QED corrections to leptonic meson decays

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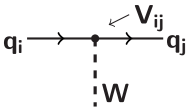
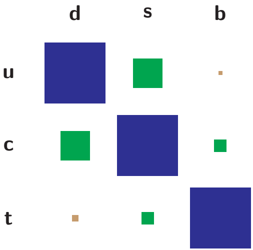
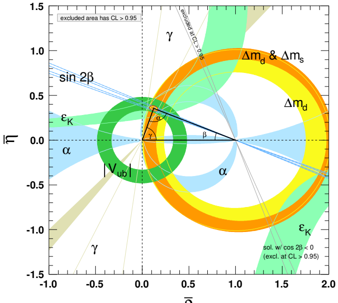
Introduction

- quark-mixing Cabibbo–Kobayashi–Maskawa (CKM) matrix

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

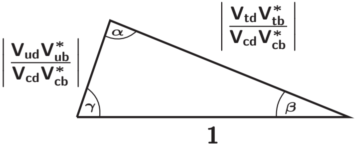






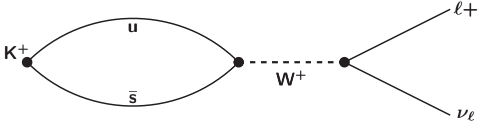
[PDG]

- Unitarity of the CKM matrix

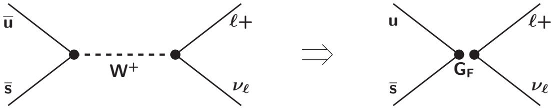


V_{us} from leptonic Kaon decays

- leptonic Kaon decay $K^+ \to \ell^+ \nu_\ell$



- effective weak Hamiltonian



- decay rate (can be measured experimentally)

$$\Gamma(K^+ \to \ell^+ \nu_\ell) = \frac{G_F^2 \, |V_{us}|^2 \, f_K^2}{8\pi} \, M_K \, m_\ell^2 \left(1 - \frac{m_\ell^2}{M_K^2}\right)^2$$

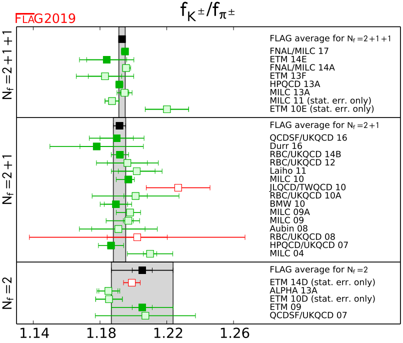
- known factors (Fermi constant G_F, masses m)
- kaon decay constant f_K, can be calculated on the lattice
- CKM matrix element V_{us}

f_K/f_π from the lattice

- pseudoscalar meson decay constant from the lattice
- axial-vector matrix element

$$\mathcal{A}_K = \langle 0|\, \bar{u}\gamma_0\gamma_5 s\, |K\rangle = M_K f_K$$

- overview Kaon/Pion decay constants



- results with precision $< \mathbf{1\%}$

Isospin Breaking Corrections

- lattice calculations usually done in the isospin symmetric limit
- two sources of isospin breaking effects
  - different masses for up- and down quark (of $\mathcal{O}((\mathbf{m_d} - \mathbf{m_u})/\Lambda_{\mathrm{QCD}})$)
  - Quarks have electrical charge (of $\mathcal{O}(\alpha)$)
- lattice calculation aiming at **1%** precision requires to include isospin breaking
- separation of strong IB and QED effects requires renormalization scheme
- definition of "physical point" in a "QCD only world" also scheme dependent

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- Euclidean path integral including QED

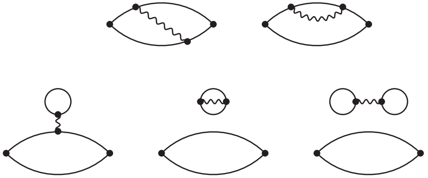
$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[\Psi, \overline{\Psi}]\mathcal{D}[U]\mathcal{D}[A]\; O\; e^{-S_F[\Psi,\overline{\Psi},U,A]}\, e^{-S_G[U]}\, e^{-S_\gamma[A]}$$

- photons in a box: finite volume corrections

Expansion around IB symmetric (eg IB corrections to meson masses)

- perturbative expansion in α [RM123 Collaboration, Phys.Rev. **D87**, 114505 (2013)]

$$\langle \mathbf{O} \rangle = \langle \mathbf{O} \rangle_{e=0} + \frac{1}{2} e^2 \frac{\partial^2}{\partial e^2} \langle \mathbf{O} \rangle \Big|_{e=0} + \mathcal{O}(\alpha^2)$$



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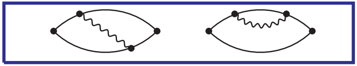
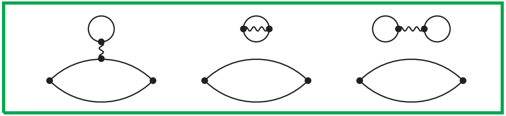


electro-quenched approximation



sea-quark effects

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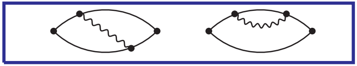

