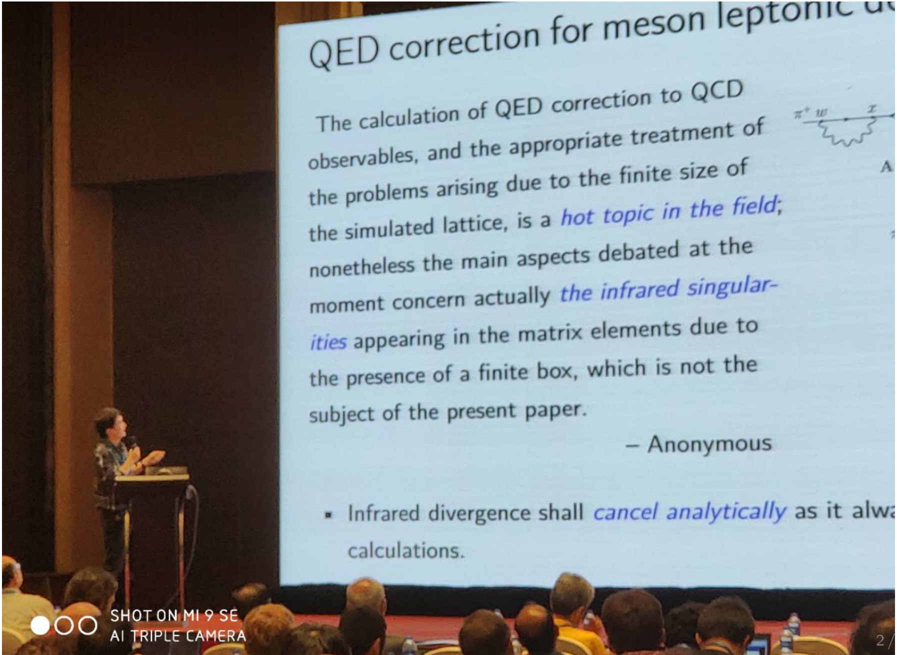
QED self energies from infinite volume reconstruction

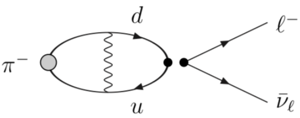
Xu Feng

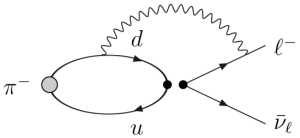
work with Norman Christ, Luchang Jin and Chris Sachrajda

Workshop on Lattice QCD, Santa Fe, 08/27/2019


QED correction for meson leptonic
The calculation of QED correction to QCD
observables, and the appropriate treatment of
the problems arising due to the finite size of
the simulated lattice, is a hot topic in the field;
nonetheless the main aspects debated at the
moment concern actually the infrared singular-
ities appearing in the matrix elements due to
the presence of a finite box, which is not the
subject of the present paper.
– Anonymous
Infrared divergence shall cancel analytically as it alw
calculations.
SHOT ON MI 9 SE
AI TRIPLE CAMERA

Long range photon on the lattice





$m_\gamma = 0 \Rightarrow$ long-range propagator enclosed in the lattice box
$\Rightarrow$ power-law finite-size effects

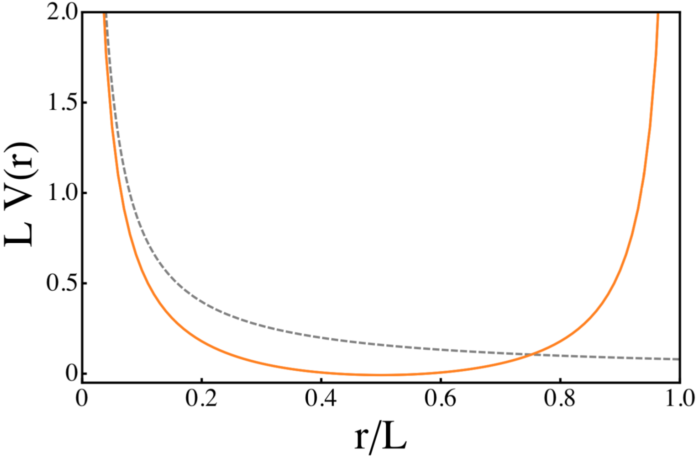
Various methods proposed to treat photon on the lattice

- $\mathrm{QED_L}$ and $\mathrm{QED_{TL}}$ [Hayakawa & Uno, 2008; S. Borsany et. al., 2015]
- Massive photon [M. Endres et. al., 2016]
- C^* boundary condition [B. Lucini et. al., 2016]
- QED_∞ and infinite-volume reconstruction $\Rightarrow$ main topic of this talk

