Radiative Corrections to Leptonic and Semileptonic Decay Rates

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In this talk I briefly review the status of two on-going RM123 + Southampton projects concerning the radiative corrections to leptonic and semileptonic decays.

1. Introduction - The story so far.
2. Radiative corrections to leptonic decays: amplitudes for $P \to \ell \bar{\nu}_\ell \gamma$ decays.
   - Lattice 2019 proceedings to appear on the arXiv tomorrow.
   - Note also a corresponding study presented by S.Meinel at Lattice 2019,

     C.Kane, C.Lehner, S.Meinel and A.Soni, arXiv:1907.00279
3. Semileptonic decays of pseudoscalar mesons.

Work done in collaboration with
G.M. de Divitiis, M. Di Carlo, A. Desiderio, R.Frezzotti, M.Garofalo, D.Giusti, M.Hansen, V.Lubicz, G.Martinelli, F.Mazzetti, F.Sanfilippo, S.Simula and N.Tantalo

- We have been developing and implementing the framework for including radiative (and strong isospin breaking) corrections to leptonic decays:

1. *QED Corrections to Hadronic Processes in Lattice QCD,*

   N.Carrasco, V.Lubicz, G.Martinelli, C.T.Sachrajda, N.Tantalo, C.Tarantino and M.Testa,

   Phys. Rev. D **91** (2015) no.7, 07450 [arXiv:1502.00257 [hep-lat]].

   - This paper develops the formalism for evaluating radiative corrections to leptonic decays of mesons.
   - In particular, we discuss the treatment and cancellation of infrared divergences.

2. *Finite-Volume QED Corrections to Decay Amplitudes in Lattice QCD*,

   V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula and N.Tantalo,

   Phys. Rev. D **95** (2017) no.3, 034504 [arXiv:1611.08497 [hep-lat]].

   - Here we show that in $\mathrm{QED_L}$ the $O(\frac{1}{L})$ corrections to leptonic decays widths are universal (i.e. structure independent) and we evaluate them.

3. *First Lattice Calculation of the QED Corrections to Leptonic Decay Rates,*
   D.Giusti, V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula, N.Tantalo and C.Tarantino,
   Phys. Rev. Lett. **120** (2018) 072001 [arXiv:1711:06537]
   - We present the first complete numerical results for $\Gamma(K_{\mu 2})/\Gamma(\pi_{\mu 2})$ at $O(\alpha)$.

4. *Light-meson leptonic decay rates in lattice QCD+QED*
   M.Di Carlo, D.Giusti, V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula and N.Tantalo,
   arXiv:1904.08731,
   - We present details of the computation in paper 3;
   - We update the analysis, in particular by improving the renormalisation;
     see talk by Matteo di Carlo at Lattice 2019
   - The paper contains a detailed discussion of how to define QCD in the presence of QED, following the preliminary discussion in my Lattice 2018 talk.

5. *Radiative corrections to decay amplitudes in lattice QCD,*
   D.Giusti, V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula and N.Tantalo,
   Lattice2018, arXiv:1811.06364 [hep-lat].
   - This Lattice 2018 talk also contains a preliminary discussion of some of the topics concerning radiative corrections to semileptonic decays; this discussion was extended at Lattice 2019.

I start with a brief review of leptonic decays of pseudoscalar mesons.

- The presence of infrared divergences requires us, at $O(\alpha)$ to consider

$$\Gamma_0(P \to \ell\bar{\nu}_\ell) + \Gamma_1(P \to \ell\bar{\nu}_\ell\gamma)\,,$$

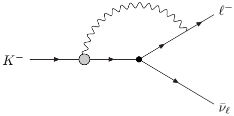
where the subscript 0,1 denotes the number of photons in the final state.

- Our initial proposal was to restrict the energy of the final-state photon to be sufficiently small ($E_\gamma < \Delta E_\gamma \simeq 20\,\text{MeV}$ say) for the structure dependence to be negligible.