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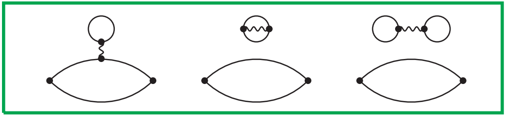
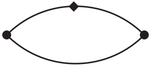


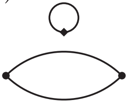
sea-quark effects

- perturbative expansion in $\Delta m_f = (m_f^0 - m_f)$ [G.M. de Divitiis *et al*, JHEP 1204 (2012) 124]

$$\langle O \rangle_{m_f} = \langle O \rangle_{m_f^0} + \Delta m_f \frac{\partial}{\partial m_f} \langle O \rangle \Big|_{m_f^0} + \mathcal{O}\left(\Delta m_f^2\right)$$



sea quark effects:
quark-disconnected diagrams



Tuning the quark masses

- isospin symmetric calculation using quark masses determined without QED
- physical quark masses including QED:

→ tune ($\mathbf{u}$,$\mathbf{d}$,$\mathbf{s}$) masses to reproduce experimental π^+, $\mathbf{K}^+$ and $\mathbf{K_0}$ mass (and check π^0 mass)

$$\mathbf{am}_{\pi^+}^{\mathrm{exp}} = \left[\mathbf{m}_{\pi}^{0} + \alpha\mathbf{m}_{\pi^+}^{\mathrm{QED}} + \Delta\mathbf{m}_{\mathbf{d}}\ \mathbf{m}_{\pi^+}^{\Delta\mathbf{m}_{\mathbf{d}}} + \Delta\mathbf{m}_{\mathbf{u}}\ \mathbf{m}_{\pi^+}^{\Delta\mathbf{m}_{\mathbf{u}}}\right]$$

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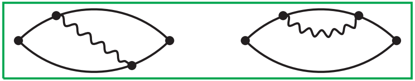

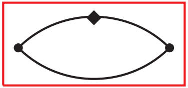

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- lattice spacing **a**: fix another mass including QED
  → e.g. Omega-Baryon (**sss**)

$$\mathbf{a} \to \mathbf{a}(\Delta \mathbf{m}_s) = \left(\mathbf{m}_\Omega^0 + \alpha \mathbf{m}_\Omega^{\mathrm{QED}} + 3\,\Delta \mathbf{m}_s\ \mathbf{m}_\Omega^{\Delta m_s}\right) / \mathbf{m}_\Omega^{\exp}$$

→ shift in **a** smaller then statistical error on lattice spacing
[T. Blum, VG,*et al.*, Phys. Rev. Lett. 121, 022003 (2018)]

Decay rate leptonic meson decays

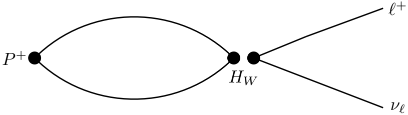
- $\mathbf{P^+}$ decay rate in rest frame ($\mathbf{P} = \{\pi, \mathbf{K}\}$)

$$\Gamma(\mathrm{P}^+ \to \ell^+ \nu_\ell) = \mathrm{K} \sum_{\mathrm{r,s}} |\mathcal{M}^{\mathrm{r,s}}|^2$$

summed over spins $\mathbf{r}, \mathbf{s}$ of final state

- matrix element

$$\mathcal{M}^{\mathrm{r,s}} = \langle \ell^+, \mathrm{r}; \nu_\ell, \mathrm{s} | \mathrm{H_W} | \mathrm{P}^+ \rangle = \overline{\mathrm{u}}^{\mathrm{r}}_{\nu_\ell} \, \widetilde{\mathcal{M}} \, \mathrm{v}^{\mathrm{s}}_\ell$$



- weak Hamiltonian $\mathbf{H_W}$
- tree-level matrix element (hadronic and leptonic part factorisable)

$$\mathcal{M}_0^{\mathrm{r,s}} = \mathrm{f_P} \, \mathrm{M_P} \, \left(\overline{\mathrm{u}}^{\mathrm{r}}_{\nu_\ell} \, \gamma_{\mathrm{L}}^{\mu} \, \mathrm{v}^{\mathrm{s}}_\ell \right) \qquad \gamma_{\mathrm{L}}^{\mu} = \gamma_\mu (1 - \gamma_5)$$

Decay rate leptonic meson decays

- tree-level decay rate

$$\Gamma^0(P^+ \to \ell^+ \nu_\ell) = \frac{G_F^2 \, |V_{us}|^2 \, f_P^2}{8\pi} \, M_P \, m_\ell^2 \left(1 - \frac{m_\ell^2}{M_P^2}\right)^2$$

- full QCD+QED decay rate

$$\Gamma = \Gamma^0 + \delta\Gamma = \Gamma^0(1 + \delta R) \qquad\qquad \delta R = \delta\Gamma / \Gamma_0$$

