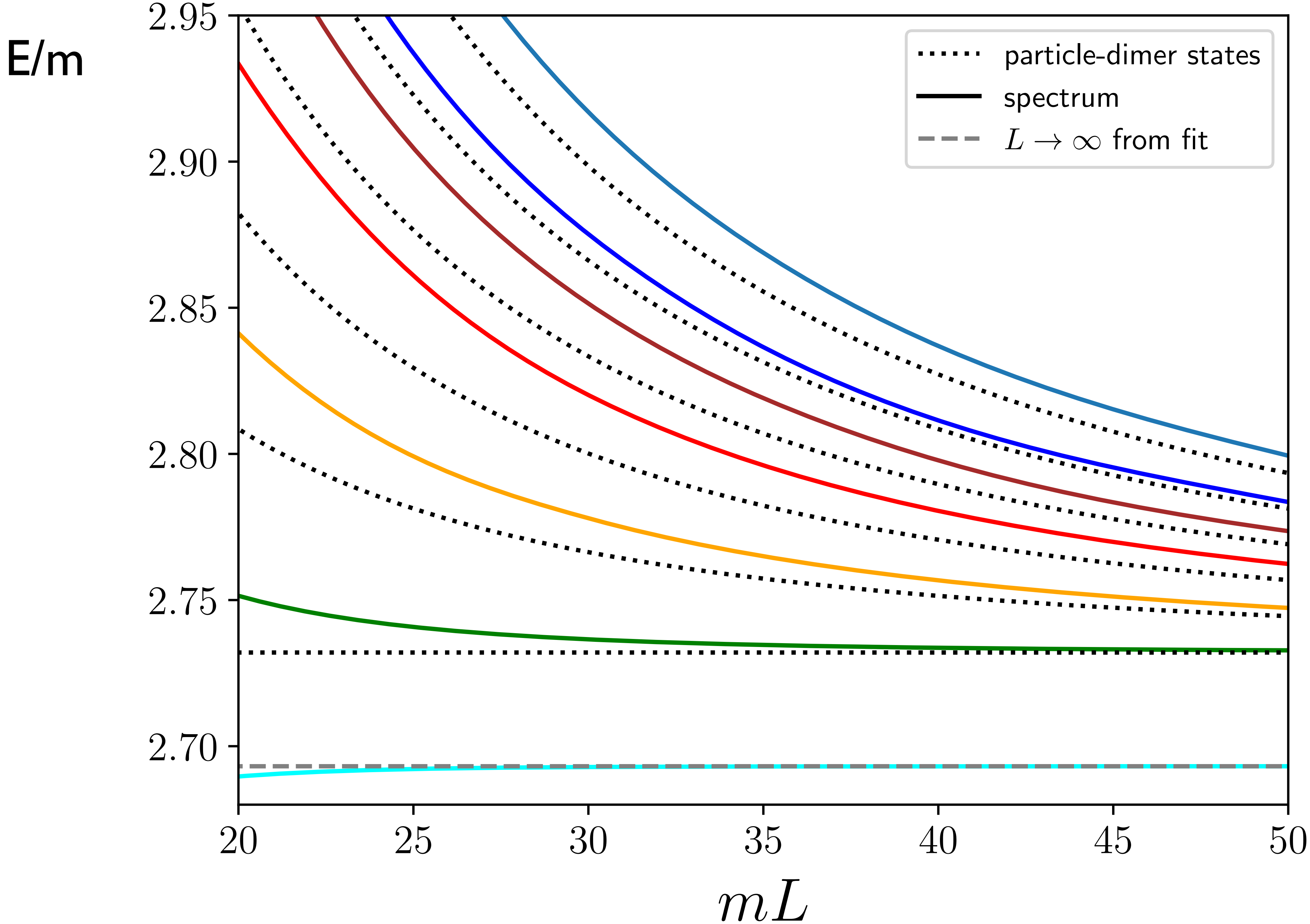


Qualitatively similar to results from NREFT [Döring et al., 18]