Remove zero mode - $\mathrm{QED_L}$

Infinite volume propagator $\Rightarrow$ finite-volume propagator

$$S_\infty(x) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ipx}}{p^2} = \frac{1}{4\pi^2 x^2} \quad \Rightarrow \quad S_L(x) = \frac{1}{VT}\sum_p{}' \frac{e^{ipx}}{p^2}, \quad p = \frac{2\pi}{L} n \neq 0$$



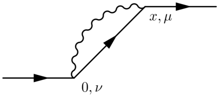
[Davoudi, Savage, PRD90 (2014) 054503]

Power-law ($1/L^n$) finite volume effect as lattice volume L increase

Starting from QED$_\infty$

XF, Luchang Jin [arXiv:1812.09817]

QED self energy



- We start with infinite volume [QED$_\infty$ method, used in HVP & HLbL]

$$\mathcal{I} = \frac{1}{2} \int d^4x\, \mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x)$$

where $\mathcal{H}_{\mu,\nu}(x)$ is the hadronic function

$$\mathcal{H}_{\mu,\nu}(x) = \mathcal{H}_{\mu,\nu}(t,\vec{x}) = \langle P | T[J_\mu(t,\vec{x}) J_\nu(0)] | P \rangle$$

$S^{\gamma}_{\mu,\nu}(x)$ is the photon propagator in the infinite volume

$$S^{\gamma}_{\mu,\nu}(x) = \frac{\delta_{\mu\nu}}{4\pi^2 x^2}$$

- Propose to replace $\mathcal{H}_{\mu,\nu}(x)$ by $\mathcal{H}^{\mathrm{lat}}_{\mu,\nu}(x)$

However, this still leads to power-law FV effects

Analysis of hadronic function

XF, Luchang Jin [arXiv:1812.09817]

- We have proposed to replace $\mathcal{H}_{\mu,\nu}(x)$ by $\mathcal{H}^{\rm lat}_{\mu,\nu}(x)$
  - $\mathcal{H}^{\rm lat}_{\mu,\nu}(x)$ mainly differs from $\mathcal{H}_{\mu,\nu}(x)$ at $x \sim L$
- The hadronic part $\mathcal{H}_{\mu,\nu}(x)$ is given by

$$\mathcal{H}_{\mu,\nu}(x) = \mathcal{H}_{\mu,\nu}(t,\vec{x}) = \langle P|T[J_\mu(t,\vec{x})J_\nu(0)]|P\rangle$$

  - $J_\mu(t,\vec{x})J_\nu(0) \quad \rightarrow \quad e^{-M\sqrt{t^2+\vec{x}}} \quad \Rightarrow \quad$ exp. suppressed
  - $\langle P|J_\mu(t,\vec{x}) \quad \rightarrow \quad e^{Mt} \quad \Rightarrow \quad$ exp. enhanced

For small $|t|$, we have exponentially suppressed FV effects:

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}) \sim e^{-M\left(\sqrt{t^2+\vec{x}^2}-t\right)} \sim e^{-M|\vec{x}|} \quad \Rightarrow \quad \mathcal{H}_{\mu,\nu}(x) - \mathcal{H}^{\rm lat}_{\mu,\nu}(x) \sim e^{-ML}$$

For large $|t|$, we shall have:

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}) \sim e^{-M\left(\sqrt{t^2+\vec{x}^2}-t\right)} \sim e^{-M\frac{\vec{x}^2}{2t}} \sim O(1)$$

Infinite volume reconstruction method

XF, Luchang Jin [arXiv:1812.09817]

Realizing at large $t > t_s$ we have ground state dominance:

$$\langle P|J_\mu(t,\vec{x})J_\nu(0)|P\rangle \sim \int \frac{d^3\vec{k}}{(2\pi)^3}\langle P|J_\mu(0)|P(\vec{k})\rangle\langle P(\vec{k})|J_\nu(0)|P\rangle e^{-E_{\vec{k}}t+Mt}e^{-i\vec{k}\cdot\vec{x}}$$

- Reconstruct $\mathcal{H}_{\mu,\nu}(t,\vec{x})$ at large t using $\mathcal{H}_{\mu,\nu}(t_s,\vec{x})$ at modest t_s

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}') \approx \int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(t_s,\vec{x}) \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} e^{-(E_{\vec{k}}-M)(t-t_s)} e^{-i\vec{p}\cdot\vec{x}'}$$

Replace

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}) \quad \Leftarrow \quad \mathcal{H}_{\mu,\nu}(t_s,\vec{x}) \quad \Leftarrow \quad \mathcal{H}^{\mathrm{lat}}_{\mu,\nu}(t_s,\vec{x})$$

The replacement only amounts for exponentially suppressed FV effects

Master formula

XF, Luchang Jin [arXiv:1812.09817]