- first order $\mathcal{O}(\alpha, m_d - m_u)$ in isospin breaking

$$\delta\Gamma = \delta K \sum_{r,s} \left|\mathcal{M}_0^{r,s}\right|^2 + 2K_0 \sum_{r,s} \Re(\mathcal{M}_0^{r,s} \; \delta\mathcal{M}^{r,s,*})$$

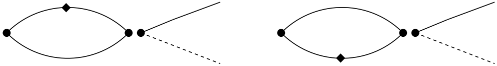
with

$$\mathcal{M}_0^{r,s} = f_P \, M_P \; \left(\overline{u}_{\nu_\ell}^r \, \gamma_L^\mu \, v_\ell^s\right) \qquad\qquad \delta\mathcal{M}^{r,s} = \overline{u}_{\nu_\ell}^r \, \delta\widetilde{\mathcal{M}} \, v_\ell^s$$

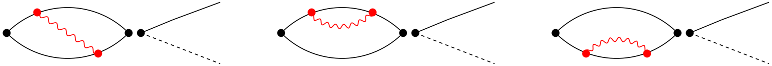
$$\Rightarrow \qquad \sum_{r,s} \Re(\mathcal{M}_0^{r,s} \; \delta\mathcal{M}^{r,s,*}) = f_P \, M_P \, \mathrm{Tr}[\not{p}_\nu \delta\widetilde{\mathcal{M}}(-\not{p}_\ell + i m_\ell)\gamma_L^\mu]$$

perturbative expansion - leptonic meson decay

- strong IB corrections $\mathcal{O}(\mathbf{m_d - m_u})$



- quark QED corrections $\mathcal{O}(\mathbf{e_q^2})$



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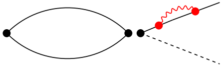
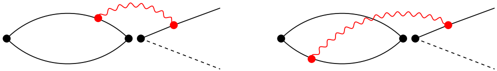


- quark-lepton QED correction $\mathcal{O}(\mathbf{e_\ell e_q})$



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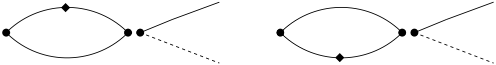

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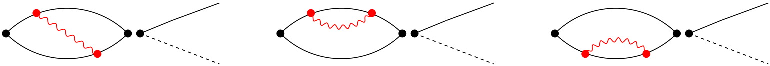

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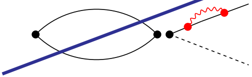
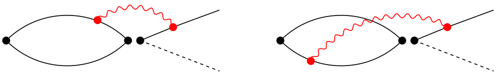


$\longrightarrow$ absorbed in renormalisation of lepton

- quark-lepton QED correction $\mathcal{O}(\mathbf{e_\ell e_q})$



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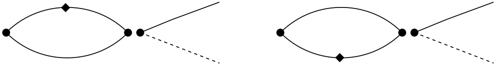

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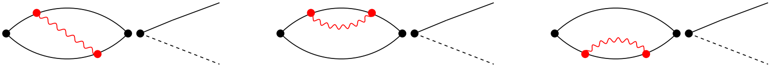

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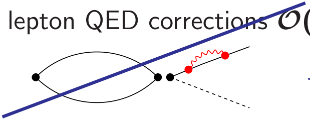

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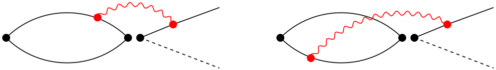

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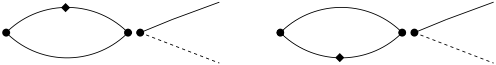

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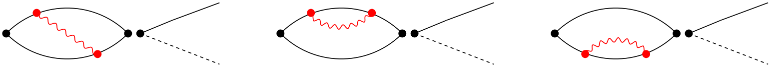
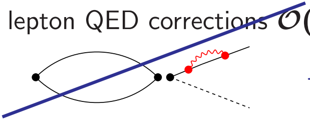
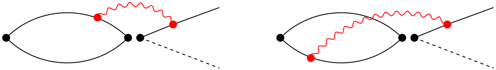


- ~~lepton QED corrections $\mathcal{O}(\mathbf{e_\ell^2})$~~



$\longrightarrow$ absorbed in renormalisation of lepton

- quark-lepton QED correction $\mathcal{O}(\mathbf{e_\ell e_q})$



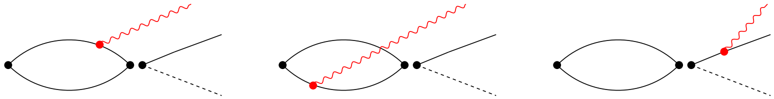
factorisable

non-factorisable

IB corrections to leptonic meson decay

- Infrared divergencies cancled by diagrams with one final state photon

$$\Gamma(K^+ \to \ell^+\nu_\ell, \alpha) + \Gamma(K^+ \to \ell^+\nu_\ell\gamma)$$



