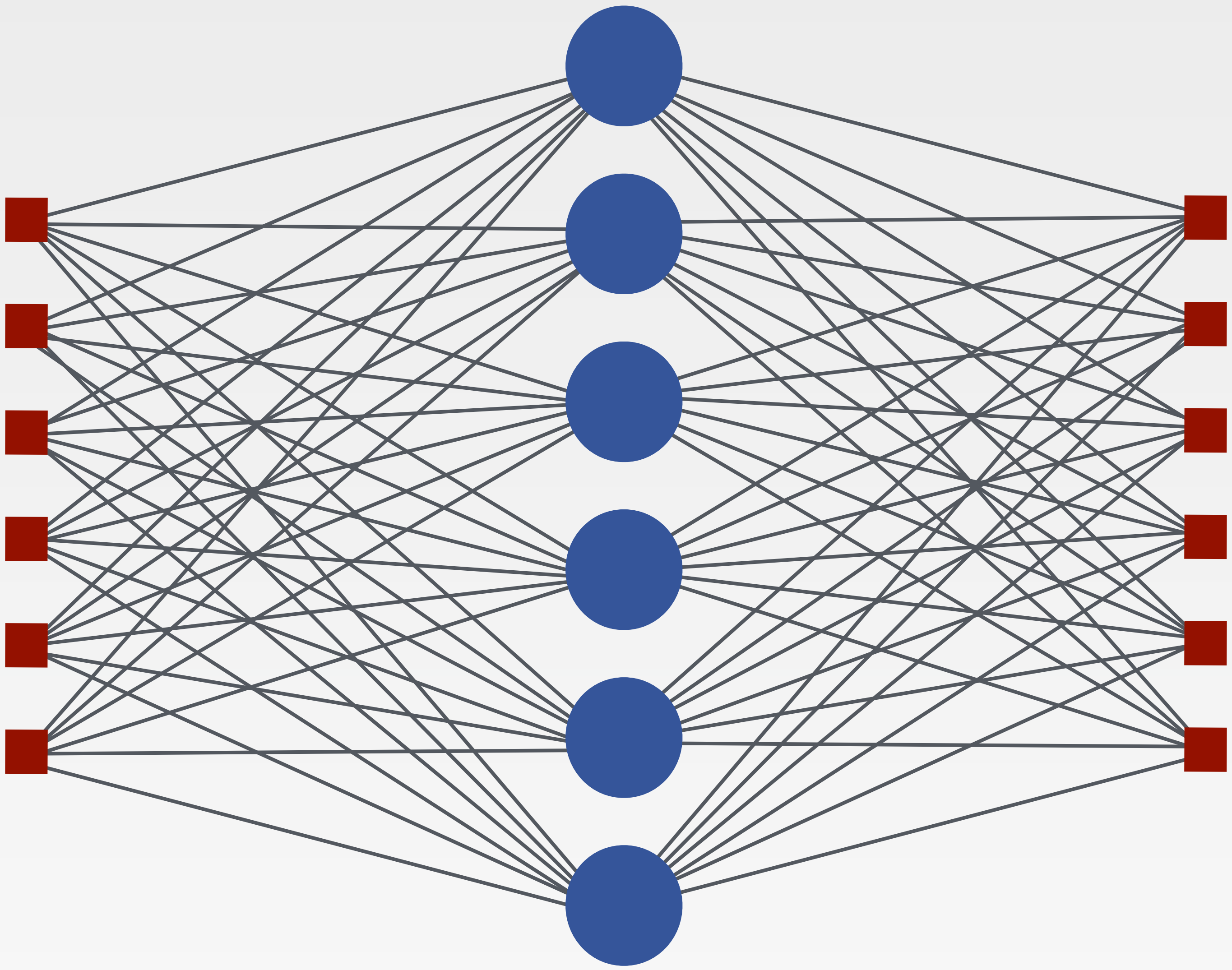

Neural network applications to spectral reconstruction

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In collaboration with

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- Motivations
- Spectral reconstruction as an inverse problem
- Case study: the light vector resonance
- Summary & perspectives

Motivations

Motivations

- Traditional lattice analysis: extract energies and matrix elements through fitting the exponential behaviour of correlators.
- **Density of state increases quickly with the volume.** Standard analysis increasingly challenging.
- Non-gapped correlators (QED, scattering, …) **Even ground state challenging at large volume.**
- Reconstruction of **long-distance effects**.
- **How to go beyond exponential fits?**

Spectral reconstruction as an inverse problem

Spectral integral equation

- In the case of 2-point functions
  Källén-Lehmann spectral representation

$$D(q^2) = \int \mathrm{d}^4x \, \langle \phi(x)\phi(0)^\dagger \rangle \, e^{-iq\cdot x} = \int_0^{+\infty} \mathrm{d}s \, \frac{\rho(s)}{q^2+s}$$

- Spectral reconstruction from lattice data:
  Find ρ given D.

- Integral operator $(S\rho)(q^2) = \int \mathrm{d}s \, \frac{\rho(s)}{q^2+s}$.
  Solution "$\rho = S^{-1}D$".
  Linear problem

Naive approach

- Arbitrary discretisation q_j^2 and s_k

$$S_{jk} = \frac{1}{q_j^2 + s_k}$$

- Distinct points: unique solution

$$\det(S) = \frac{\prod_{k>j}(q_j^2 - q_k^2)(s_j - s_k)}{\prod_{j,k}(q_j^2 + s_k)}$$

- In practice: extremely ill-conditioned matrix (typically $\kappa \simeq \mathcal{O}(10^{100})$)

Appropriateness of machine learning

- Machine-learning approach: **fit a parametrisation of the inverse transform** on a relevant functional space.
- Advantage here:
  trivial to construct random problem-solution pairs.
- Neural networks are just a specific parametrisation.
- Existing work: [Kades *et al.*, arXiv:1905.04305]

Challenges

- Determine an appropriate neural architecture.
- Establish a strategy for generating training and validation datasets.
- Design a Monte-Carlo strategy to validate predictions on synthetic lattice data.
- *Investigate generalisation of trained models.*
- *Compare against existing methods.*