To sum up, we split the integral $\mathcal{I}$ into two parts

$$
\begin{aligned}
\mathcal{I} &= \mathcal{I}^{(s)} + \mathcal{I}^{(l)} \\
\mathcal{I}^{(s)} &= \frac{1}{2}\int_{-t_s}^{t_s}\int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(x)S^{\gamma}_{\mu,\nu}(x) \\
\mathcal{I}^{(l)} &= \int_{t_s}^{\infty}\int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(x)S^{\gamma}_{\mu,\nu}(x) \\
&= \int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(t_s,\vec{x})L_{\mu,\nu}(t_s,\vec{x})
\end{aligned}
$$

where $L_{\mu,\nu}(t_s,\vec{x})$ is known

$$
L_{\mu,\nu}(t_s,\vec{x}) = \frac{\delta_{\mu\nu}}{2\pi^2}\int_0^{\infty} dp\,\frac{\sin(p|\vec{x}|)}{2(p+E_p-M)|\vec{x}|}e^{-pt_s}
$$

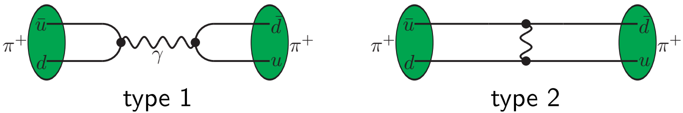
At $t \le t_s$,

$$
\mathcal{H}_{\mu,\nu}(t,\vec{x}) \quad \Leftarrow \quad \mathcal{H}^{\mathrm{lat}}_{\mu,\nu}(t,\vec{x})
$$

Ground-state dominance can be verified by the t_s dependence

Example: Pion mass splitting

$m_{\pi^+} - m_{\pi^0}$:



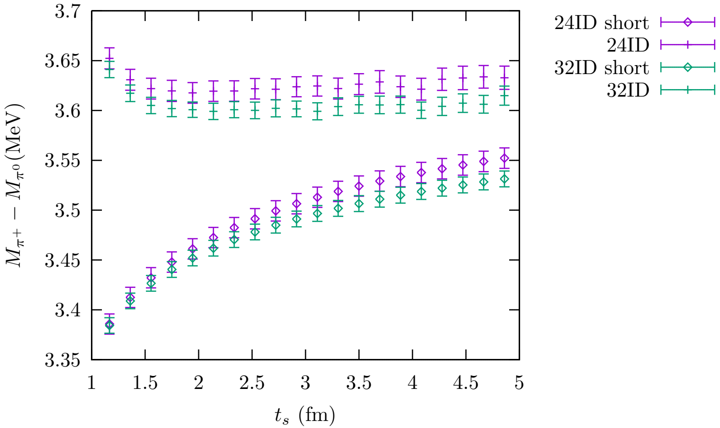
Isospin breaking effects: EM (α_e) + strong $\left(\frac{m_u - m_d}{\Lambda_{\rm QCD}}\right)$ contributions

- Strong IB breaking appears at $O\left(\left(\frac{m_u - m_d}{\Lambda_{\rm QCD}}\right)^2\right) \Rightarrow$ dominated by EM effect
- Previous calculation by RM123, 2013

$$M^2_{\pi^+} - M^2_{\pi^0} = 1.44(13)_{\rm stat}(16)_{\rm chiral} \times 10^3\ \mathrm{MeV}^2$$

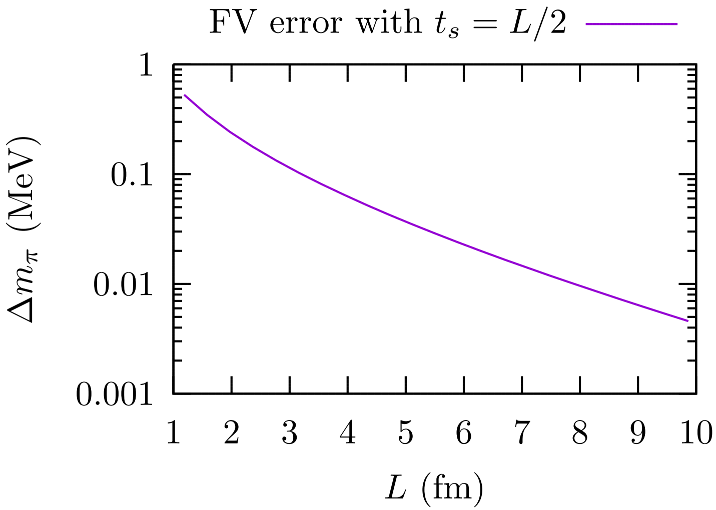
including type 2 diagram only

Using infinite-volume reconstruction



- 24ID: 142 MeV, a^{-1}=1.015 GeV, L=4.7 fm, $N_{\mathrm{conf}} = 91$
- 32ID: 142 MeV, a^{-1}=1.015 GeV, L=6.2 fm, $N_{\mathrm{conf}} = 56$
- ground state saturation at $t_s \gtrsim 1.5$ fm
- stat. error $\lesssim 0.3\%$, including both type 1 and type 2 diagrams
- residual FV effects $\Rightarrow$ $L = 4.7$ fm not large enough for phyiscal m_π