- pioneering work to calculate IB correction to decay rate by RM123
  - formalism developed in [N. Carrasco *et al*, Phys.Rev. **D91**, 074506 (2015)]
  - finite volume effects [V. Lubicz *et al*, Phys. Rev. **D95**, 034504 (2017)]
  - first lattice results [M. Di Carlo *et al*, arXiv:1904.08731], [D. Giusti *et al*, Phys. Rev. Lett. **120**, 072001 (2018)]
- this work: calculation directly at the physical point

Lattice Setup

- $N_f = 2 + 1$ Möbius Domain Wall Fermions
- near physical quark masses
- inverse lattice spacing $a^{-1} = 1.730(4)$ GeV
- $48^3 \times 96$ with $L_s = 24$
- valence light quarks: physical mass z-Möbius DWF with $L_s = 10$

- Feynman gauge and QED_L for photon propagators

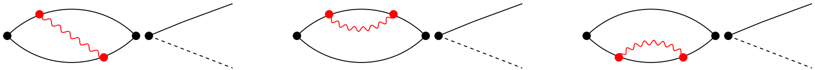
$$\Delta_{\mu\nu}(x - y) = \langle A_\mu(x) A_\nu(y) \rangle = \delta_{\mu\nu} \frac{1}{N} \sum_{k, \vec{k} \neq 0} \frac{e^{ik \cdot (x-y)}}{\hat{k}^2}$$

- use stochastic photon fields $A_\mu(x)$ to estimate $\Delta_{\mu\nu}(x - y)$
  [D. Giusti et al. Phys.Rev. D95 (2017) 114504]
- em vertices using local vector currents $\gamma_\mu A_\mu = \not{A}$

- all results shown in this talk are very preliminary

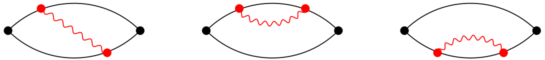
factorisable QED diagrams

- factorisable diagrams QED correction



$\rightarrow$ hadronic and leptonic part can be factorised (as in tree-level)

- only need to calculate



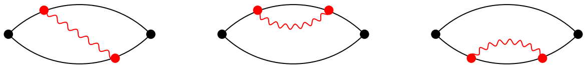
- IB correction from factorisable diagrams

$$\delta^{qq}\mathcal{M}^{rs} = \left(\bar{u}^r_{\nu_\ell}\,\gamma^\mu_L\,v^s_\ell\right)\delta\mathcal{A}$$

with

$$\delta\mathcal{A} = \delta\langle 0|\bar{q}_1\gamma_0\gamma_5 q_2|P^+\rangle$$

IB correction from factorisable diagrams



- correlators w/o QED (example: Kaon)

$$C^0_{PP}(t) = \langle 0|(\bar{s}\gamma_5 u)(\bar{u}\gamma_5 s)|0\rangle = A_0\, e^{-m_0 t} \qquad A_0 = \frac{\phi_0^2}{2m_0}$$

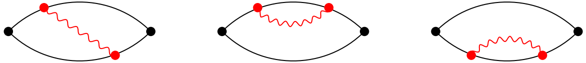
$$C^0_{AP}(t) = \langle 0|(\bar{s}\gamma_0\gamma_5 u)(\bar{u}\gamma_5 s)|0\rangle = B_0\, e^{-m_0 t} \qquad B_0 = \frac{\phi_0 \mathcal{A}_0}{2m_0}$$

- $\mathcal{O}(\alpha)$ QED corrections

$$\frac{\delta C_{PP}(t)}{C^0_{PP}(t)} = \frac{\delta A}{A_0} - \delta m\, t = 2\frac{\delta\phi}{\phi_0} - \frac{\delta m}{m_0} - \delta m\, t$$

$$\frac{\delta C_{AP}(t)}{C^0_{AP}(t)} = \frac{\delta B}{B_0} - \delta m\, t = \frac{\delta\phi}{\phi_0} + \frac{\delta\mathcal{A}}{\mathcal{A}_0} - \frac{\delta m}{m_0} - \delta m\, t$$

IB correction from factorisable diagrams



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$$\mathbf{C^0_{AP}(t) = \langle 0|(\bar{s}\gamma_0\gamma_5 u)(\bar{u}\gamma_5 s)|0\rangle = B_0\, e^{-m_0 t}} \qquad \mathbf{B_0 = \frac{\phi_0 \mathcal{A}_0}{2m_0}}$$

- $\mathcal{O}(\alpha)$ QED corrections

$$\frac{\delta C_{PP}(t)}{C^0_{PP}(t)} = \frac{\delta A}{A_0} - \delta m\, t = 2\frac{\delta\phi}{\phi_0} - \frac{\delta m}{m_0} - \delta m\, t$$

$$\frac{\delta C_{AP}(t)}{C^0_{AP}(t)} = \frac{\delta B}{B_0} - \delta m\, t = \frac{\delta\phi}{\phi_0} + \boxed{\frac{\delta \mathcal{A}}{\mathcal{A}_0}} - \frac{\delta m}{m_0} - \delta m\, t$$