- It is convenient to organise the calculation in the form

$$\Gamma_0 + \Gamma_1(\Delta E_\gamma) = \lim_{V\to\infty}\left(\Gamma_0 - \Gamma_0^{\text{pt}}\right) + \lim_{V\to\infty}\left(\Gamma_0^{\text{pt}} + \Gamma_1(\Delta E_\gamma)\right).$$

  - "pt" implies the meson P is treated as "point-like".
  - Each of the two terms on the right-hand side is infrared finite.
  - The second term on the r.h.s. can be calculated in perturbation theory.
  - Γ_0, on the other-hand, must be computed in a lattice computation, e.g.



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- In this section, I report on recent progress on the development of the techniques for the lattice computation of the amplitude for $P \to \ell \bar{\nu}_\ell \gamma$ and a demonstration of its practicability.
- It is then natural to organise the calculation as:

$$\Gamma_0 + \Gamma_1 = \lim_{V\to\infty}\left(\Gamma_0 - \Gamma_0^{\rm pt}\right) + \lim_{V\to\infty}\left(\Gamma_0^{\rm pt} + \Gamma_1^{\rm pt}\right) + \lim_{V\to\infty}\left(\Gamma_1 - \Gamma_1^{\rm pt}\right)$$

  so that each of the three terms on the r.h.s. is infrared finite.
- The non-perturbative evaluation of Γ_1 has the important practical implication that the method can be applied to the decays of heavy mesons.
  - For example, since $m_{B^*} - m_B \simeq 45$ MeV, for heavy mesons there is another small scale present, which limits the scope and precision of the perturbative calculations for soft photons.
- Given the remarkable, sub-percent, precision of the determinations of f_P in lattice QCD simulations, such QED corrections are necessary to fully exploit the results for the extraction of CKM matrix elements.

- The starting point for the evaluation of the amplitude for the $P\to \ell\bar\nu\gamma$ decay and hence for the corresponding Γ_1 is

$$\begin{aligned} H^{\alpha r}(k,p) &= \varepsilon^r_\mu(k)\int d^4y\, e^{ik\cdot y}\langle 0|\,T\left\{J^\alpha_W(0)J^\mu_{\rm em}(y)\right\}|P(p)\rangle \\ &\equiv \varepsilon^r_\mu(k)H^{\alpha\mu}(k,p)=\varepsilon^r_\mu(k)\left(H^{\mu\alpha}_{\rm pt}(k,p)+H^{\mu\alpha}_{\rm SD}(k,p)\right).\end{aligned}$$

- The point-like contribution is readily given by

$$H^{\mu\alpha}_{\rm pt}(k,p)=f_P\left[g^{\mu\alpha}+\frac{(2p-k)^\mu(p-k)^\alpha}{2p\cdot k-k^2}\right].$$

- The *Structure Dependent* contribution can conveniently be written in terms of four form-factors:

$$\begin{aligned} H^{\mu\alpha}_{\rm SD}(k,p) &= H_1\,(k^2g^{\mu\alpha}-k^\mu k^\alpha)+H_2\left[(p\cdot k-k^2)k^\mu-k^2(p-k)^\mu\right](p-k)^\alpha \\ &\quad -i\frac{F_V}{m_P}\varepsilon^{\mu\alpha\gamma\beta}k_\gamma p_\beta+\frac{F_A}{m_P}\left[(p\cdot k-k^2)g^{\mu\alpha}-(p-k)^\mu k^\alpha\right].\end{aligned}$$

  - For an on-shell photon ($k^2=0,\varepsilon^r(k)\cdot k=0$), only F_V and F_A contribute to $H^{\mu\alpha}(k,p)$.
  - Our task therefore is to compute $F_{V,A}$.

- In our first paper we estimated how big the SD contributions to Γ_1 might be using ChPT.
N.Carrasco et al., arXiv:1502.00257

- We define

$$R_1^A(\Delta E) = \frac{\Gamma_1^A(\Delta E)}{\Gamma_0^{\alpha,\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)}\,, \qquad A = \{\mathrm{SD},\mathrm{INT}\}\,,$$

where SD and INT refer to the structure dependent and interference (between SD and pt) contributions respectively.