Isotropic approximation: am=2, $\mathcal{K}_{\mathrm{df},3}$=0

Extend to larger L, below 1+1+1 threshold

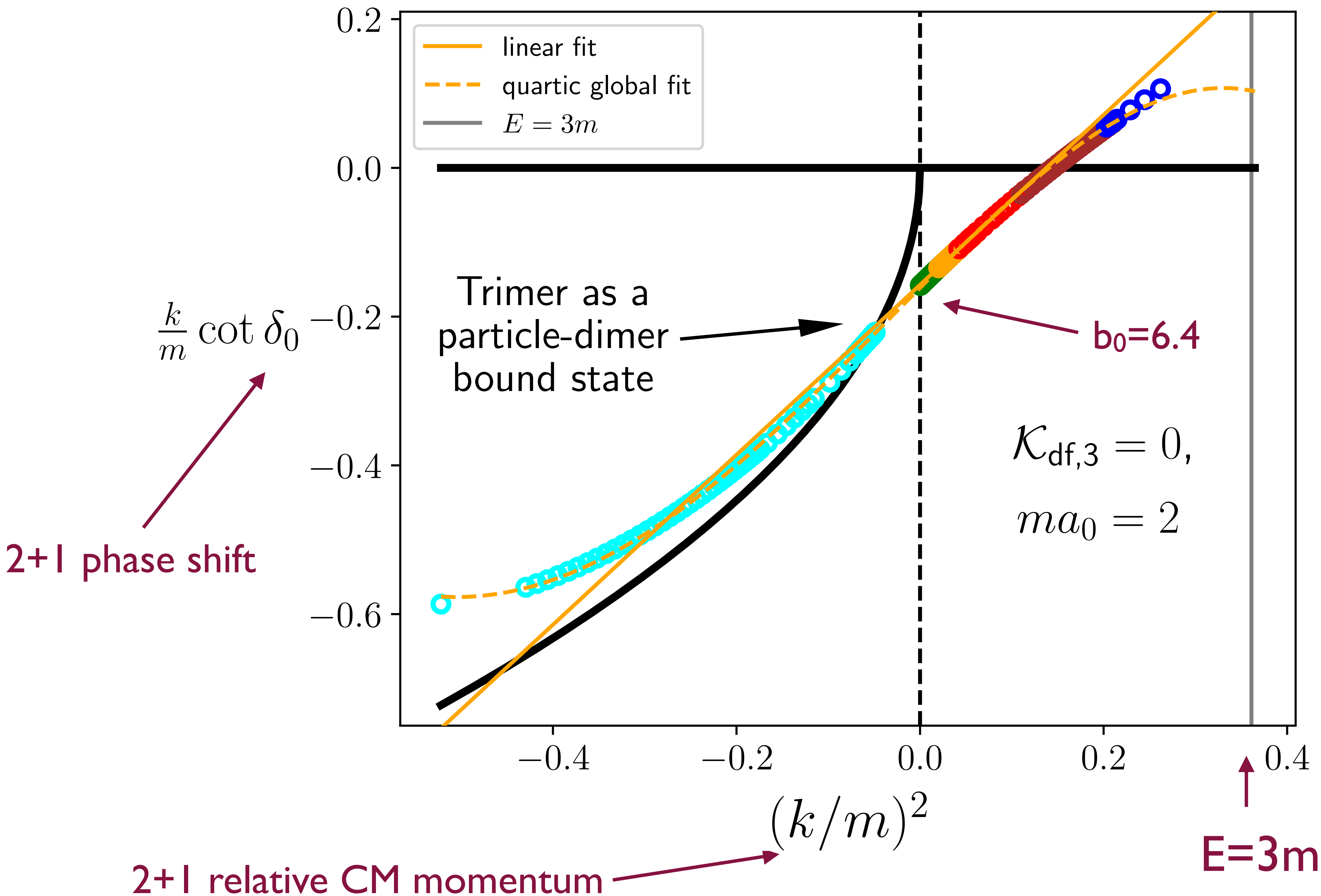


Dominantly 2+1 states

trimer!

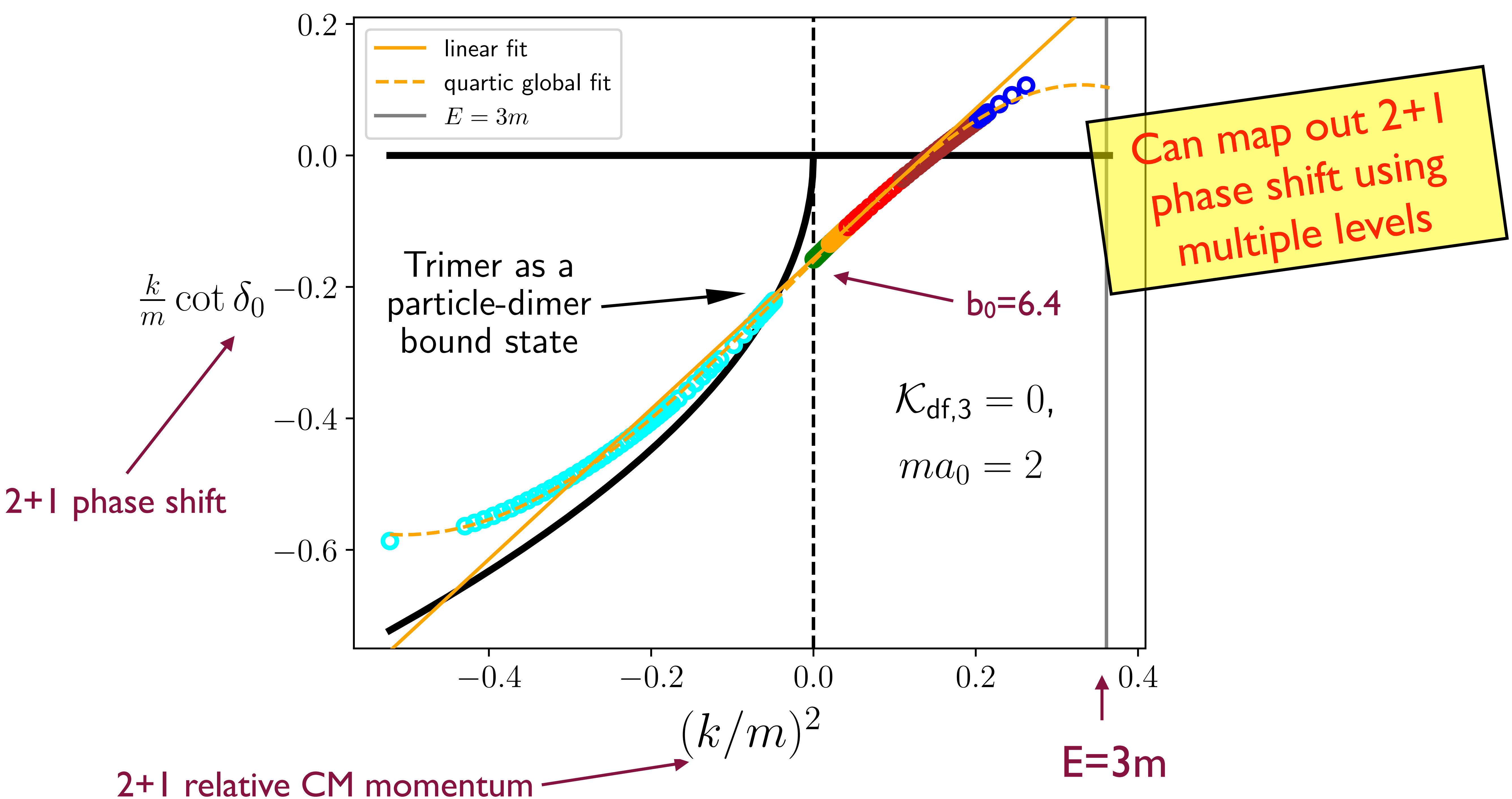
Isotropic approximation: a=2, $\mathcal{K}_{\mathrm{df},3}$=0