IB correction from factorisable diagrams

- correlators w/o QED (including backward propagating signal)

$$C^0_{PP}(t) = 2\,A_0\,e^{-m_0\frac{T}{2}}\cosh\left[m_0\left(\frac{T}{2}-t\right)\right]$$

$$C^0_{AP}(t) = 2\,B_0\,e^{-m_0\frac{T}{2}}\sinh\left[m_0\left(\frac{T}{2}-t\right)\right]$$

- $\mathcal{O}(\alpha)$ QED corrections

$$\frac{\delta C_{PP}(t)}{C^0_{PP}(t)} = \frac{\delta A}{A_0} - \delta m\frac{T}{2} + \delta m\left(\frac{T}{2}-t\right)\tanh\left[m_0\left(\frac{T}{2}-t\right)\right]$$

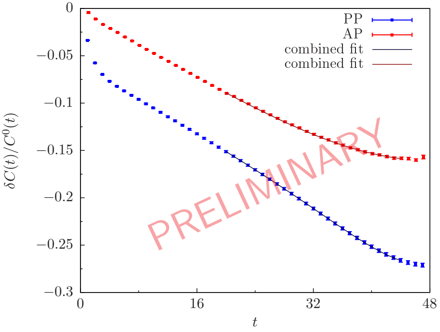
$$\frac{\delta C_{AP}(t)}{C^0_{AP}(t)} = \frac{\delta B}{B_0} - \delta m\frac{T}{2} + \delta m\left(\frac{T}{2}-t\right)\coth\left[m_0\left(\frac{T}{2}-t\right)\right]$$

results QED factorisable diagrams

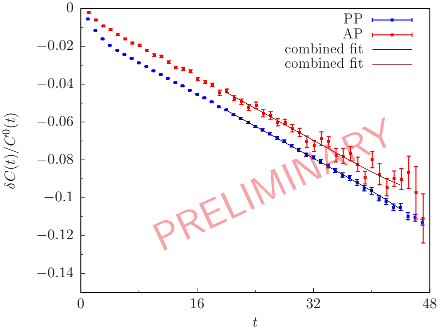
- 20 configurations on the physical point ensemble
- combined fit to $\delta C_{PP}(t)/C^0_{PP}(t)$ and $\delta C_{AP}(t)/C^0_{AP}(t)$

  $\longrightarrow$ three parameters δm, $\delta A/A_0$, $\delta B/B_0$

pion



kaon



non-factorisable diagrams

- non-factorisable diagrams

- include lepton in lattice calculation
- neutrino can be done anlytically
- amputated weak Hamiltonian and matrix element

$$\overline{\mathbf{H}}_{\mathbf{W}}^{\alpha} = (\gamma_\mu^{\mathbf{L}}\ell)^\alpha\,(\overline{\mathbf{q}}_1\gamma_\mu^{\mathbf{L}}\mathbf{q}_2) \qquad \overline{\mathcal{M}}^{\mathbf{r},\alpha} = \langle \ell^+, \mathbf{r}|\overline{\mathbf{H}}_{\mathbf{W}}^{\alpha}|\mathbf{P}^+\rangle = (\widetilde{\mathcal{M}}\mathbf{v}_\ell^{\mathbf{r}})^\alpha$$

- Euclidean three-point function (in full QCD+QED)

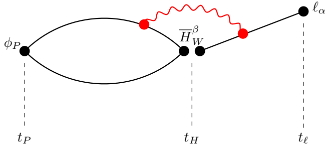
$$\mathbf{C}^{\alpha\beta}(\mathbf{t}_\ell, \mathbf{t}_{\mathbf{H}}, \mathbf{t}_{\mathbf{P}}) = \left\langle \overline{\ell}^{\alpha}(\mathbf{t}_\ell)\,\overline{\mathbf{H}}_{\mathbf{W}}^{\beta}(\mathbf{t}_{\mathbf{H}})\,\phi_{\mathbf{P}}^{\dagger}(\mathbf{t}_{\mathbf{P}})\right\rangle$$

- at $\mathcal{O}(\alpha)$

$$\mathbf{C}^{\alpha\beta}(\mathbf{t}_\ell, \mathbf{t}_{\mathbf{H}}, \mathbf{t}_{\mathbf{P}}) = \mathbf{C}_0^{\alpha\beta}(\mathbf{t}_\ell, \mathbf{t}_{\mathbf{H}}, \mathbf{t}_{\mathbf{P}}) + \delta\mathbf{C}^{\alpha\beta}(\mathbf{t}_\ell, \mathbf{t}_{\mathbf{H}}, \mathbf{t}_{\mathbf{P}}) + \mathcal{O}(\alpha^2)$$

lattice setup non-factorisable diagrams

- lattice calculation



- correlation function

$$\delta^{\ell q} C^{\alpha\beta}(t_\ell, t_H, t_P) = \sum_{y,z} \sum_{\vec{x}_P, \vec{x}_H} \mathrm{Tr}\left[\gamma_5 S^u(x_P, y) \not{A}(y) S^u(y, x_H) \gamma_\mu^L S^s(x_H, x_P)\right] \times \left(\gamma_\mu^L S^\ell(x_H, z) \not{A}(z) S^\ell(z, x_\ell)\right)_{\alpha\beta}$$