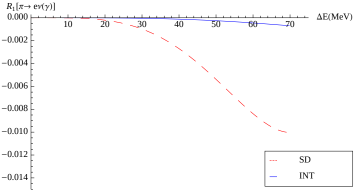
- At $O(p^4)$ in chiral perturbation theory,

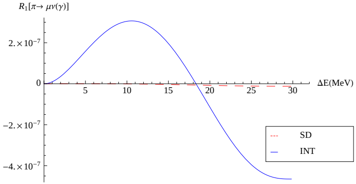
$$F_V = \frac{m_P}{4\pi^2 f_P} \quad \text{and} \quad F_A = \frac{8m_P}{f_P}\,(L_9^r + L_{10}^r)\,,$$

where $P = \pi$ or K and L_9^r, L_{10}^r are Gasser-Leutwyler coefficients.

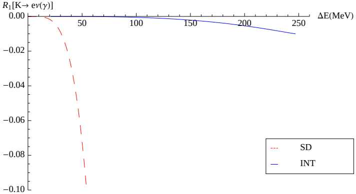
- The numerical values of these constants have been taken from the review by M.Bychkov and G.D'Ambrosio in the PDG. F_V and F_A are 0.0254 and 0.0119 for the pion and 0.096 and 0.042 for the Kaon (for the pion these values of the form factors, obtained from direct measurements, can be found in the supplement to the PDG.)

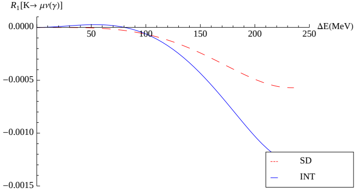
Pion





Kaon





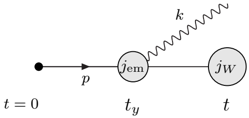
- For decays into a muon, the SD contributions are negligible at about 20 MeV and for decays into an electron they are small.

Estimates of SD contributions to $\Gamma_1(\Delta E)$ (cont)

- For heavy-light mesons we don't have such ChPT calculations.
- For the B-meson in particular we have another small scale $< \Lambda_{\text{QCD}}$, $m_{B^*} - m_B \simeq 45$ MeV so that we may expect that we will have to go to smaller ΔE in order to be able to neglect SD effects.
- Calculations based on the extreme approximation of single pole dominance suggest that this is likely to be the case.

  D. Becirevic, B. Haas and E. Kou, arXiv:0907.1845 [hep-ph]
- Now amenable to direct calculations.

Extracting the form-factors



- A convenient kinematical variable is $x_\gamma = 2p\cdot k/m_P^2$. $\quad (0 \leq x_\gamma < 1)$

- The Euclidean correlation functions which are to be evaluated are of the form:

$$C^{\alpha r}(t,\vec{p},\vec{k}) = \varepsilon_\mu^r(k)\int d^4y\int d^3x\,\langle 0|T\left\{j_W^\alpha(t,\vec{0})j_{\mathrm{em}}^\mu(y)\right\}\phi_P^\dagger(0,\vec{x})\,|0\rangle\; e^{E_\gamma t_y - i\vec{k}\cdot\vec{y}+i\vec{p}\cdot\vec{x}}$$

- We now remove the external factors defining:

$$R^{\alpha r}(t,\vec{p},\vec{k}) = \frac{2E}{e^{-t(E-E_\gamma)}\langle P(\vec{p})|\phi_P^\dagger(0)|0\rangle}C^{\alpha r}(t,\vec{p},\vec{k}) = \varepsilon_\mu^r(k)\,H^{\mu\alpha}(k,p)+\cdots$$

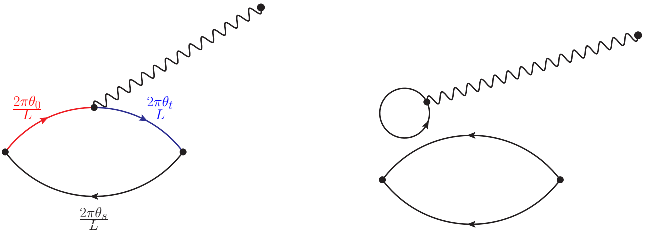
  where the $\cdots$ represent the contributions from excited states.