Case study: the light vector resonance

Generalities

- Vector-vector spectral representation

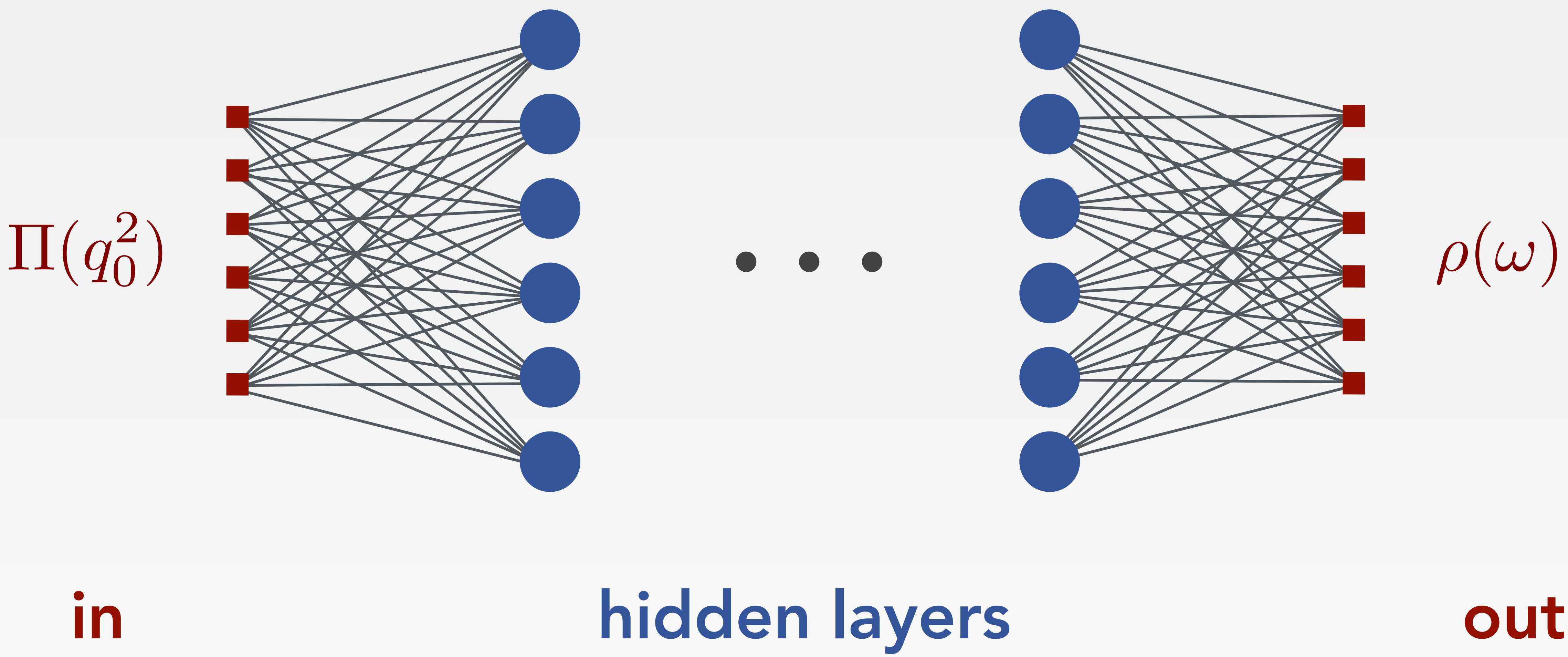
$$C_{\mu\nu}(t) = \int \mathrm{d}^3\mathbf{x}\ \langle 0|\, \mathrm{T}[J_\mu(t,\mathbf{x})J_\nu(0)]\, |0\rangle$$

$$\Pi_{\mu\nu}(q_0) = \int \mathrm{d}t\, C_{\mu\nu}(t)e^{-iq_0 t} = \delta_{\mu\nu}q_0^2\Pi(q_0^2)$$

$$\Pi(q_0^2) = \int_0^{+\infty} \mathrm{d}s\, \frac{\rho(s)}{q_0^2 + s}$$

$$\rho(s) = \frac{1}{3s}\int \mathrm{d}\alpha \textstyle\sum_\mu |\langle 0|\, J_\mu(0)\, |\alpha\rangle|^2 \delta(\omega_\alpha^2 - s)$$

Neural network architecture



Size	$\frac{N_T}{2} - 1$	N_h	N_ω
Step	$\frac{2\pi}{T}$		$\delta\omega$

● : $\tanh(x)$ ■ : x

Data preconditioning

- **Momentum space input layer**
  Reduce the hierarchy of weights, accelerate training.
  Just a change of basis on the input layer.

- **Linear scaling enforcement**
  Always normalise in and out by the input average.
  Quality of prediction insensitive to overall scale.

Training

- Loss function: 2-norm difference

$$L(\Pi, \rho) = \frac{1}{N} \sum_j \|F(\Pi_j) - \rho_j\|_2^2$$

F: neural net

- Training: stochastically minimise L on a training set
  Monitor loss through an independent validation set.

- Mini-batch size 4

- ADAM minimiser with “reduce-on-plateau” adaptive learning rate.

Training/Validation set

- Training set: 5000 random Breit-Wigner examples

$$300\ \mathrm{MeV} \leq M \leq 1000\ \mathrm{MeV}$$
$$120\ \mathrm{MeV} \leq \Gamma \leq 300\ \mathrm{MeV}$$

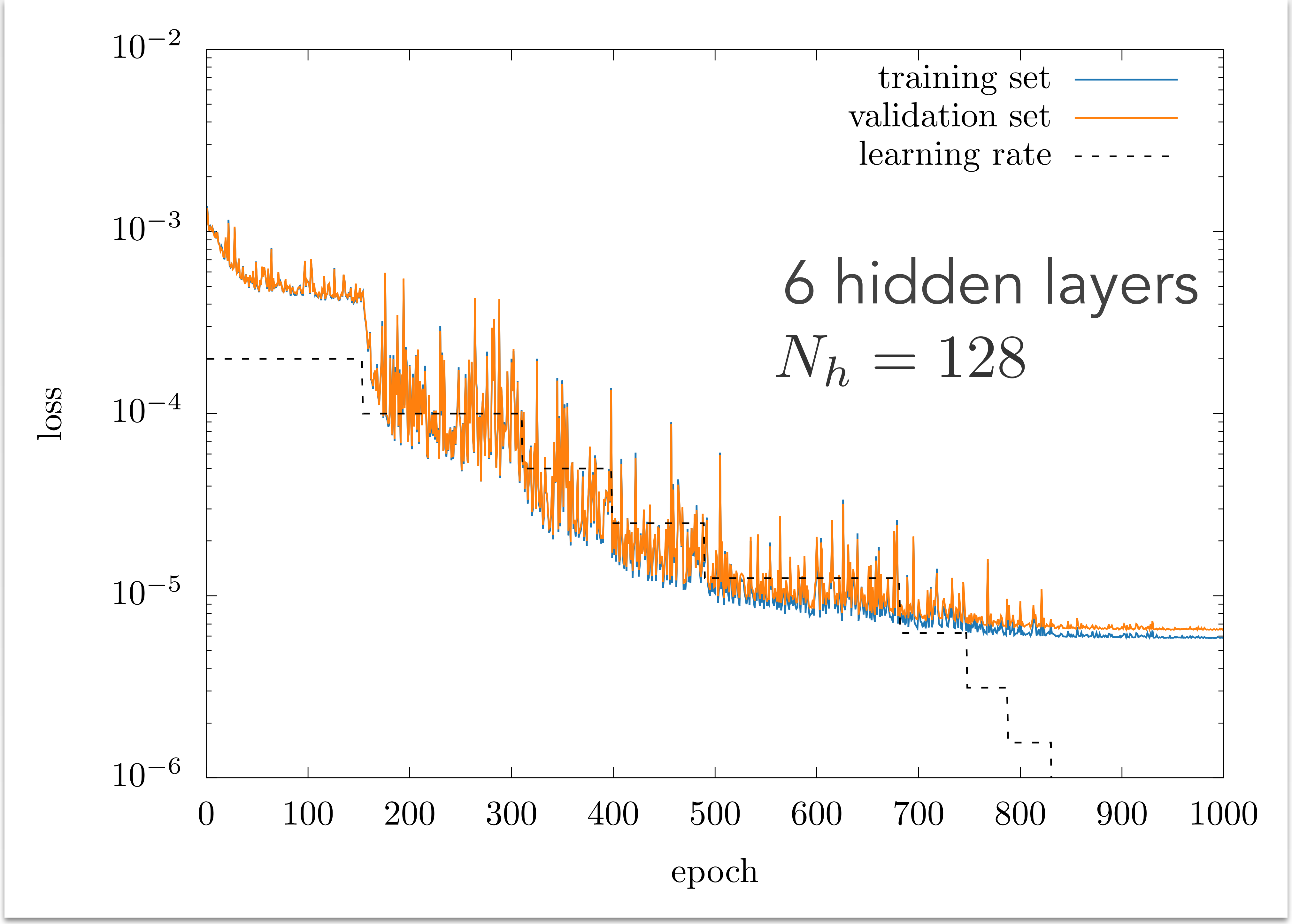
- Validation set: same, with 1000 examples.

- Loss function value not very meaningful.