2+1 EFT: solve QC2 for nondegenerate particles



Isotropic approximation: a=2, $\mathcal{K}_{\mathrm{df},3}$=0

2+1 EFT: solve QC2 for nondegenerate particles



Isotropic approximation: a=2, $\mathcal{K}_{\mathrm{df},3}=0$

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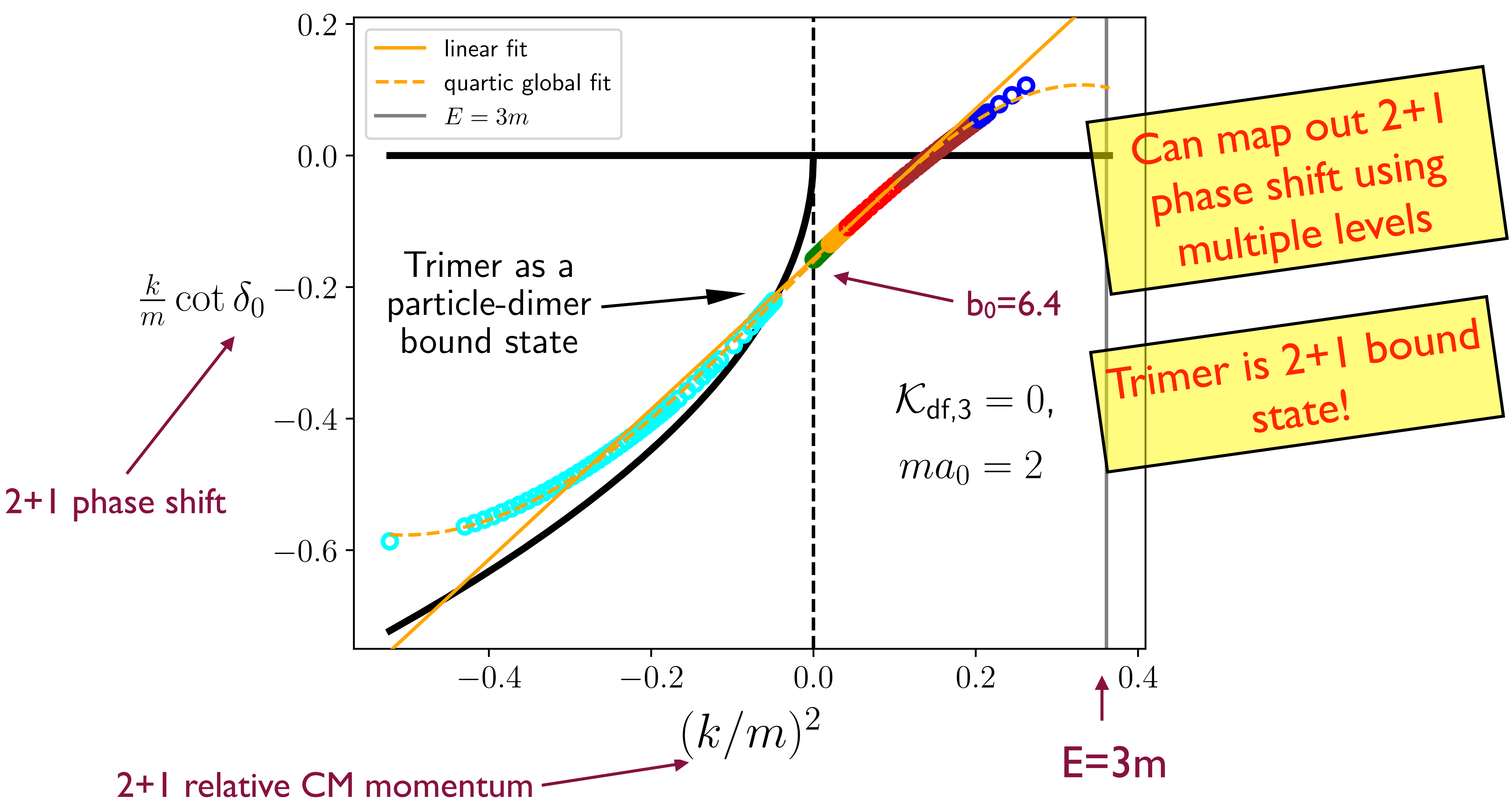
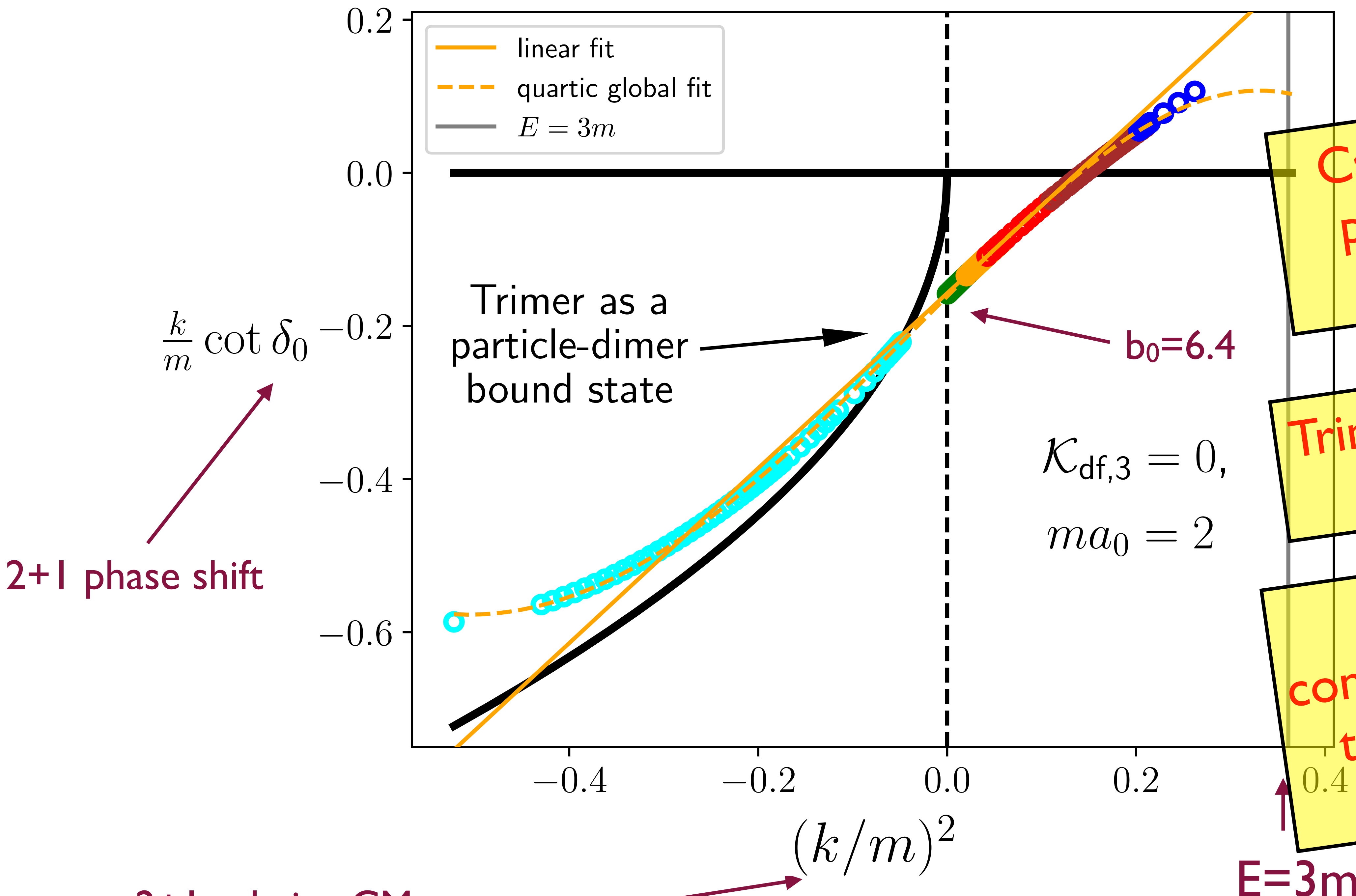