- The choice of polarisation vectors can be used to project out the form factors. For example, taking $\vec{p}=(0,0,|\vec{p}|)$ and $\vec{k}=(0,0,E_\gamma)$ we can choose for the basis of polarisation vectors $\varepsilon_{1,2}^\mu = (0,\mp\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0)$ so that

$$R_A^{jr}(t,\vec{p},\vec{k}) \to \frac{\varepsilon_r^j m_P}{2}x_\gamma\left[F_A(x_\gamma)+\frac{2f_P}{m_P x_\gamma}\right], \quad R_V^{jr}(t,\vec{p},\vec{k}) \to \frac{i\left(E_\gamma\vec{\varepsilon}_r\times\vec{p} - E\,\vec{\varepsilon}_r\times\vec{k}\right)^j}{m_P}F_V(x_\gamma)$$

  where the subscripts A,V on R^{jr} label the components of the weak current.

Calculation of the amplitude



- For each choice of quark masses we have 5 values of each $\theta \Rightarrow$ 100 sets of momenta.
- The results presented below have been obtained in the electro-quenched approximation.
- All results shown below are preliminary.

Plateaus

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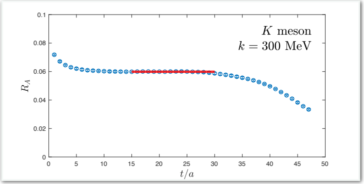
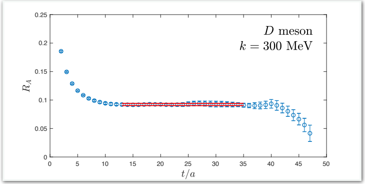
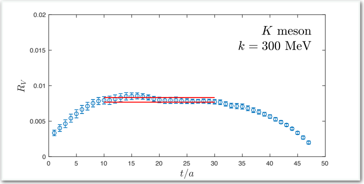
The plateaus for the ratio's are are generally good:

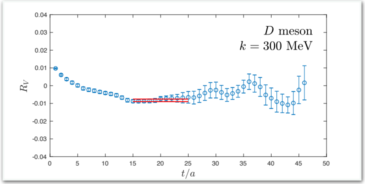
R_A



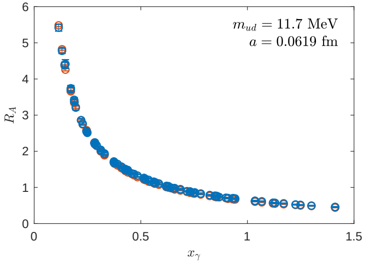


R_V

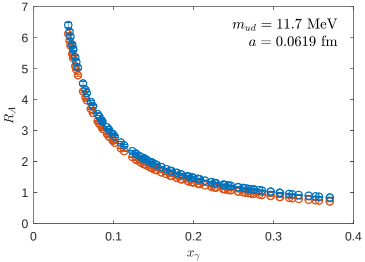




- These plots are for $m_\pi \simeq 360\,$MeV; m_K and m_D are close to their physical values.



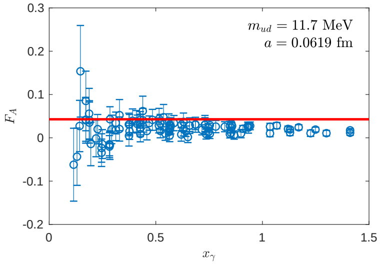
Kaon

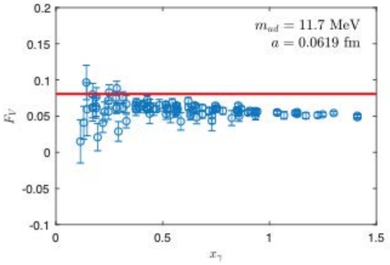


D_s

- $R_A = F_A + \dfrac{2f_P}{m_P x_\gamma}$; $\quad R_A^{\text{pt}} = \dfrac{2f_P}{m_P x_\gamma}$
- For light mesons the SD form-factors are small.