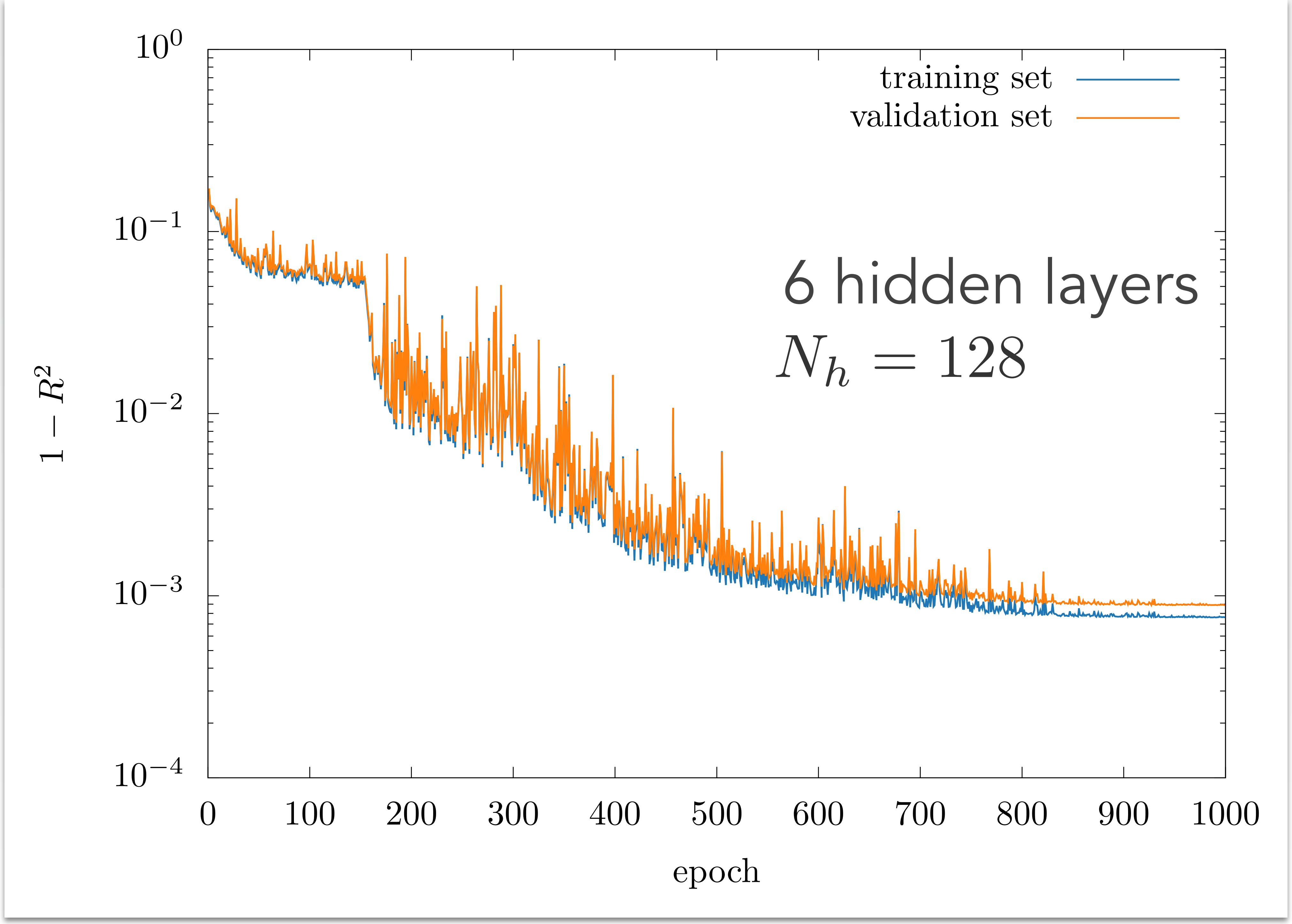
- R^2-score more useful

$$R^2 = 1 - \frac{\|F(\Pi) - \rho\|_2^2}{\mathrm{Var}(\rho)}$$

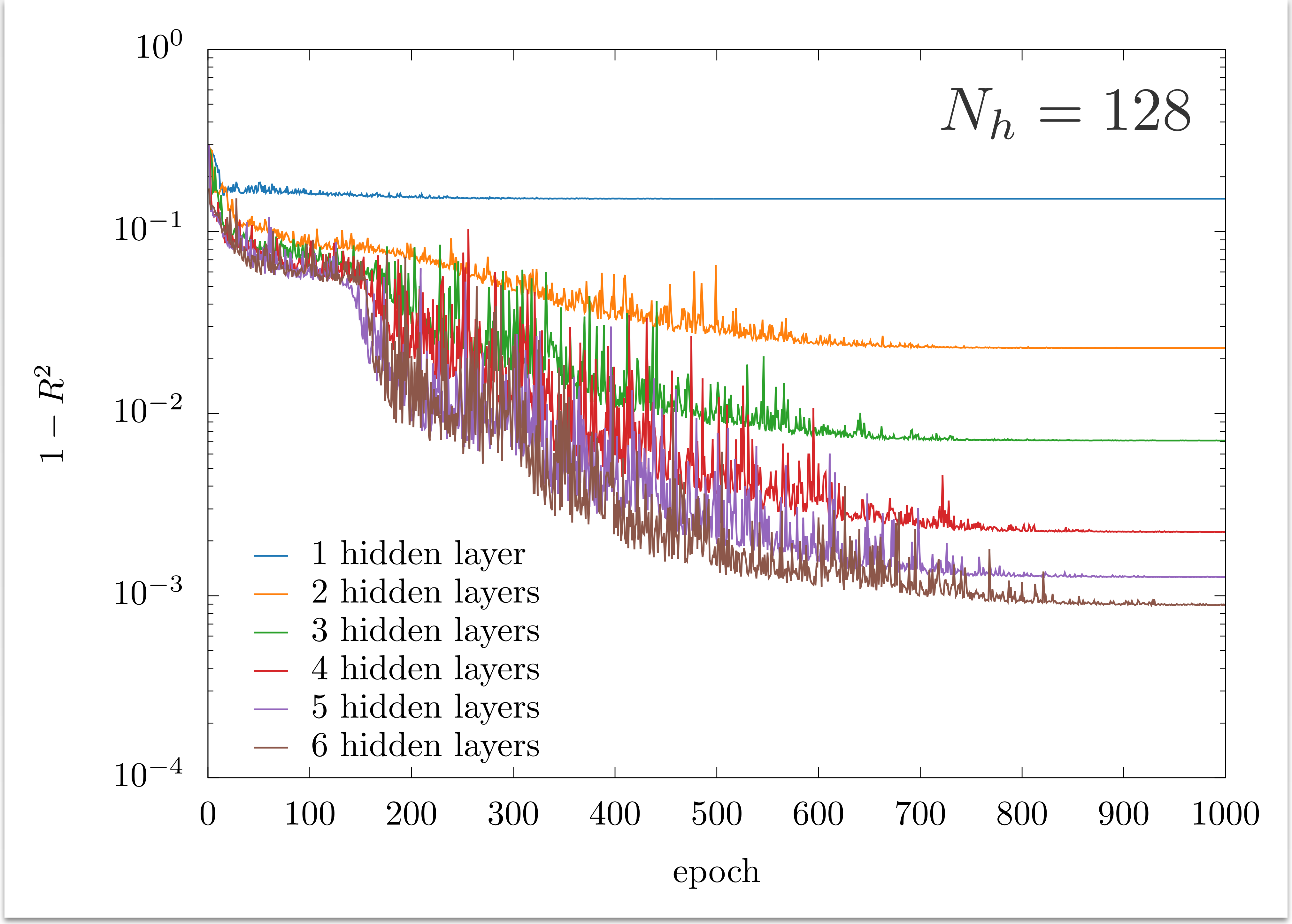
Learning curve example



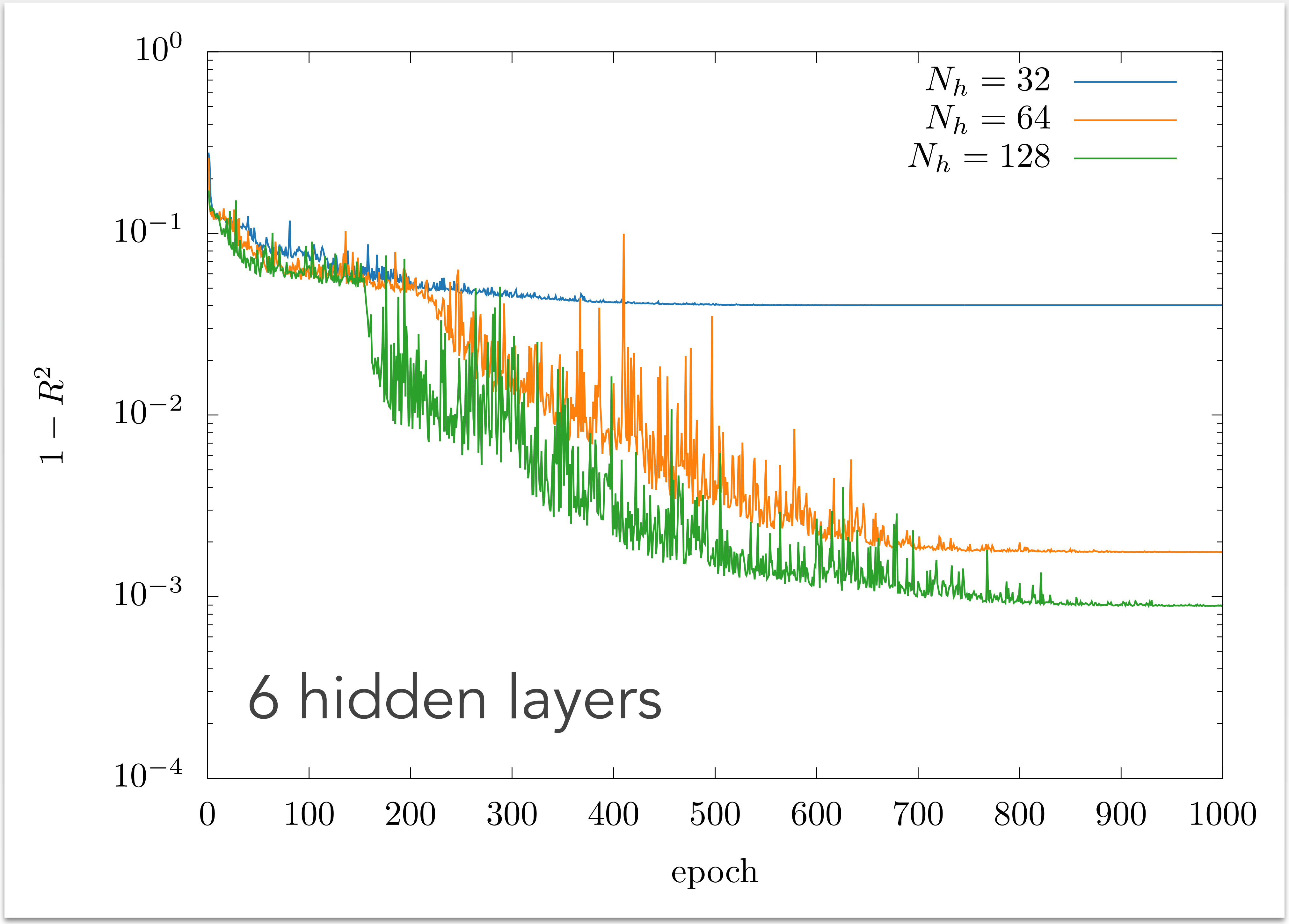
Learning curve example