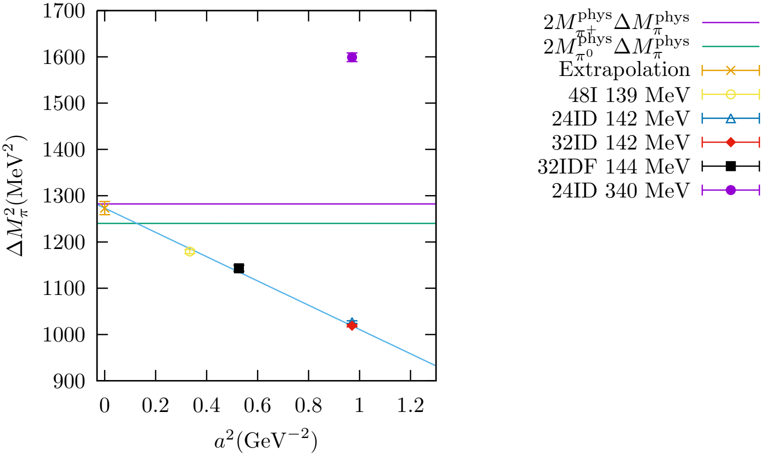
FV effects from scalar QED



FV error exponentially suppressed

Pion mass splitting

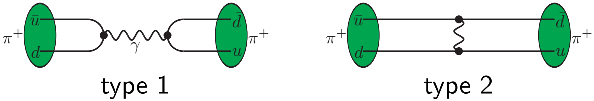
$$\Delta M_\pi^2(a, M_\pi) = \Delta M_\pi^2(0, M_\pi^{\mathrm{phys}}) + c_1 a^2 + c_2 \left(M_\pi^2 - (M_{\pi^+}^{\mathrm{phys}})^2 \right)$$



$$\Delta M_\pi^2(0, M_{\pi^+}^{\mathrm{phys}}) = 1.275(15) \times 10^3 \ \mathrm{MeV}^2$$

10 times more accurate than previous

Calculation of both type 1 and 2 diagrams



$$C_1(x-y) = \mathrm{Tr}\left[\gamma_5 S(t_i;x)\gamma_\mu S(x;t_i)\right]\mathrm{Tr}\left[\gamma_5 S(t_f;y)\gamma_\nu S(y;t_f)\right]$$
$$C_2(x-y) = \mathrm{Tr}\left[\gamma_5 S(t_f;x)\gamma_\mu S(x;t_i)\,\gamma_5 S(t_i;y)\gamma_\nu S(y;t_f)\right]$$

A general form can be written as

$$C(x-y) = H_1(x)H_2(y)$$

Double FFT allows for a spatial volume average of hadronic part $C(x)$

$$\begin{aligned} C(x) &= \frac{1}{V}\sum_{\vec{y}} H_1(x+y)H_2(y) \\ &= \frac{1}{V}\sum_{\vec{y}}\left(\frac{1}{V}\sum_{\vec{p}}\tilde{H}_1(t_x,\vec{p})e^{i\vec{p}\cdot(\vec{x}+\vec{y})}\right)\left(\frac{1}{V}\sum_{\vec{q}}\tilde{H}_2(t_y,\vec{q})e^{i\vec{q}\cdot\vec{y}}\right) \\ &= \frac{1}{V}\left(\frac{1}{V}\sum_{\vec{p}}\tilde{H}_1(t_x,\vec{p})\tilde{H}_2(t_y,-\vec{p})e^{i\vec{p}\cdot\vec{x}}\right) \end{aligned}$$

[Double FFT also proposed by Murphy & Detmold at lattice 2018]

Photon propagator is exact

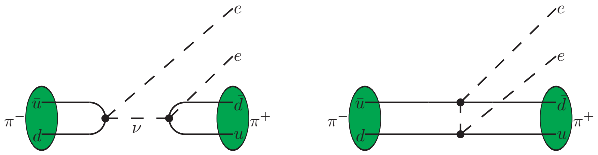
Some side products

Applications to

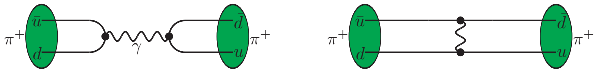
1. $0\nu 2\beta$ decay $\pi^- \to \pi^+ ee$
2. Rare K decay $K \to \pi\nu\bar{\nu}$ [Chris' talk on Monday]
3. $\pi^0 \to \gamma\gamma$, $\pi^0 \to e^+e^-$, $K_L \to \mu^+\mu^-$ [Norman' talk on Monday]

Similarity between $\pi^- \to \pi^+ ee$ and π^+-π^0 mass splitting

- $\pi^- \to \pi^+ ee$:



- $m_{\pi^+} - m_{\pi^0}$:



Different feature: $\pi^- \to \pi^+ ee$ invovles also axial vector current

- For vector current, LO & NLO FV corrections are universial

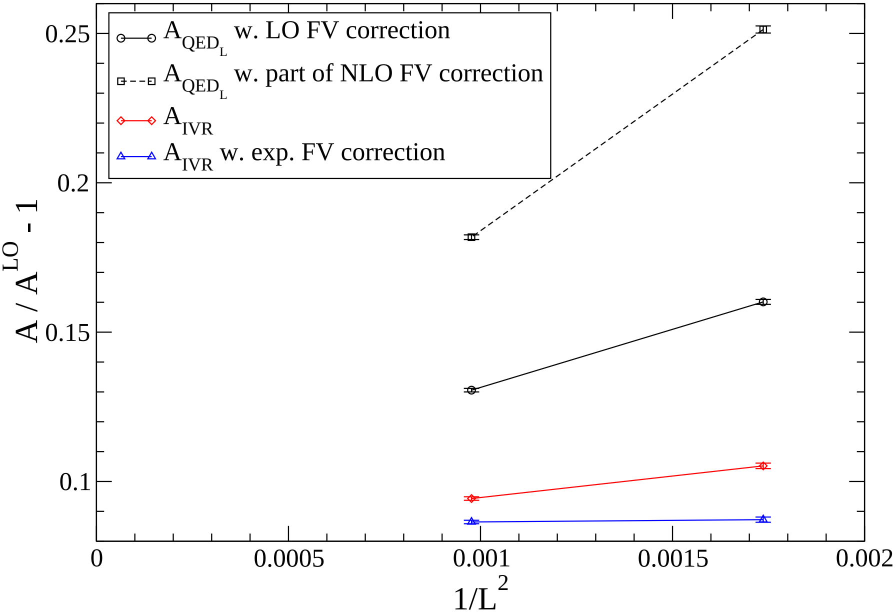
  $\Rightarrow$ described by scalar QED

  [Antonin's talk]

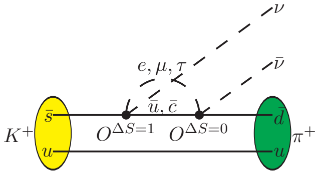
- For axial vector current, intermediate particle is a scalar state or $\pi\pi$

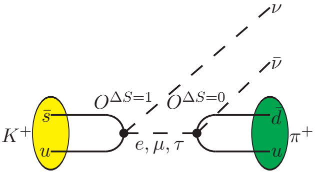
  $\Rightarrow$ non-universial FV effects

Preliminary results for $\pi^- \to \pi^+ ee$

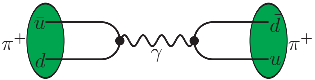


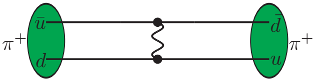
- Rare kaon decay





- $m_{\pi^+} - m_{\pi^0}$:





Different feature:

- In $m_{\pi^+} - m_{\pi^0}$, intermediate state always heavier than initial/final state
- For rare K decay, the intermediate state can be lighter

Treat with exponentially growing behavior

[Chris' talk on Monday]

- We start with a model

$$\mathcal{I} = \int d^4x\, \mathcal{H}(x) S_\ell(x)$$

where initial/final state given by K and ground intermediate state by π

$$\mathcal{H}(x) = \mathcal{H}(t,\vec{x}) = \langle K|T[O(t,\vec{x})O(0)]|K\rangle$$

- In Minkowski space

$$\mathcal{I}^{(M)} = \sum_n \frac{1}{2E_\ell}\langle K|O(0)|n\rangle\langle n|O(0)|K\rangle \frac{1}{E_n + E_\ell - M_K - i\varepsilon}$$

the amplitude includes both real and imaginary contribution

- In Euclidean space

$$\begin{aligned}\mathcal{I}^{(s)} &= \int_0^{t_s} dt \int d^3x\, \mathcal{H}(x) S_\ell(x) \\ &= \sum_n \frac{1}{2E_\ell}\langle K|O(0)|n\rangle\langle n|O(0)|K\rangle \frac{1 - e^{-(E_n+E_\ell-M_K)t_s}}{E_n + E_\ell - M_K}\end{aligned}$$

$\mathcal{I}^{(s)}$ can't approach $\mathcal{I}^{(M)}$ in the $t_s \to \infty$ limit