Form factors compared to ChPT





- Red lines correspond to the lowest non-trivial order of ChPT

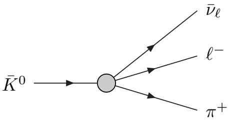
$$F_A = \frac{8m_K}{f_K}(L_9^r + L_{10}^r) \simeq \frac{8m_K}{f_K}\,(1.7\times 10^{-3}); \qquad F_V = \frac{8m_K}{4\pi^2 f_K}$$

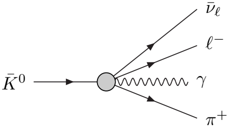
- Similar results were found for the other mesons: π, D, D_s.
- With improved statistics and analysis, we will also be able to extract the momentum dependence.

- The computation of the amplitude/rate for $P \to \ell\bar{\nu}\gamma$ decays is important in itself, and also for the evaluation of the $O(\alpha)$ corrections to the rate for leptonic decays (and hence the corresponding CKM matrix element).
  - This is particularly important for the decays of heavy mesons where the small value of the hyperfine splitting $\Rightarrow$ point-like approximation for small but measurable ΔE is unlikely to be justified.
- We have seen that even with moderate statistics it is possible to extract the form factors with good precision (at least in the electro-quenched approximation) and to study the momentum dependence of the form-factors.
- In the near future we will be able to compare precise theoretical predictions with experimental measurements.
- Note a corresponding study presented by S.Meinel at Lattice 2019,

C.Kane, C.Lehner, S.Meinel and A.Soni, arXiv:1907.00279

- For illustration we consider $K_{\ell 3}$ decays, but the discussion is general:





- A particularly appropriate measurable quantity to compute is

$$\frac{d^2\Gamma}{dq^2 ds_{\pi\ell}}$$

where $q^2 = (p_K - p_\pi)^2$ and $s_{\pi\ell} = (p_\pi + p_\ell)^2$.

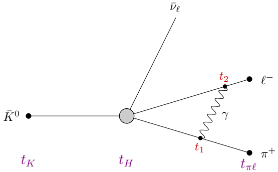
- Following the same procedure as for leptonic decays we write:

$$\frac{d^2\Gamma}{dq^2 ds_{\pi\ell}} = \lim_{V\to\infty}\left(\frac{d^2\Gamma_0}{dq^2 ds_{\pi\ell}} - \frac{d^2\Gamma_0^{\rm pt}}{dq^2 ds_{\pi\ell}}\right) + \lim_{V\to\infty}\left(\frac{d^2\Gamma_0^{\rm pt}}{dq^2 ds_{\pi\ell}} + \frac{d^2\Gamma_1(\Delta E)}{dq^2 ds_{\pi\ell}}\right)$$

  - Infrared divergences cancel separately in each of the two terms.

- If the amplitude for real photons is computed non-perturbatively, then the above formula is modified as for leptonic decays to three terms on the r.h.s..

- A general feature when evaluating long-distance contributions in Euclidean space is the presence of unphysical terms growing exponentially with the time separation. Consider for illustration the following contribution to the correlator:

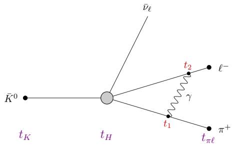


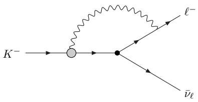
- Upon integrating over $t_{1,2}$ there are terms in the momentum sum proportional to

$$e^{-(E^{\rm int}_{\pi\ell}-E^{\rm ext}_{\pi\ell})(t_{\pi\ell}-t_H)}$$

  where $E^{\rm ext}_{\pi\ell}$ and $E^{\rm int}_{\pi\ell}$ are the external and internal energies of the pion-lepton pair.
- If there are states with $E^{\rm int}_{\pi\ell} < E^{\rm ext}_{\pi\ell}$ then there are unphysical exponentially growing contributions in $t_{\pi\ell}-t_H$.
- The energy non-conserving matrix elements with initial or final states having energy $E^{\rm int}$ can also be calculated to aid in the subtraction of the exponentially growing terms.
  - IV reconstruction method to be developed for the SL decays.