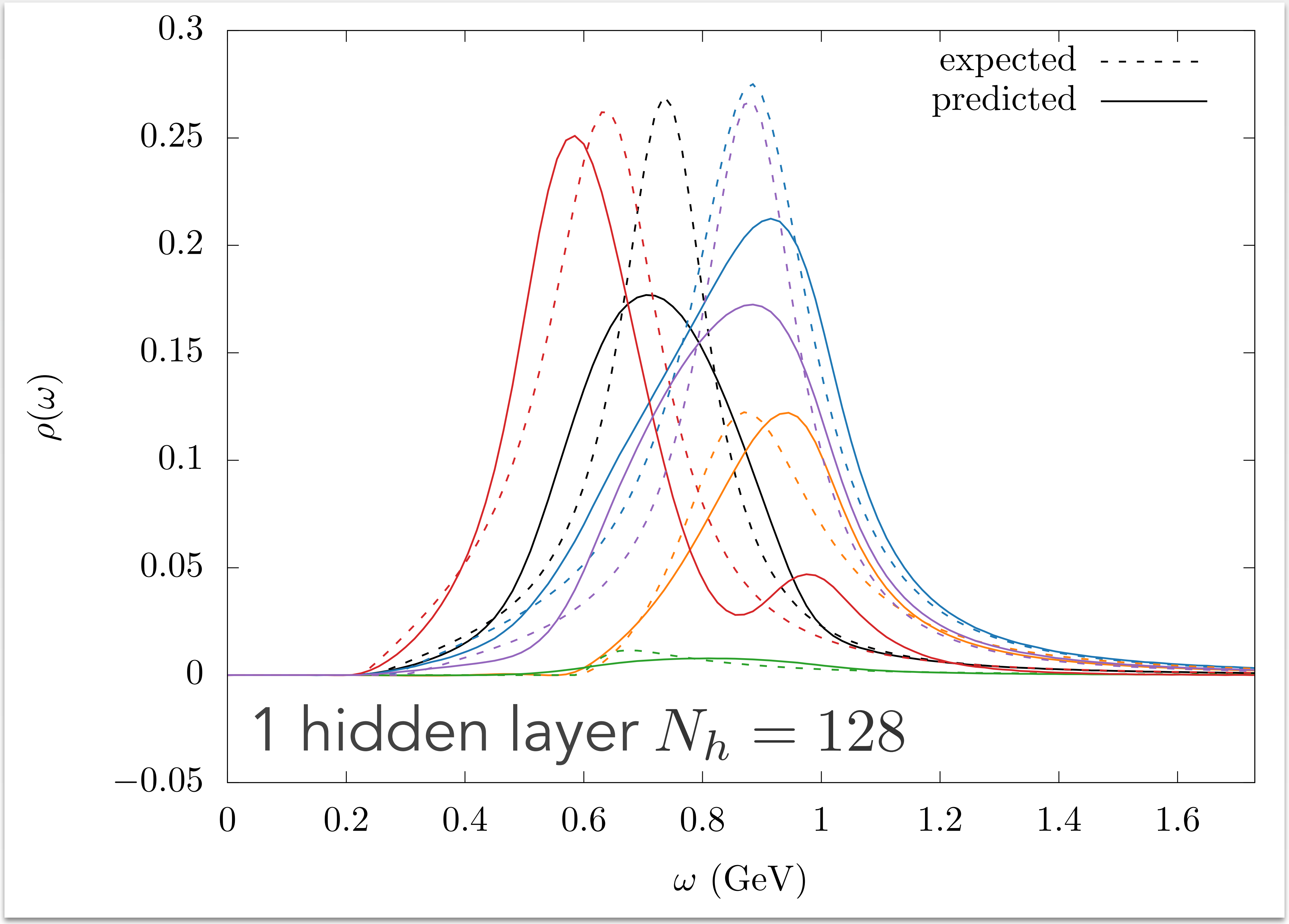
Learning curves vs. hidden layers



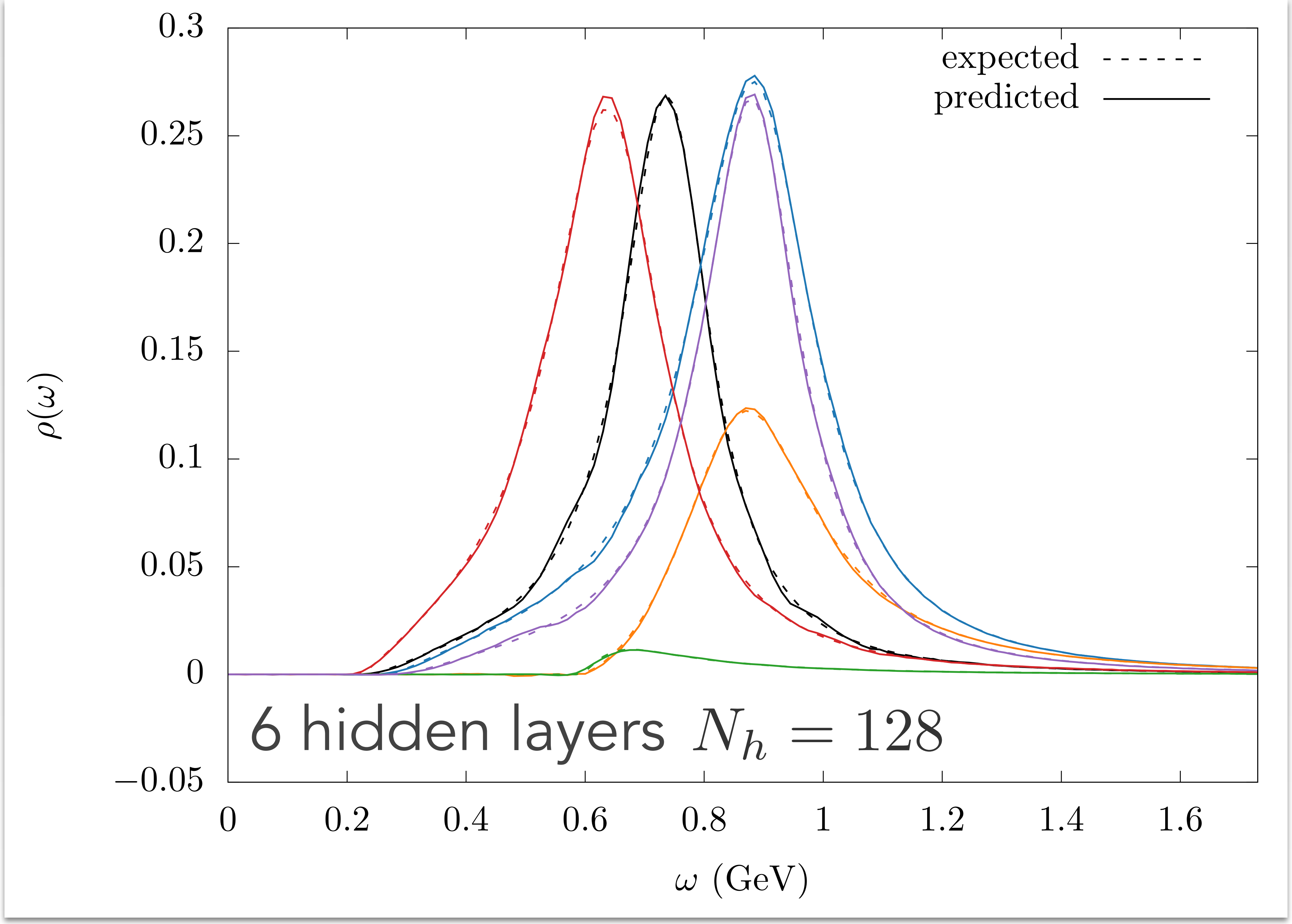
Learning curves vs. hidden layer size



Validation set sample



Validation set sample



Synthetic lattice data