Isotropic approximation: a=2, $\mathcal{K}_{\rm df,3}$=0

2+1 EFT: solve QC2 for nondegenerate particles

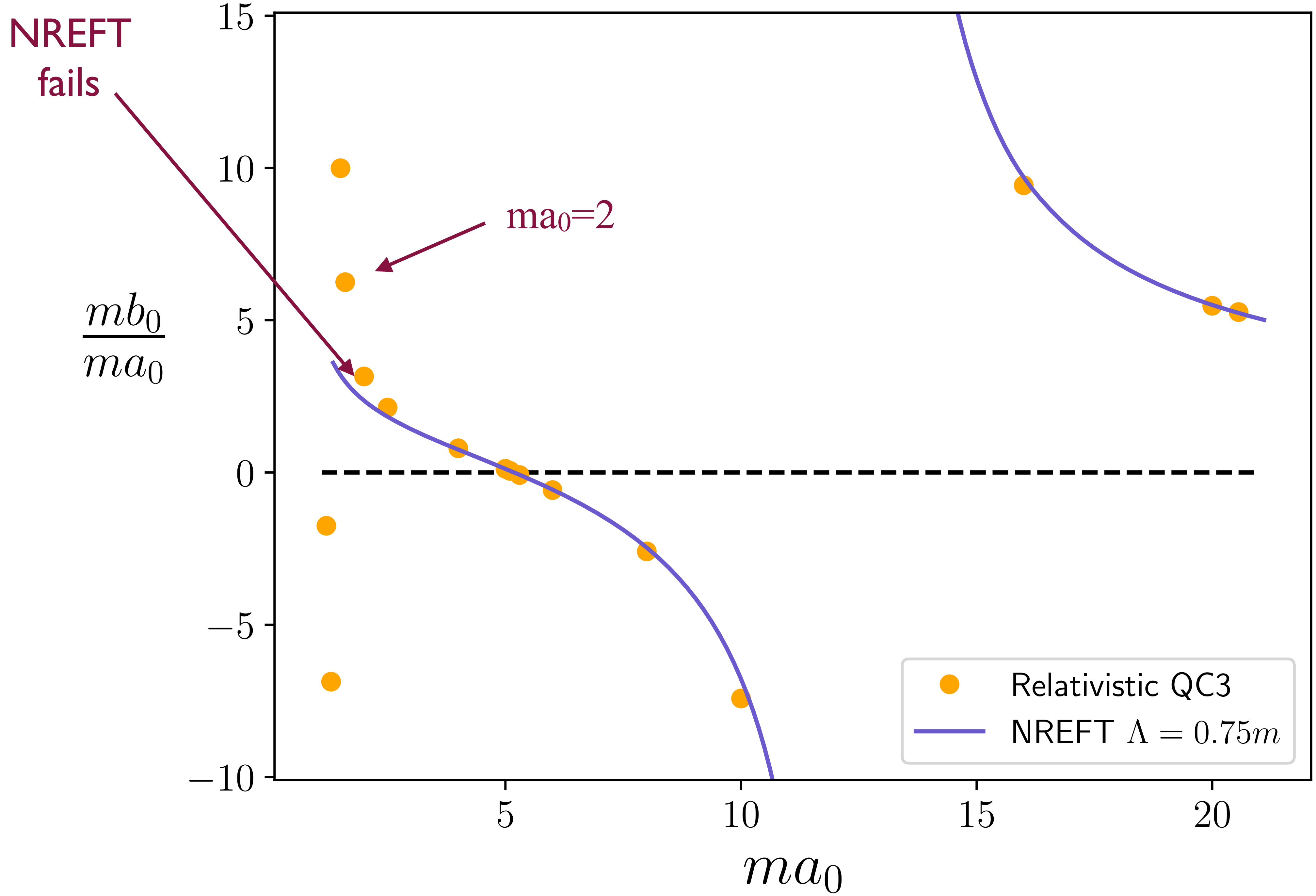


Can map out 2+1 phase shift using multiple levels

Trimer is 2+1 bound state!

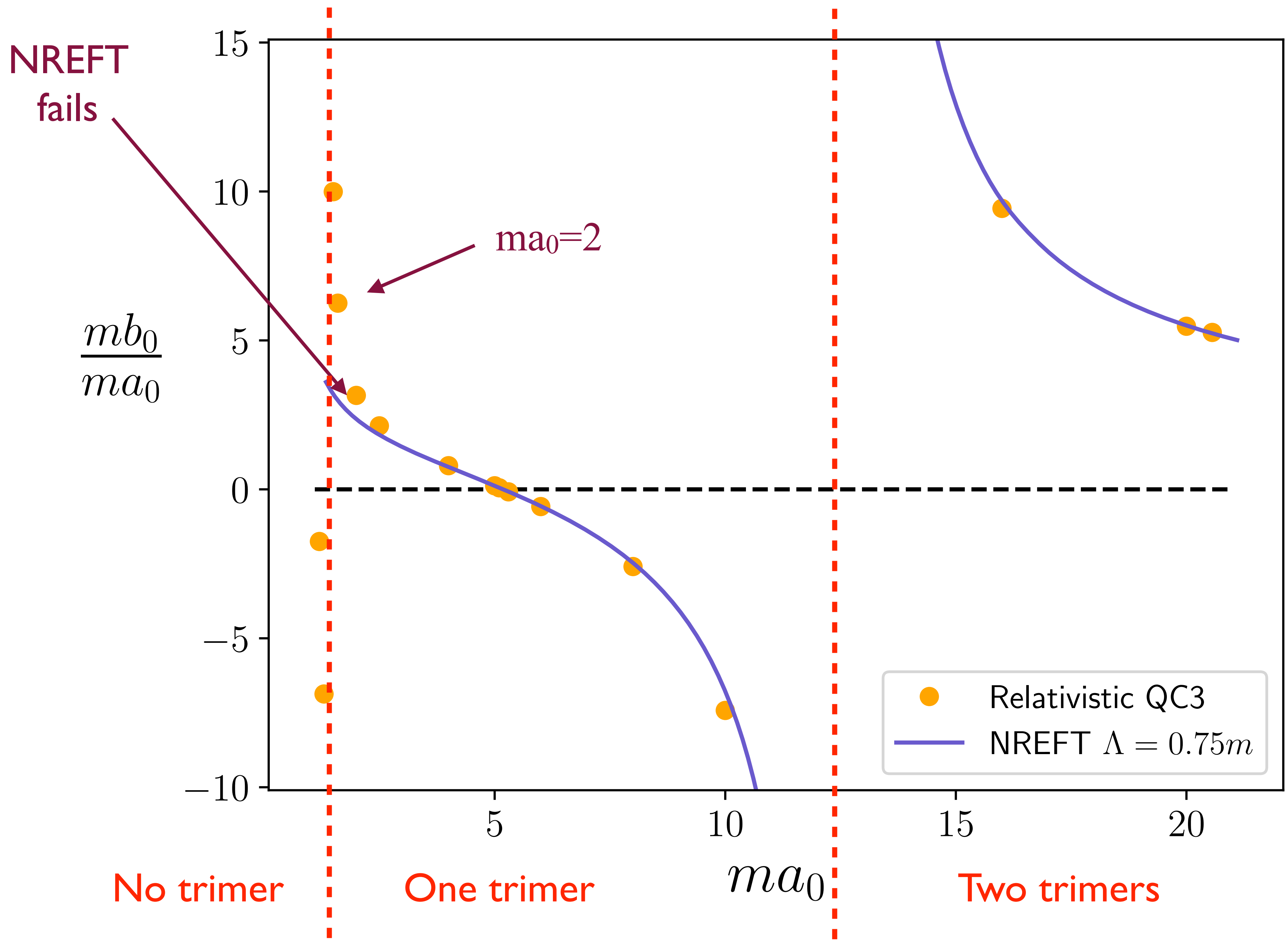
Quantization condition is useful as tool for studying infinite-volume!

Dimer properties vs a_0



Dimer properties vs a_0

Appearance of series of Efimov trimers!