- recap: QED correction to decay rate

$$\delta\Gamma \sim \sum_{r,s} \Re(\mathcal{M}_0^{r,s}\ \delta\mathcal{M}^{r,s,*})$$

$$\sum_{r,s} \Re(\mathcal{M}_0^{r,s}\ \delta\mathcal{M}^{r,s,*}) = f_P\, M_P\, \mathrm{Tr}[\not{p}_\nu \delta\widetilde{\mathcal{M}}(-\not{p}_\ell + i m_\ell)\gamma_L^\mu]$$

extract correction to decay rate

- spectral representation

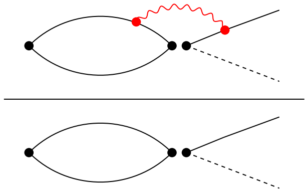
$$\delta^{\ell q} C^{\alpha\beta}(t_\ell, t_H, t_P) = \frac{\phi_0 \left[\delta^{\ell q}\widetilde{\mathcal{M}}\left(-\not{p}_\ell + i m_\ell\right)\right]_{\alpha\beta}}{4 E_\ell M_P}\, e^{-(t_H - t_P) M_P}\, e^{-(t_\ell - t_H) E_\ell}$$

- include the lepton trace

$$\mathrm{Tr}\left[\not{p}_\nu \delta^{\ell q} C(t_H) \gamma_L^\mu\right] \propto \mathrm{Tr}\left[\not{p}_\nu \delta^{\ell q}\widetilde{\mathcal{M}}\left(-\not{p}_\ell + i m_\ell\right)\gamma_L^\mu\right]$$

- $\delta^{\ell q}\Gamma / \Gamma_0$ can be obtained from long distance behaviour of

$$R(t_\ell, t_H, t_P) = \frac{\mathrm{Tr}\left[\not{p}_\nu \delta^{\ell q} C(t_\ell, t_H, t_P)\gamma_L^\mu\right]}{\mathrm{Tr}\left[\not{p}_\nu C^0(t_\ell, t_H, t_P)\gamma_L^\mu\right]}$$



extract correction to decay rate

part of correction to decay rate

- spectral representation

$$\delta^{\ell q} C^{\alpha\beta}(t_\ell, t_H, t_P) = \frac{\phi_0 \left[\delta^{\ell q} \widetilde{\mathcal{M}} \left(-\not{p}_\ell + i m_\ell\right)\right]_{\alpha\beta}}{4 E_\ell M_P} e^{-(t_H - t_P) M_P} e^{-(t_\ell - t_H) E_\ell}$$

- include the lepton trace

$$\mathrm{Tr}\left[\not{p}_\nu \delta^{\ell q} C(t_H) \gamma_L^\mu\right] \propto \mathrm{Tr}\left[\not{p}_\nu \delta^{\ell q} \widetilde{\mathcal{M}} \left(-\not{p}_\ell + i m_\ell\right) \gamma_L^\mu\right]$$

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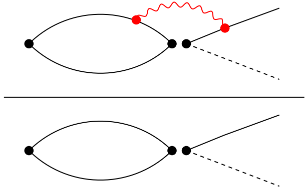
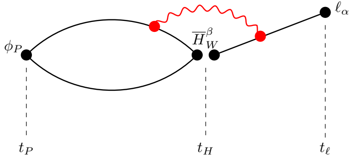


lattice setup non-factorisable diagrams

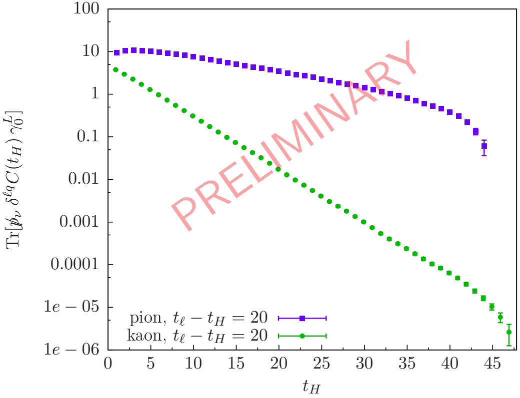
- lattice calculation



- 7 configurations on the physical point ensemble
- **96** source positions for $\mathbf{t_P}$ (**Z2**-Wall sources)
  $\rightarrow$ bin 8 consecutive source positions
- $\mathbf{t_\ell - t_H}$ fixed (varied between $\mathbf{12 - 40}$ to check for systematic effects)
- lepton: free Domain Wall Fermion with muon mass as pole mass
- twisted boundary conditions for muon for energy/momentum conservation