- Correlator based on Gunaris-Sakurai parametrisation with physical parameters.

- Phenomenological variance model (work in progress)

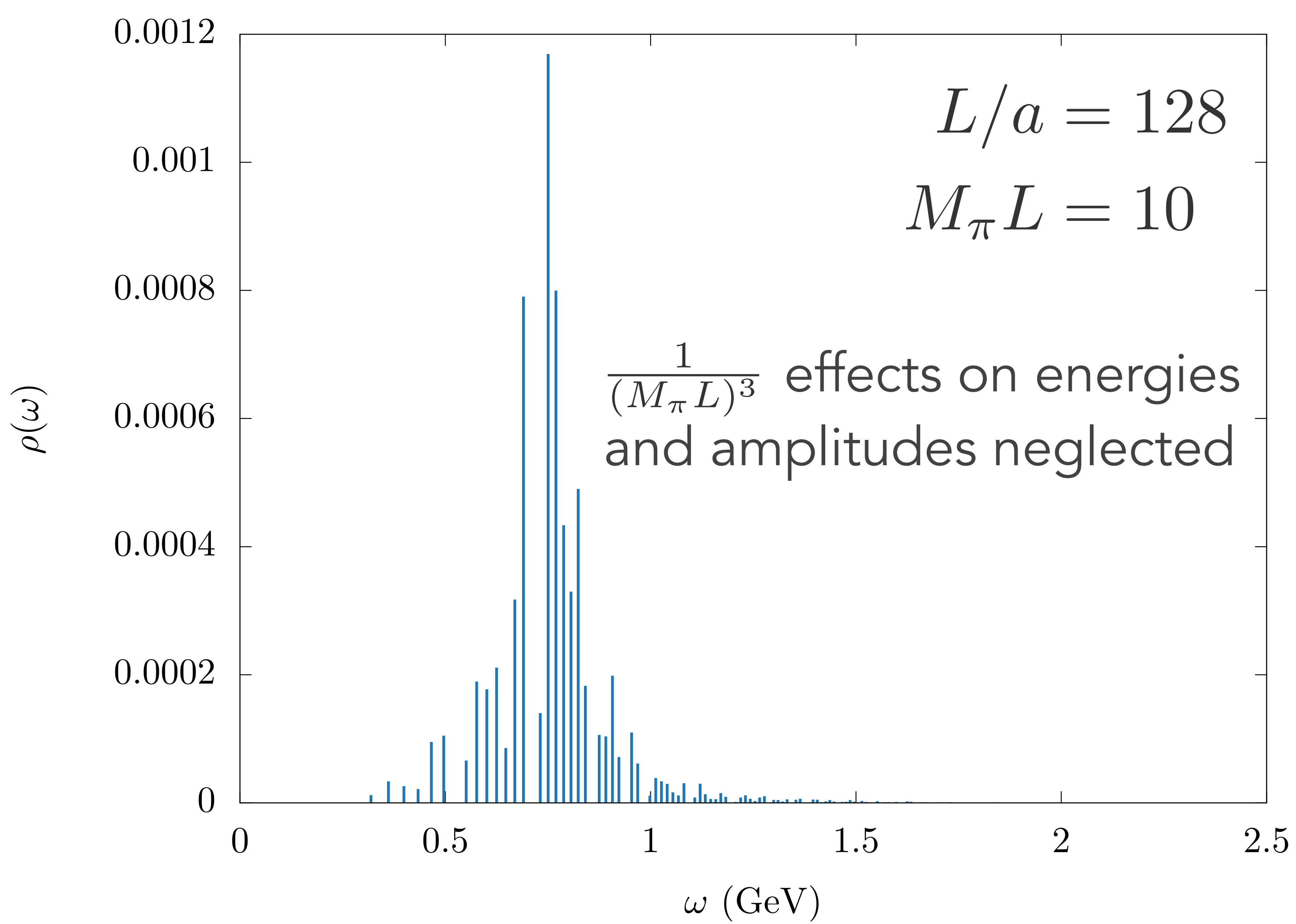
$$C_{jk} = \frac{1}{1 + \frac{1-\alpha}{\alpha}(j-k)^2} \qquad \rho_j = \frac{1}{j^2}\frac{\beta}{1+\gamma^2(j-1)^2}$$