Exponentially growing terms in time (cont.)





- The presence of exponentially growing terms is a generic feature in the evaluation of long-distance contributions. They must be identified and subtracted.
  - The number of such terms depends on $s_{\pi\ell}$ and on the chosen (twisted) boundary conditions.
  - For kaon decays, in some corners of phase space, there may in principle also be multi-hadron intermediate states corresponding to growing exponentials, but these are expected to be small.
    - For example $K \to \pi\pi\ell\nu \to \pi\ell\nu(\gamma)$ only contributes at high order (p^6) in ChPT and is present due to the Wess-Zumino-Witten term in the action.
    - More importantly, we can restrict the values of $s_{\pi\ell}$ to a range below the multi-hadron threshold.
- D and B decays: the large number of such terms which need to be subtracted in most of phase space, makes it *very difficult* to implement the method.
- No such exponentially growing terms are present for leptonic decays.

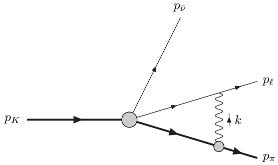
- For leptonic decays of the pseudoscalar meson P, in QED$_{\rm L}$ the finite-volume effects take the form:

$$\Gamma_0^{\rm pt}(L) = C_0(r_\ell) + \tilde{C}_0(r_\ell)\log(m_P L) + \frac{C_1(r_\ell)}{m_P L} + \dots,$$

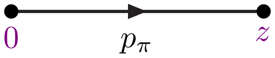
  where $r_\ell = m_\ell/m_P$.

- The exhibited L-dependent terms are *universal*, i.e. independent of the structure of the meson!
  - We have calculated these coefficients (using the QED$_{\rm L}$ regulator of the zero mode).
- The leading structure-dependent FV effects in $\Gamma_0 - \Gamma_0^{\rm pt}$ are of $O(1/L^2)$.
- The following scaling law is useful in the choice of terms to be evaluated. If the integrand/summand $\to \frac{1}{(k^2)^{\frac{n}{2}}}$ as $k \to 0$ then we have the scaling law:

$$\int \frac{dk_0}{(2\pi)} \left(\frac{1}{L^3} \sum_{\vec{k}\neq 0} - \int \frac{d^3k}{(2\pi)^3} \right) \frac{1}{(k^2)^{\frac{n}{2}}} \Rightarrow O\left(\frac{1}{L^{4-n}}\right)$$



- For illustration consider the above diagram. The infrared divergences occur from the terms in the integrand which $\sim 1/k^4$ and the $O(1/L)$ corrections from those which $\sim 1/k^3$.
  - The $O(1/L^2)$ corrections are structure dependent and so we cannot evaluate them analytically.
  - Since the leading term as $k \to 0 \sim 1/k^4$, we need to expand the propagators and vertices (including those from the weak Hamiltonian) to $O(k)$ in order to determine the ir divergence and the $O(1/L)$ FV correction.
- In studying the universality of the $O(1/L)$ corrections, the use of the e.m. Ward Identities is particularly useful (or equivalently the construction of a gauge invariant effective theory).
  - For illustration, I now present a simple, instructive and important example.



- We define the pion propagator $\Delta_\pi(p_\pi)$ by:

$$\begin{aligned} C_{\pi\pi}(p_\pi) &= \int d^4z\, e^{-ip_\pi\cdot z}\,\langle 0|T\{\phi_\pi(z)\phi_\pi^\dagger(0)\}|0\rangle \\ &\equiv \left|\langle 0|\phi_\pi(0)|\pi(p_\pi)\rangle\right|^2 \Delta_\pi(p_\pi) \\ &\equiv \left|\langle 0|\phi_\pi(0)|\pi(p_\pi)\rangle\right|^2 \frac{Z_\pi(p_\pi^2)}{p_\pi^2+m_\pi^2}\,. \end{aligned}$$

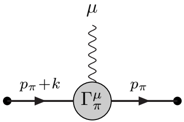
  - Z_π parametrises the structure dependence of the pion propagator.