Results QED diagrams

- $\mathrm{Tr}\left[\not{p}_\nu \delta^{\ell q} \mathbf{C}(\mathbf{t_H}) \gamma_0^L\right] \propto \mathrm{Tr}\left[\not{p}_\nu \delta^{\ell q} \widetilde{\mathcal{M}} \left(-\not{p}_\ell + \mathbf{i}\mathbf{m}_\ell\right) \gamma_0^L\right]$



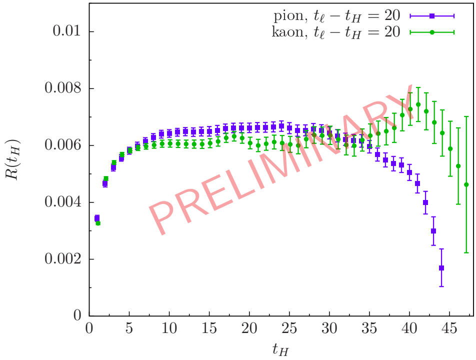
- decays exponentially with pion/kaon mass

Results ratio

- ratio of QED diagram over tree-level diagram

$$\mathbf{R(t_H)} = \frac{\text{Tr}\left[\not{p}_\nu \delta^{\ell \mathbf{q}} \mathbf{C(t_H)} \gamma_0^{\mathbf{L}}\right]}{\text{Tr}\left[\not{p}_\nu \mathbf{C^0(t_H)} \gamma_0^{\mathbf{L}}\right]} \quad \longrightarrow \quad \delta^{\ell \mathbf{q}} \Gamma / \Gamma_0$$

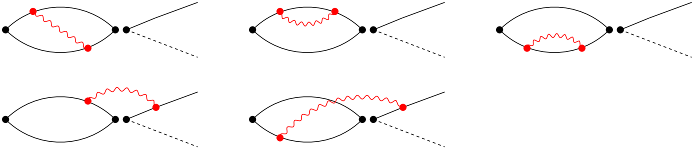
- result physical point ensemble



Summary

- lattice determinations of $\mathbf{f_K}$, $\mathbf{f_\pi}$ have reached precision of $\lesssim$ **1%**
  $\rightarrow$ isospin breaking correction become important
  $\rightarrow$ necessary to improve determination of CKM matrix elements

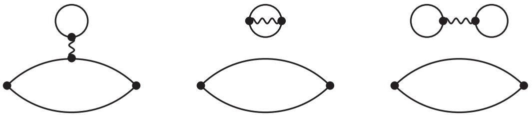
- preliminary results for QED corrections to leptonic meson decays
  - physical point ensemble
  - factorisable and non-factorisable QED corrections to decay rate



  - low statistics, but results look encouraging

Outlook

- work in progress
  - increase statistics and complete analysis
  - disconnected diagrams & sea-quark effects, e.g.



- future work
  - renormalisation of the weak Hamiltonian including QED
  - diagrams with final state photon
  - semi-leptonic meson decays $\mathbf{K} \to \pi\ell\nu$

Thank you

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 757646.

Backup

The zero-mode of the photon field

- zero-mode of the photon field
  shift symmetry of the of the photon action $\mathbf{A}_\mu(\mathbf{x}) \to \mathbf{A}_\mu(\mathbf{x}) + \mathbf{c}_\mu$
  $\to$ cannot be constrained by gauge fixing

- different prescriptions of QED:
- $\mathrm{QED}_{\mathbf{TL}}$: remove the zero-mode of the photon field, i.e. $\tilde{\mathbf{A}}_\mu(\mathbf{k}=0)=0$
  [A. Duncan, E. Eichten, H. Thacker, Phys.Rev.Lett. **76**, 3894 (1996)]
- $\mathrm{QED}_{\mathbf{L}}$: remove all the spatial zero-modes, i.e. $\tilde{\mathbf{A}}_\mu(\mathbf{k_0}, \vec{\mathbf{k}}=0)=0$
  [S. Uno and M. Hayakawa, Prog. Theor. Phys. **120**, 413 (2008)]
- $\mathrm{QED}_{\mathbf{m}}$: use a massive photon and take $\mathbf{m}_\gamma \to \mathbf{0}$
  [M. Endres *et al.*,Phys. Rev. Lett. **117** (2016) 072002]
- $\mathrm{QED}_{\mathbf{C}}$: $\mathbf{C}^*$ boundary conditions in spatial direction, i.e. fields are periodic up to charge conjugation [B. Luchini *et al.* JHEP **02** (2016) 076]

all-to-all propagators and meson fields

- all-to-all propagator [J. Foley *et al*, Comput.Phys.Commun. **172**, 145-162 (2005)]

$$D^{-1}(x,y) = \sum_{i=1}^{N_l+N_h} v_i(x) w_i^\dagger(y) = \sum_{i=1}^{N_l} v_i(x) w_i^\dagger(y) + \sum_{i=N_l+1}^{N_l+N_h} v_i(x) w_i^\dagger(y)$$