C_{jk} : correlation between $\Pi(q_j^2)$ and $\Pi(q_k^2)$

ρ_j : relative error on $\Pi(q_j^2)$

- Realistic for $\alpha = 99.9\%$ $\beta = \mathcal{O}(1\%)$ $\gamma = 0.0039$

Finite-volume spectrum



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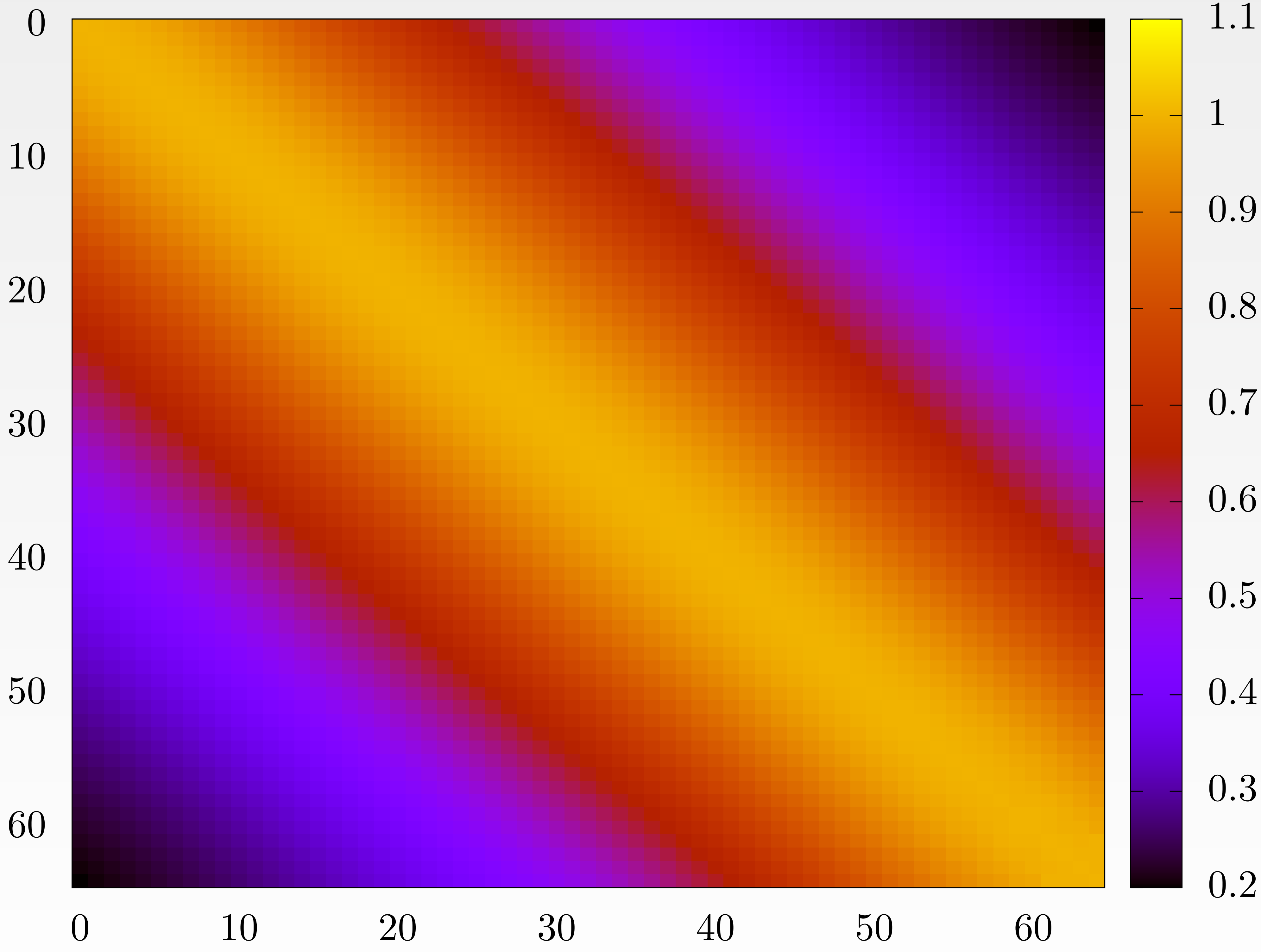
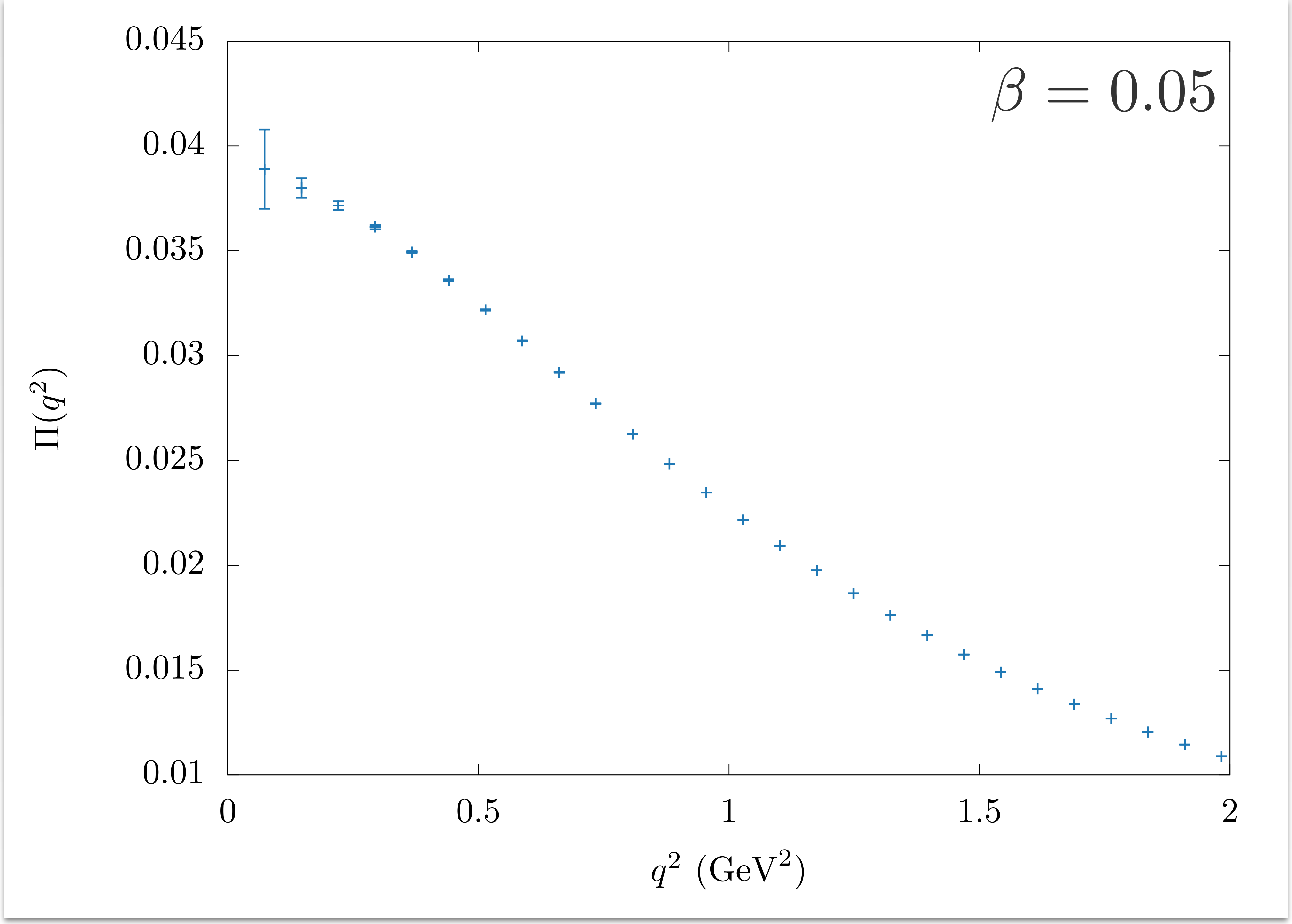
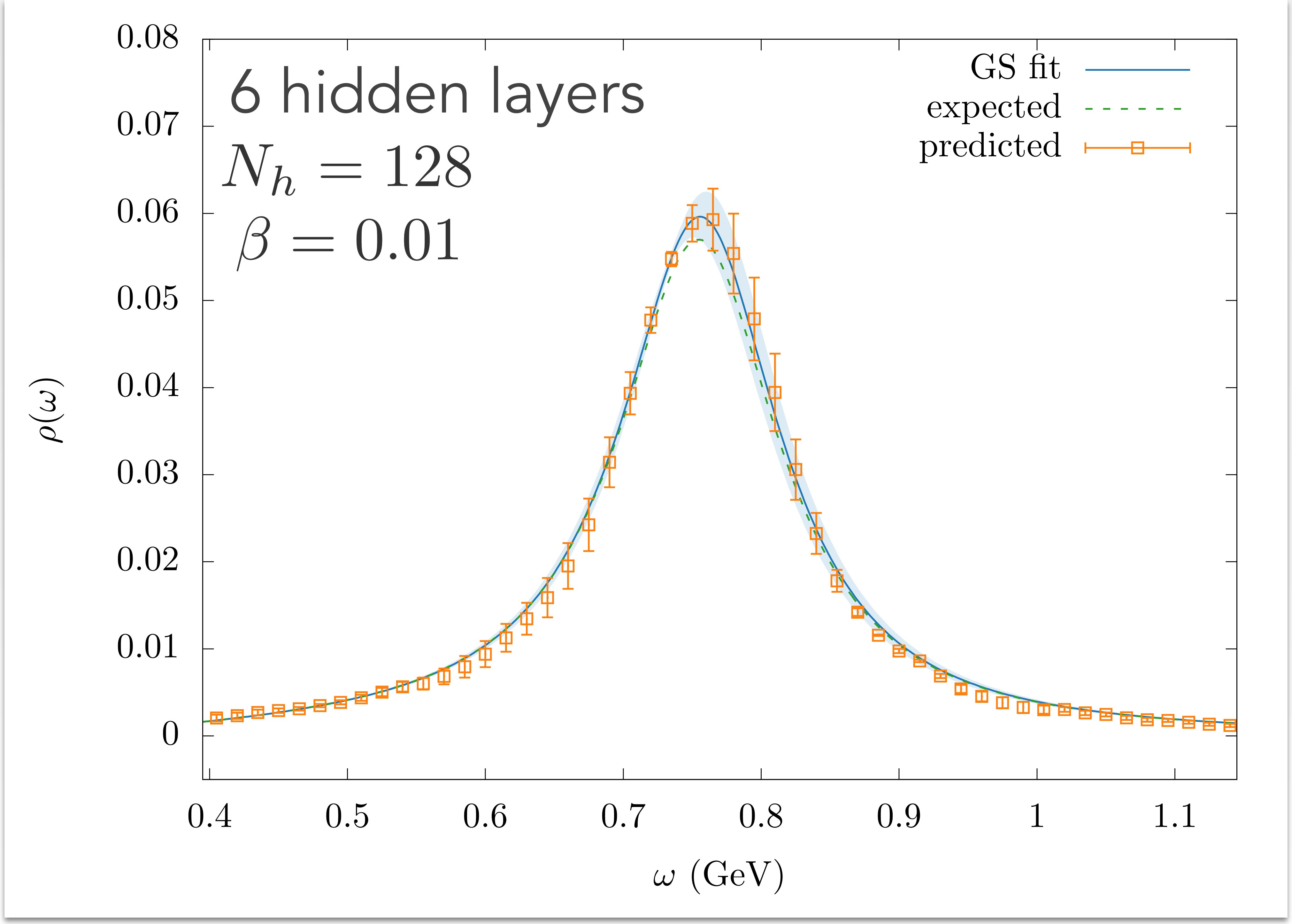