- Define $\mathcal{I}^{(l)}$ as

$$
\begin{aligned}
\mathcal{I}^{(l)} &= \mathcal{I}^{(M)} - \mathcal{I}^{(s)} \\
&= \sum_n \frac{1}{2E_\ell} \langle K|O(0)|n\rangle\langle n|O(0)|K\rangle \left(\frac{1}{E_n + E_\ell - M_K - i\varepsilon} - \frac{1 - e^{-(E_n+E_\ell-M_K)t_s}}{E_n + E_\ell - M_K} \right) \\
&= \sum_n \frac{1}{2E_\ell} \langle K|O(0)|n\rangle\langle n|O(0)|K\rangle \frac{e^{-(E_n+E_\ell-M_K)t_s}}{E_n + E_\ell - M_K - i\varepsilon}
\end{aligned}
$$

$\langle K|O(0)|n\rangle\langle n|O(0)|K\rangle$ known from ground-state dominance

- $\mathcal{I}^{(l)}$ can be written as

$$
\mathcal{I}^{(l)} = \int d^3x\, \mathcal{H}(t_s,\vec{x}) L(t_s,\vec{x})
$$

with

$$
L(t_s,\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_\ell} \frac{1}{E_\pi + E_\ell - M_K - i\varepsilon} e^{i\vec{p}\cdot\vec{x}} e^{-E_\ell t_s}
$$

If $M_\pi = M_K$, $L(t_s,\vec{x})$ reproduces the expression for self energy

Application (III): non-QCD decays

A lot of interesting decays with non-QCD final states

- photonic decays: $\pi^0 \to \gamma\gamma$, $\eta_c \to \gamma\gamma$, $\chi_{c0} \to \gamma\gamma$, $K_L \to \gamma\gamma$
- leptonic decays: $\pi^0 \to e^+e^-$, $K_L \to \mu^+\mu^-$ [Norman's talk on Monday]
- radiative leptonic decays: $\pi^0 \to \gamma e^+e^-$, $B \to \gamma\mu^+\mu^-$, $B \to \gamma\ell\nu$
  [Christopher, Stefan & Soni, 2018 and also Shoji's talk]

Conventional method

- Study momenta dependence of the form factor, e.g. $F_{\pi\gamma\gamma}(m_\pi^2, p_1^2, p_2^2)$
  [XF, S. Aoki, S. Hashimoto, et al, PRL109, 182001, 2012]

Can the calculation be simpler?

How about calculating the on-shell amplitude in coordinate space

$$A = \int d^4x \quad \underbrace{\omega(x)}_{\text{Non-QCD}} \quad \underbrace{H(x)}_{\text{Hadronic}}$$

Take $\pi^0 \to \gamma\gamma$ as an example

- Step 1 - Calculate hadronic matrix element in coordinate space

$$\mathcal{H}_{\mu\nu}(x) = \langle 0 | T[J_\mu(x) J_\nu(0)] | \pi^0(q) \rangle$$

- Step 2 - Choose on-shell momentum

$$\mathcal{F}_{\mu\nu}(q,p,p') = \int d^4x\, e^{-ipx} \mathcal{H}_{\mu\nu}(x)$$

with

$$p = (im_\pi/2, \vec{p}), \quad p' = (im_\pi/2, -\vec{p}), \quad q = (im_\pi, \vec{0}), \quad |\vec{p}| = m_\pi/2.$$

We have

$$\mathcal{F}_{\mu\nu}(q,p,p') = \varepsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta F_{\pi\gamma\gamma}(m_\pi^2, 0, 0)$$

- Step 3 - Obtain a Lorentz scalar amplitude

$$\begin{aligned} \mathcal{I} &= \varepsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta \int d^4x\, e^{-ipx} \mathcal{H}_{\mu\nu}(x) \\ &= \varepsilon_{\mu\nu\alpha\beta} q_\beta \int d^4x\, e^{-ipx} \left(-i \frac{\partial}{\partial x_\alpha}\right) \mathcal{H}_{\mu\nu}(x) \\ &= m_\pi \int d^4x\, e^{-ipx}\, \varepsilon_{\mu\nu\alpha 0} \frac{\partial \mathcal{H}_{\mu\nu}(x)}{\partial x_\alpha} \end{aligned}$$

Take $\pi^0 \to \gamma\gamma$ as an example

- Step 4 - Average over the spatial direction for $\vec{p}$

$$
\begin{aligned}
\mathcal{I} &= m_\pi \int dt\, e^{m_\pi t/2} \int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}}\, \varepsilon_{\mu\nu\alpha 0} \frac{\partial \mathcal{H}_{\mu\nu}(x)}{\partial x_\alpha} \\
&= 2 \int dt\, e^{m_\pi t/2} \int d^3\vec{x}\, \frac{\sin(m_\pi |\vec{x}|/2)}{|\vec{x}|}\, \varepsilon_{\mu\nu\alpha 0} \frac{\partial \mathcal{H}_{\mu\nu}(x)}{\partial x_\alpha} \\
&= \int dt\, e^{m_\pi t/2} \int d^3\vec{x}\, \frac{-m_\pi |\vec{x}| \cos(m_\pi |\vec{x}|/2) + 2\sin(m_\pi |\vec{x}|/2)}{|\vec{x}|^3}\, \varepsilon_{\mu\nu\alpha 0} x_\alpha \mathcal{H}_{\mu\nu}(x)
\end{aligned}
$$