- all-to-all vectors $v_i(x)$, $w_i(x)$

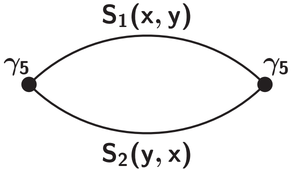
low-modes (eigenvectors $\phi(x)$)

$$v_i(x) = \phi_i(x)$$
$$w_i(x) = \phi_i(x)/\lambda_i$$

high-modes (from stochastic solves)

$$v_i(x) = D^{-1}_{\mathrm{defl}}(x,y)\eta_i(y)$$
$$w_i(x) = \eta_i(x)$$

- e.g. pseudoscalar two-point function



$$C(y_0 - x_0) = \sum_{\vec{x},\vec{y}} \mathrm{Tr}\left[\gamma_5 S_1(x,y)\, \gamma_5 S_2(y,x)\right]$$

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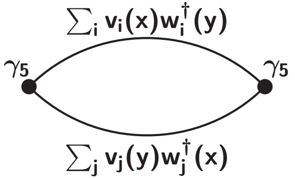

$$\mathbf{C(y_0 - x_0) = \sum_{\vec{x},\vec{y}} \mathrm{Tr}\Big[\gamma_5 \sum_i v_i(x) w_i^\dagger(y)\, \gamma_5 \sum_j v_j(y) w_j^\dagger(x)\Big]}$$

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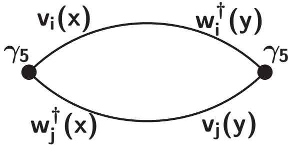

$$C(y_0 - x_0) = \sum_{i,j} \mathrm{Tr}\Big[\sum_{\vec{y}} w_i^\dagger(y)\gamma_5 v_j(y) \ \sum_{\vec{x}} w_j^\dagger(x)\gamma_5 v_i(x)\Big]$$

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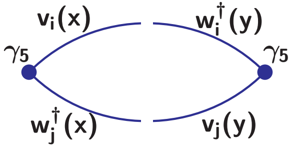


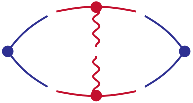
Meson field

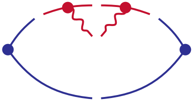
$$\Pi_{ji}(x_0;\Gamma) = \sum_{\vec{x}} w_j^\dagger(x)\Gamma v_i(x)$$

$$C(y_0 - x_0) = \sum_{i,j} \mathrm{Tr}\left[\Pi_{ij}(y_0;\gamma_5)\ \Pi_{ji}(x_0;\gamma_5)\right]$$

QED meson fields

- $\not{A}$-meson fields $\Pi_{ji}(x_0; \not{A}) = \sum_{\vec{x}} w_j^\dagger(x) i\gamma_\mu A^\mu(x) v_i(x)$

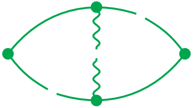


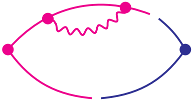


- sequential propagators on **v**-vectors & sequential $\not{A}$-meson fields

$$\tilde{v}_i(x) = \sum_{y} S(x,y) i\gamma_\mu A^\mu(y) v_i(y) \qquad \Pi_{ji}(x_0; \Gamma S \not{A}) = \sum_{\vec{x}} w_j^\dagger(x) \Gamma \tilde{v}_i(x)$$

$$\tilde{\tilde{v}}_i(x) = \sum_{y} S(x,y) i\gamma_\mu A^\mu(y) \tilde{v}_i(y) \qquad \Pi_{ji}(x_0; \Gamma S \not{A} S \not{A}) = \sum_{\vec{x}} w_j^\dagger(x) \Gamma \tilde{\tilde{v}}_i(x)$$

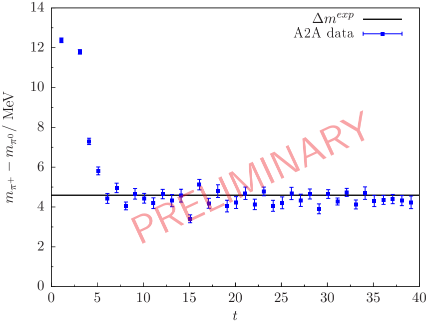




- sequential $\not{A}$-meson fields give smaller statistical errors
- $\not{A}$-meson fields can be used for disconnected diagrams (sea-quark effects)
  [see talk by J. Richings, Lattice 2019]

QED corrections to meson masses

- **2000** low-modes for light quark, **96** $\times$ **12** time-diluted, spin-color diagonal stochastic sources for high-modes with sequential $\not{A}$-meson fields
- difference of charged and neutral pion mass



- Finite volume corrections $\text{QED}_\mathbf{L}$ [BMW Collaboration, Science **347** (2015) 1452–1455]

$$\mathbf{m^2(L) \sim m^2 \left\{1 - q^2\alpha\left[\frac{\kappa}{mL}\left(1 + \frac{2}{mL}\right)\right]\right\}} \quad \text{with} \quad \boldsymbol{\kappa = 2.837297}$$