- Expanding the propagator for small values of k and $\varepsilon_\pi^2 = p_\pi^2 + m_\pi^2$ we obtain:

$$\Delta_\pi(p_\pi+k) = \frac{1-2z_{\pi_1}p_\pi\cdot k-\varepsilon_\pi^2 z_{\pi_1}+O(k^2,\varepsilon_\pi^4,\varepsilon_\pi^2 k)}{\varepsilon_\pi^2+2p_\pi\cdot k+k^2}\,,$$

where the structure dependent parameter z_{π_1} is given by:

$$z_{\pi_1} = \left.\frac{dZ_\pi^{-1}(p_\pi^2)}{dp_\pi^2}\right|_{p_\pi^2=-m_\pi^2}\,.$$



- Similarly we define the amputated $\pi\gamma\pi$ vertex Γ^μ_π, by amputating the propagators and matrix elements of the interpolating operators in the correlation function:

$$C^\mu_\pi(p_\pi,k) = i\int d^4z\,d^4x\,e^{-ip_\pi\cdot z}\,e^{-ik\cdot x}\langle 0|\,T\left\{\phi_\pi(z)\,j^\mu(x)\,\phi^\dagger_\pi(0)\right\}|0\rangle\,.$$

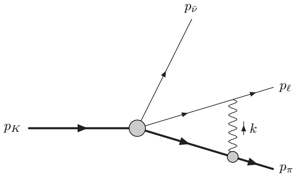
- We now expand Γ_π for small k (and the ε_π).
- The key result is obtained from the Ward Identity:

$$k_\mu\Gamma^\mu_P(p_\pi,k) = \left\{\Delta^{-1}_\pi(p_\pi+k) - \Delta^{-1}_\pi(p_\pi)\right\}\,,$$

which relates the first-order expansion coefficients and yields

$$Z_\pi(p_\pi+k)\,\Gamma^\mu_\pi(p_\pi,k) = Q_\pi\,(2p_\pi+k)^\mu + O(k^2,\varepsilon^2_\pi)\,.$$

$\Rightarrow$ no $O(1/L)$ correction from pion propagator and $\pi\gamma\pi$ vertex.



- We have seen that, as a result of the Ward Identity, we do not need the derivatives of the pion form-factors to obtain the $O(1/L)$ corrections.
- However, we also need to expand the weak-vertex which, in QCD without QED, is a linear combination of two form-factors $f^\pm(q^2)$.

- Off-shell, the $K\pi\ell\bar\nu$ weak vertex is a linear combination of two functions $F^\pm(p_\pi^2, p_K^2, 2p_K\cdot p_\pi)$ (which reduce on-shell to the form-factors $f^\pm(q^2)$).
- The WI relates the $K\pi\ell\bar\nu$ and $K\pi\ell\bar\nu\gamma$ vertices and does lead to a partial, but not complete, cancelation of $O(\frac{1}{L})$ terms.
- The $O(\frac{1}{L})$ corrections are found to depend on $\frac{df^\pm(q^2)}{dq^2}$, as well as on $f^\pm(q^2)$.
  - Such derivative terms are generic (a consequence of the Low theorem); absent only in particularly simple cases, such as leptonic decays.
- These corrections are "universal" in the sense that the coefficients are physical and can be computed in lattice simulations.
  - There are no corrections of the form $\frac{df^\pm}{dm_\pi^2}$ or $\frac{df^\pm}{dm_K^2}$, which would not be physical.