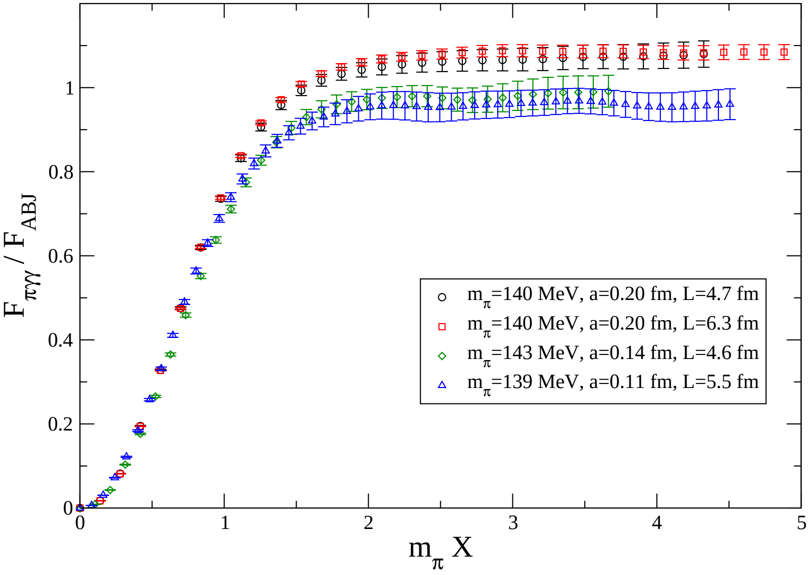
- Step 5 - Master formula

$$
F_{\pi^0\gamma\gamma}(m_\pi^2, 0, 0) = \frac{\mathcal{I}}{[\varepsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta][\varepsilon_{\mu\nu\rho\sigma} p_\rho q_\sigma]} = \frac{2}{m_\pi^4}\mathcal{I} = \int d^4x\, \omega(x) H(x)
$$

Weight function $\omega(x)$ is known analytically

Key quantity required from lattice QCD is $H(x) = \varepsilon_{\mu\nu\alpha 0} x_\alpha \mathcal{H}_{\mu\nu}(x)$

Perform the integral in the region of $\sqrt{t^2 + \vec{x}^2} < X$



Branching ratio given by Kroll-Wada formula

$$\frac{\Gamma_{\pi\to\gamma e^+ e^-}}{\Gamma_{\pi\to\gamma\gamma}} = \frac{\alpha}{3\pi}\int_r^1 \frac{d\rho}{\rho}(1-\rho)^3\left(1-\frac{r}{\rho}\right)^{\frac{1}{2}}\left(2+\frac{r}{\rho}\right)\frac{F^2_{\pi^0\gamma\gamma}(m_\pi^2, s, 0)}{F^2_{\pi^0\gamma\gamma}(m_\pi^2, 0, 0)}$$

where $r = 4m_e^2/m_\pi^2$, $\rho = s/m_\pi^2$.

$$\frac{F^2_{\pi^0\gamma\gamma}(m_\pi^2, s, 0)}{F^2_{\pi^0\gamma\gamma}(m_\pi^2, 0, 0)} = 1 + 2\left(\frac{F(\rho,0)}{F(0,0)} - 1\right) + \left(\frac{F(\rho,0)}{F(0,0)} - 1\right)^2$$

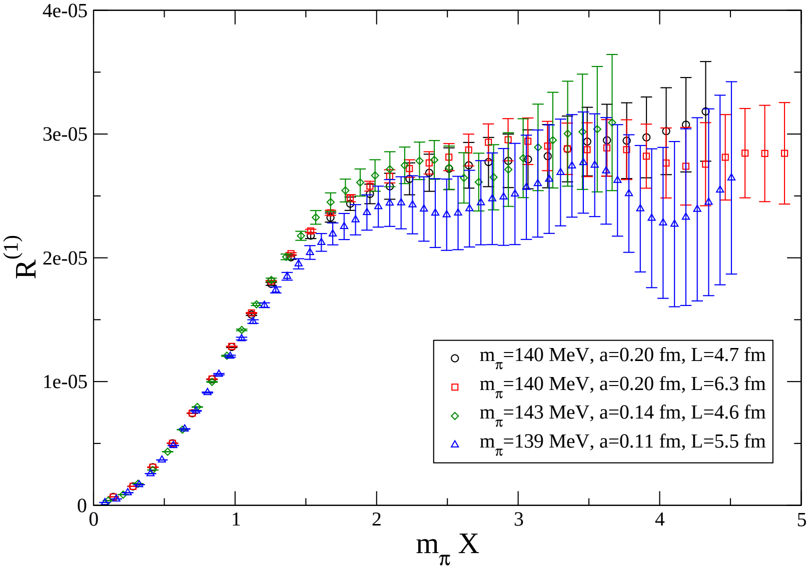
Correspondingly, the branching ratio can be written as

$$\frac{\Gamma_{\pi\to\gamma e^+ e^-}}{\Gamma_{\pi\to\gamma\gamma}} = R^{(0)} + R^{(1)} + R^{(2)}$$

where $R^{(0)} = 0.01185$ is irrelevant for QCD correction.