Phillips curve in toy N+D / Tritium system

Choose parameters so that $m_{\text{dimer}} : m = M_D : M$ and vary $\mathscr{K}_{\text{df},3}$

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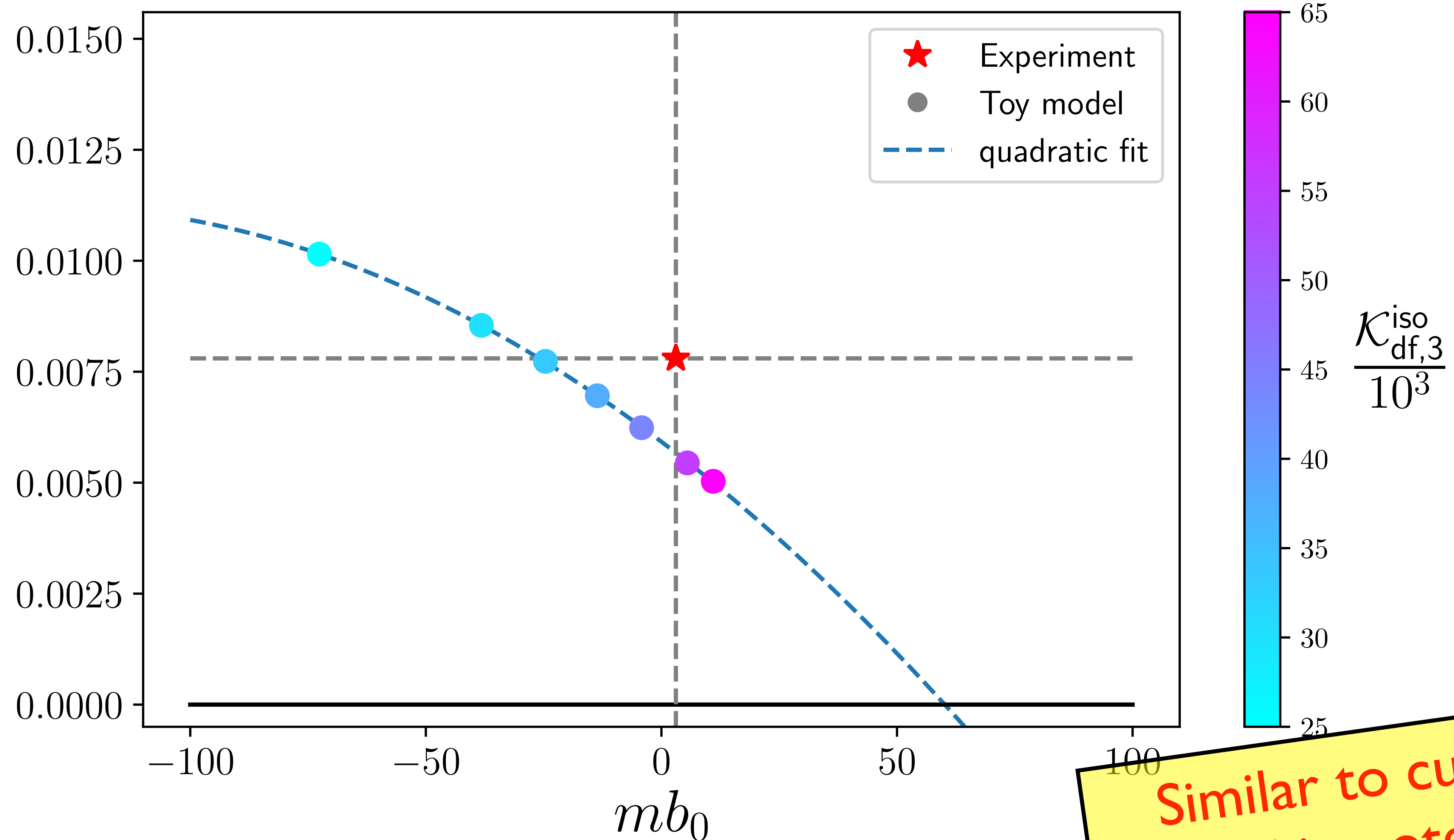