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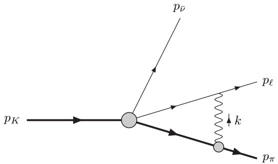

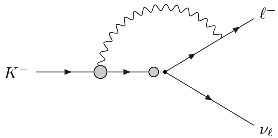

- For leptonic decays the corrections are proportional to f_K (computed in QCD simulations) and there is no scope for terms analogous to $\frac{df^\pm(q^2)}{dq^2}$.
  - Again there are no $O(\frac{1}{L})$ terms proportional to $\frac{df_K}{dm_K^2}$.
- For semileptonic decays, we have calculated the integrands/summands necessary to evaluate the coefficients of the $O(\frac{1}{L})$ corrections analytically using the Poisson summation formula, but have not yet evaluated the corrections themselves.
- In the ignorance of the analytic coefficients, the subtraction of the $O(\frac{1}{L})$ effects can be performed by fitting data obtained at different volumes.
  - Such a procedure for our numerical study of leptonic decays (where the $O(\frac{1}{L})$ corrections are known and can be subtracted) results in the doubling of the uncertainty in the theoretical prediction extrapolated to the physical point in the infinite volume limit; disappointing, but not a major problem.

- The starting point for our study is the relation:

$$\frac{d^2\Gamma}{dq^2 ds_{\pi\ell}} = \lim_{V\to\infty}\left(\frac{d^2\Gamma_0}{dq^2 ds_{\pi\ell}} - \frac{d^2\Gamma_0^{\rm pt}}{dq^2 ds_{\pi\ell}}\right) + \lim_{V\to\infty}\left(\frac{d^2\Gamma_0^{\rm pt}}{dq^2 ds_{\pi\ell}} + \frac{d^2\Gamma_1(\Delta E)}{dq^2 ds_{\pi\ell}}\right)$$

  and the second term on the r.h.s. needs to be calculated in perturbation theory.
  - This has not been fully done (yet?).
- The second term has been calculated in the *soft-photon* approximation in which all terms proportional to k^n $(n \geq 1)$ are dropped in the numerator.

  De Boer, Kitahari, Nišandžić, arXiv:1803.05881
  - This work was motivated by the $R(D)$ and $R(D^*)$ anomalies in semileptonic B-decays and the suggestion that radiative corrections not present in *photos* may be the explanation. Appears to be not true.
- The soft-photon approximation is sufficient to make both terms on the r.h.s. infrared finite, but not to eliminate the $O(\frac{1}{L})$ corrections in the first term.

$$\frac{d^2\Gamma}{dq^2 ds_{\pi\ell}} = \lim_{V\to\infty}\left(\frac{d^2\Gamma_0}{dq^2 ds_{\pi\ell}} - \frac{d^2\Gamma_0^{\rm pt}}{dq^2 ds_{\pi\ell}}\right) + \lim_{V\to\infty}\left(\frac{d^2\Gamma_0^{\rm pt}}{dq^2 ds_{\pi\ell}} + \frac{d^2\Gamma_1(\Delta E)}{dq^2 ds_{\pi\ell}}\right)$$

- The calculation of the second term on the r.h.s. has also been performed at lowest non-trivial order in ChPT. Cirigliano, Giannotti, Neufeld, arXiv:0807.4507

- We are developing the framework for the computation of radiative corrections to semileptonic $K_{\ell 3}$ decays.
  - This builds on our theoretical framework, and its successful implementation, in computations of radiative corrections to leptonic decays.
- Important points to note:
  1. An appropriate observable to study is $\frac{d^2\Gamma}{dq^2 ds_{\pi\ell}}$.
  2. The presence in general of unphysical exponentially growing terms in $t_{\pi\ell} - t_H$ which need to be subtracted.
  3. The universality of the $O(\frac{1}{L})$ corrections, which do however depend on the form-factors $f^{\pm}(q^2)$ and on their derivatives w.r.t. q^2.
     - Generic feature, absent only for simple processes such as leptonic decays.
- Things still to do include:
  1. To evaluate the coefficients on the $O(\frac{1}{L})$ corrections.
     - Otherwise these corrections can be fitted numerically.
     - If the $O(\frac{1}{L})$ corrections are not to be evaluated analytically, then the soft-photon approximation may be the most convenient one for the term which is added and subtracted.
  2. To test and implement the method numerically.