Synthetic lattice data



Prediction validation



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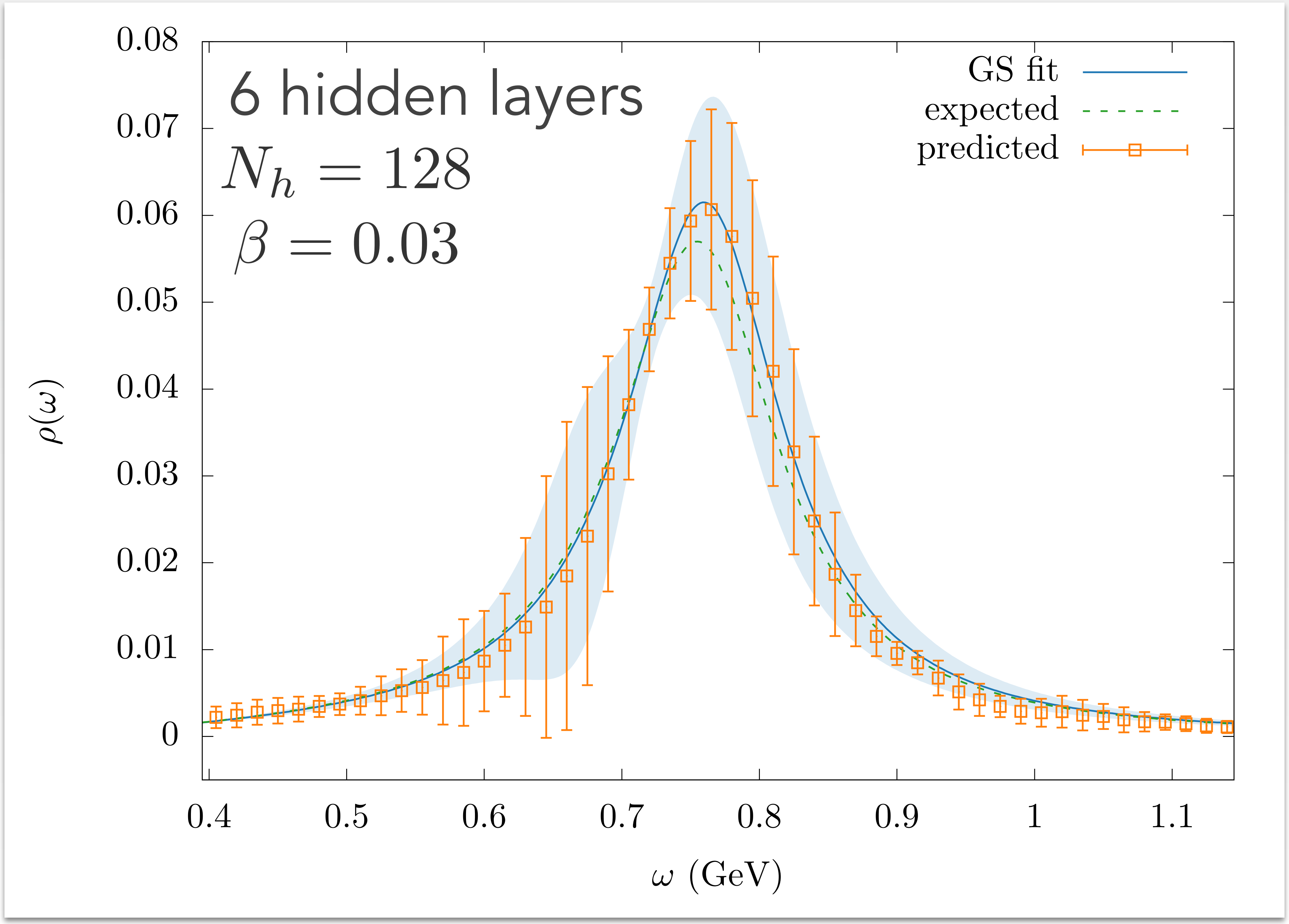