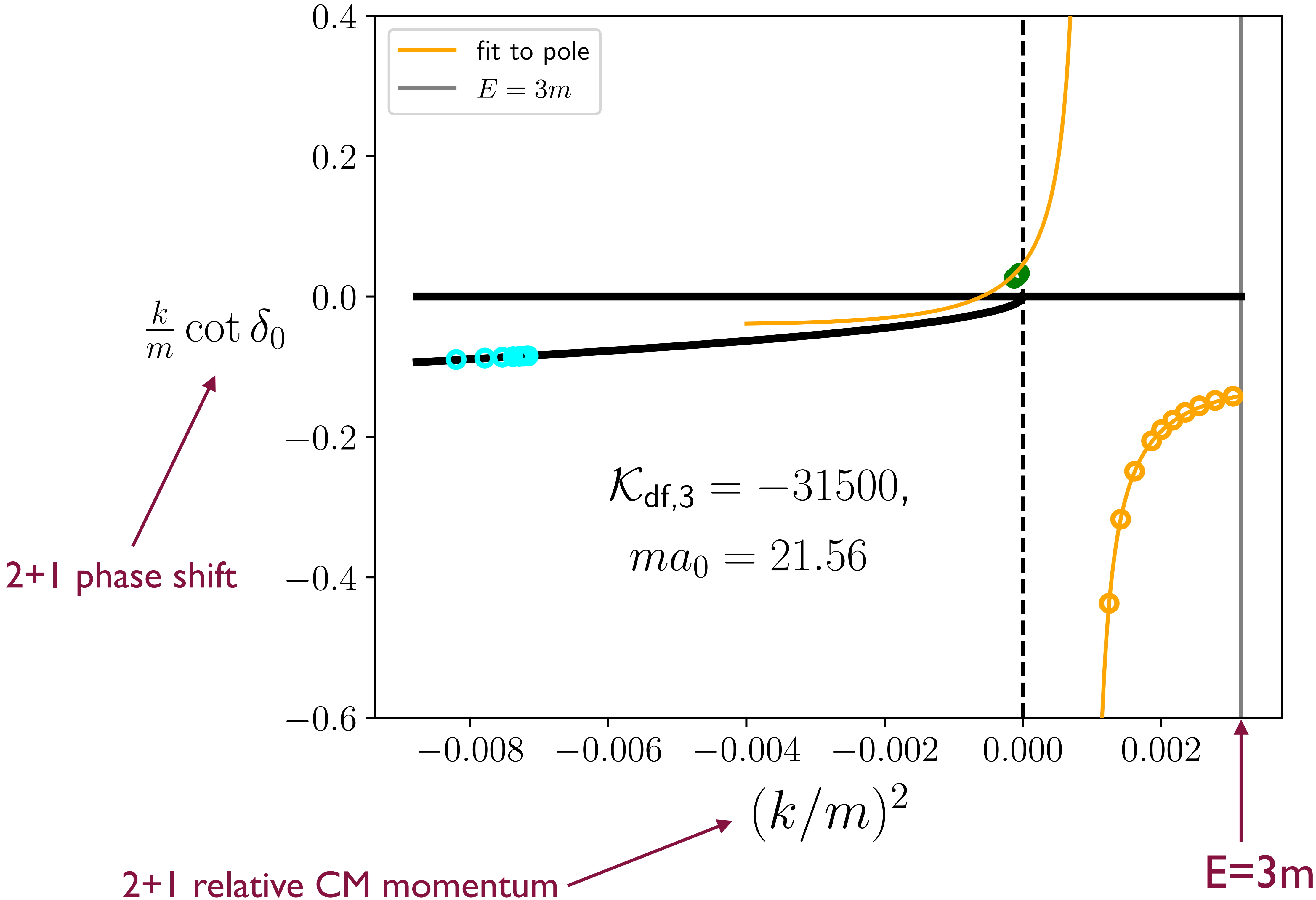
trimer binding energy

particle-dimer scattering length

Similar to curve found in potential models and chiral EFT

Toy N+D / Tritium system

Choose parameters so that $m_{trimer} : m_{dimer} : m = M_T : M_D : M$



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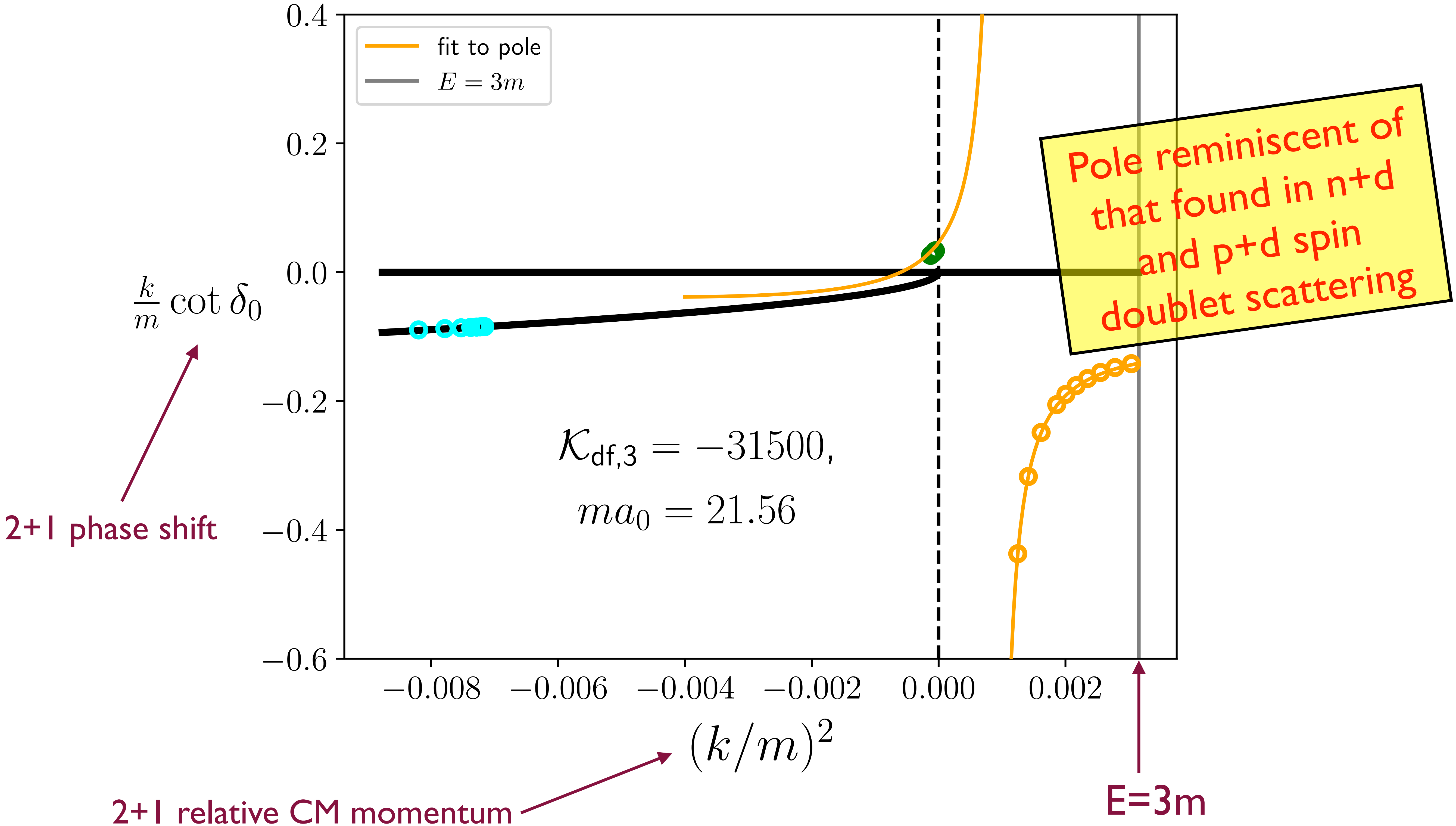