$$R^{(1)} = \int d^4x\, \omega^{(1)}(t, |\vec{x}|)\varepsilon_{\mu\nu\alpha 0} x_\alpha \mathcal{H}_{\mu\nu}(x)$$

Results for $\pi^0 \to \gamma e^+ e^-$

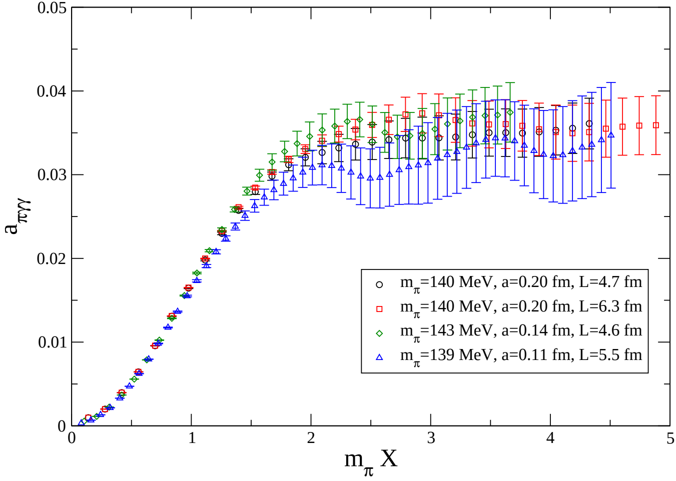


Results for form factor slope

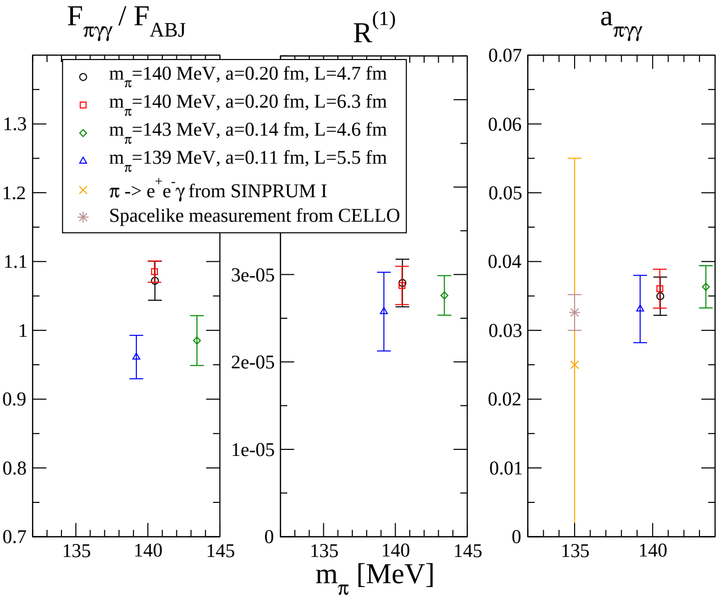
Taylor expansion of form factor $F(\rho, 0)$ at small ρ is

$$\frac{F(\rho,0)}{F(0,0)} = 1 + a_{\pi\gamma\gamma}\rho + \cdots$$

$$a_{\pi\gamma\gamma} = \int d^4x\, \omega^{(a)}(t, |\vec{x}|)\varepsilon_{\mu\nu\alpha 0} x_\alpha \mathcal{H}_{\mu\nu}(x)$$



Short summary for $\pi^0 \to \gamma\gamma$ and $\pi^0 \to \gamma e^+ e^-$



Conclusion

- Many interesting processes can be calculated in coordinate space

$$A = \int d^4x\, \omega(x) H(x)$$

  - $\omega(x)$ sometimes complicated, but can be known analytically
  - The task for lattice QCD is to evaluate $H(x)$, basically 2pt, 3pt and 4pt function

- For self energies, by using infinite-volume reconstruction, finite-volume effects can be exponentially suppressed

$$\mathcal{I} = \mathcal{I}^{(s)} + \mathcal{I}^{(l)}$$

- We can also treat with the case where the intermediate state is a single stable hadron and lighter than initial/final state

- Apply the method to leptonic and semileptonic decays

[Talks also given by Vera and Chris]