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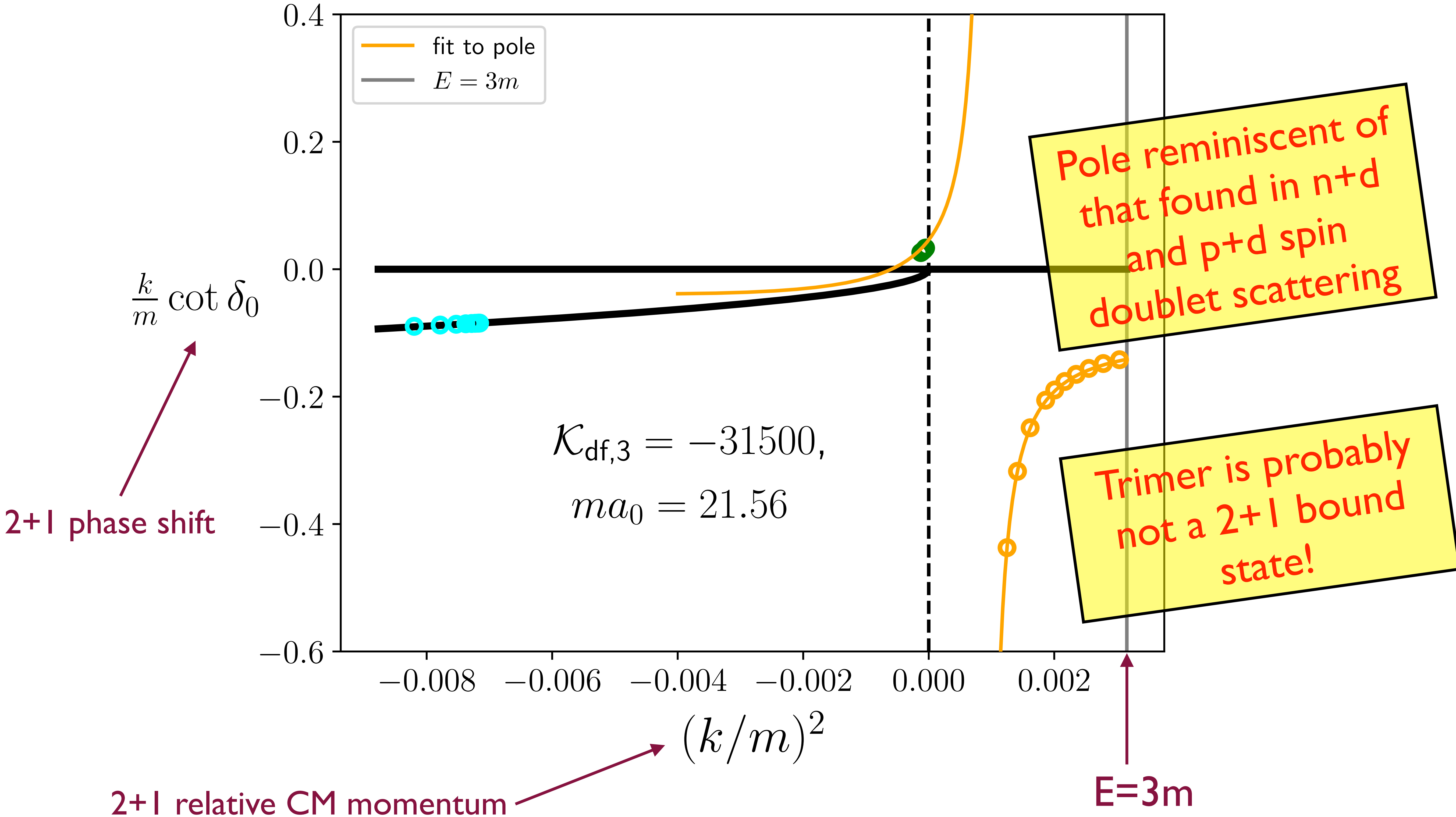

Conclusions & Outlook

Status

- Simplest case is ready to use!
  - Identical spinless particles with a Z_2 symmetry: applies to $3\pi^+$
  - LQCD results for $3\pi^+$ from [Hanlen & Hörz, 19]
  - Applied to results from φ^4 theory [Romero-López et al., 18]
- Reasonable understanding of relationship between approaches [BHREV19]
- Unitarity of parametrization of $\mathcal{M}_3$ has been demonstrated [BHSS19], and equivalence to B-matrix parametrization shown [Jackura et al, 19]
  - BHS parametrizations may be useful to analyze scattering data

To-do list for QC3

- Generalize formalism to broaden applications ("straightforward")
  - Degenerate particles with isospin, for, e.g., $\omega \to 3\pi$ in isosymmetric QCD
  - Nondegenerate particles with spin for, e.g., N(1440)
  - Determination of Lellouch-Lüscher factors to allow application to $K \to 3\pi$ etc
- Understand appearance of unphysical solutions (wrong residue) for some values of parameters—observed in [BHS18; BRS19]
  - May be due to truncation, or due to exponentially suppressed effects, or both
  - Can investigate the latter by varying the cutoff function [BBHRS, in progress]
- Develop physics-based parametrizations of $\mathcal{K}_{df,3}$ to describe resonances
  - Use relation of $\mathcal{K}_{df,3}$ to alternative K matrices derived in [Jackura, SS, et al., 19]?
  - Need to learn how to relate $\mathcal{K}_{df,3}$ to $\mathcal{M}_3$ above threshold
- Move on to QC4 !?

Thank you!
Questions?

Backup slides