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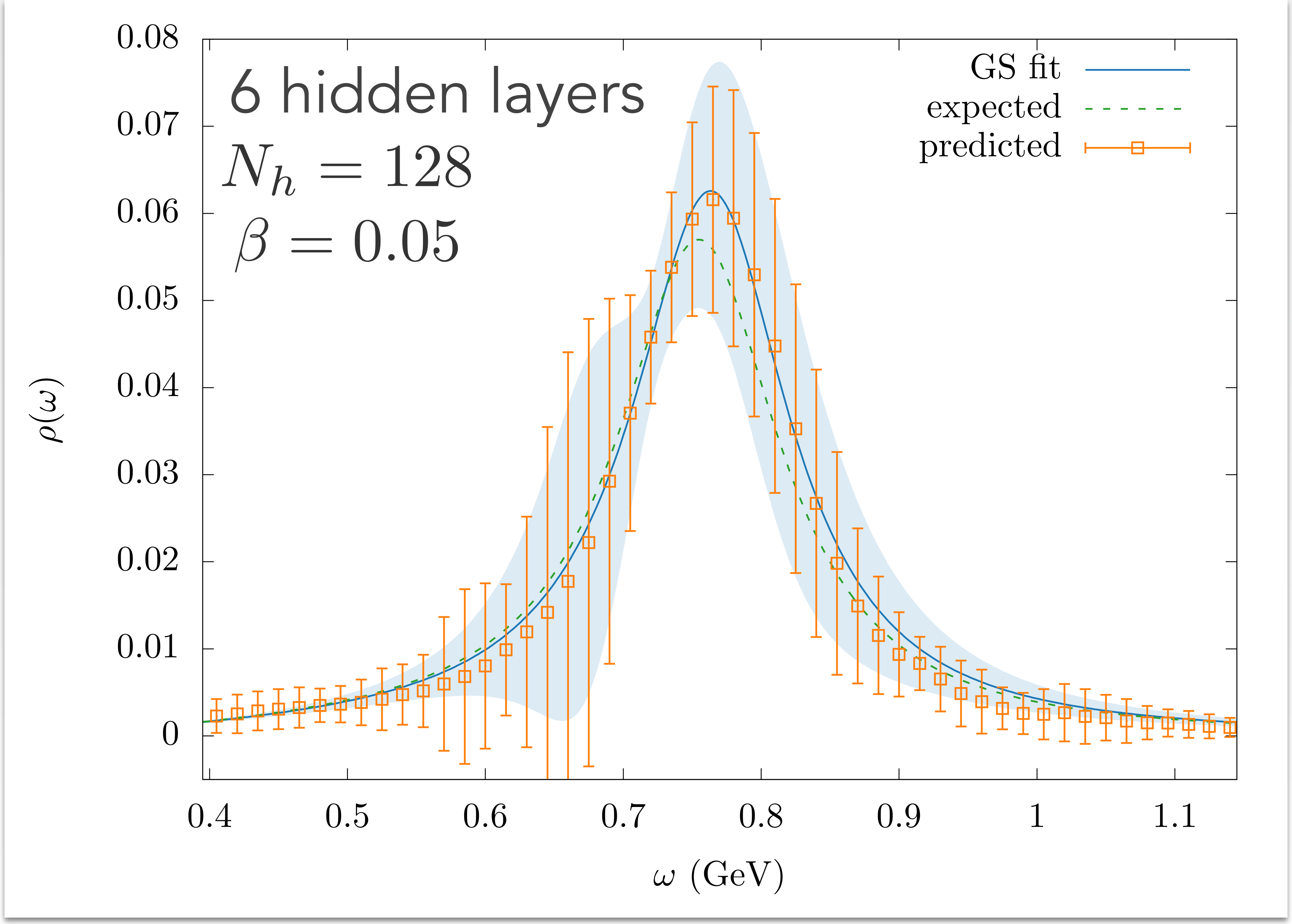

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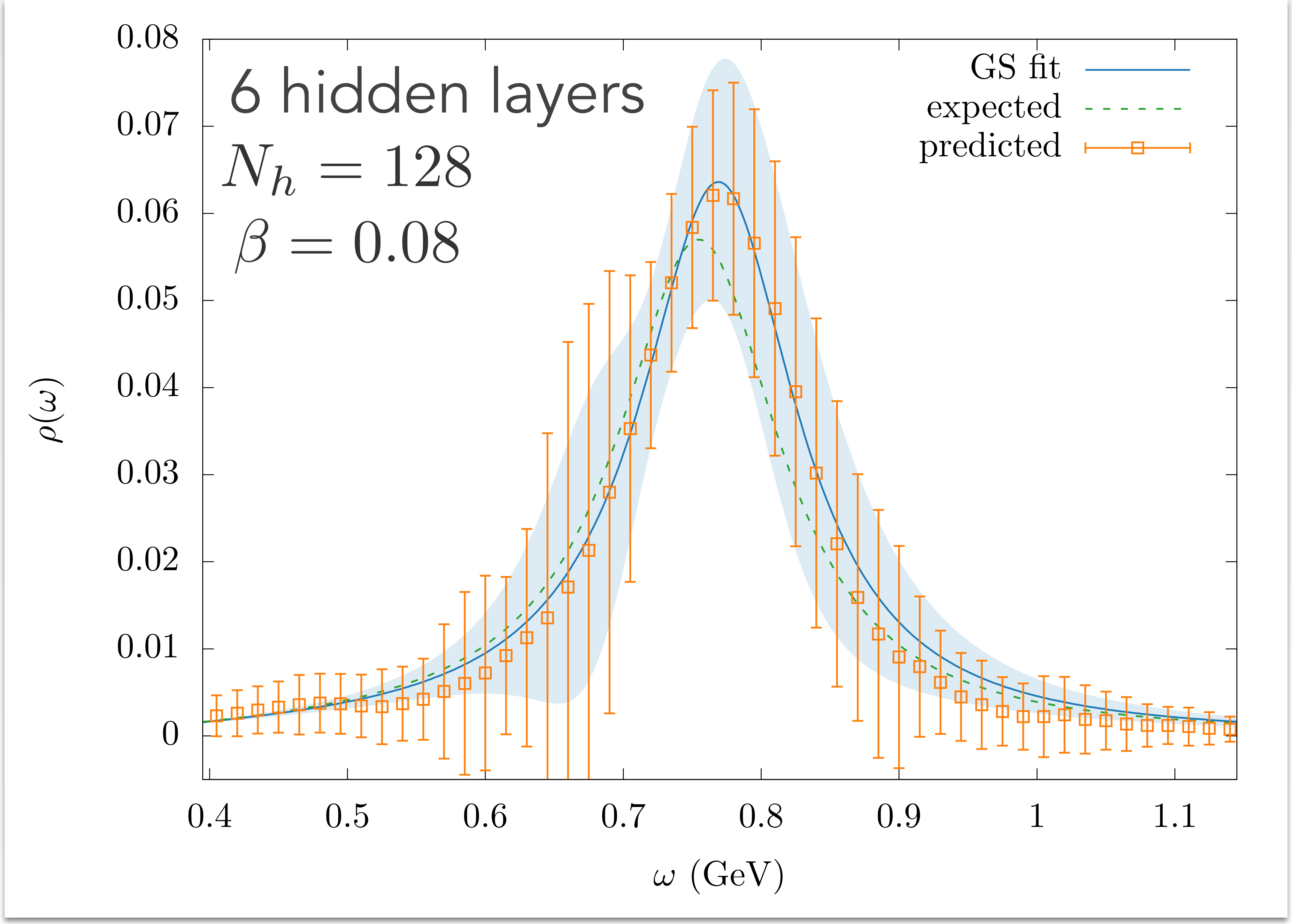
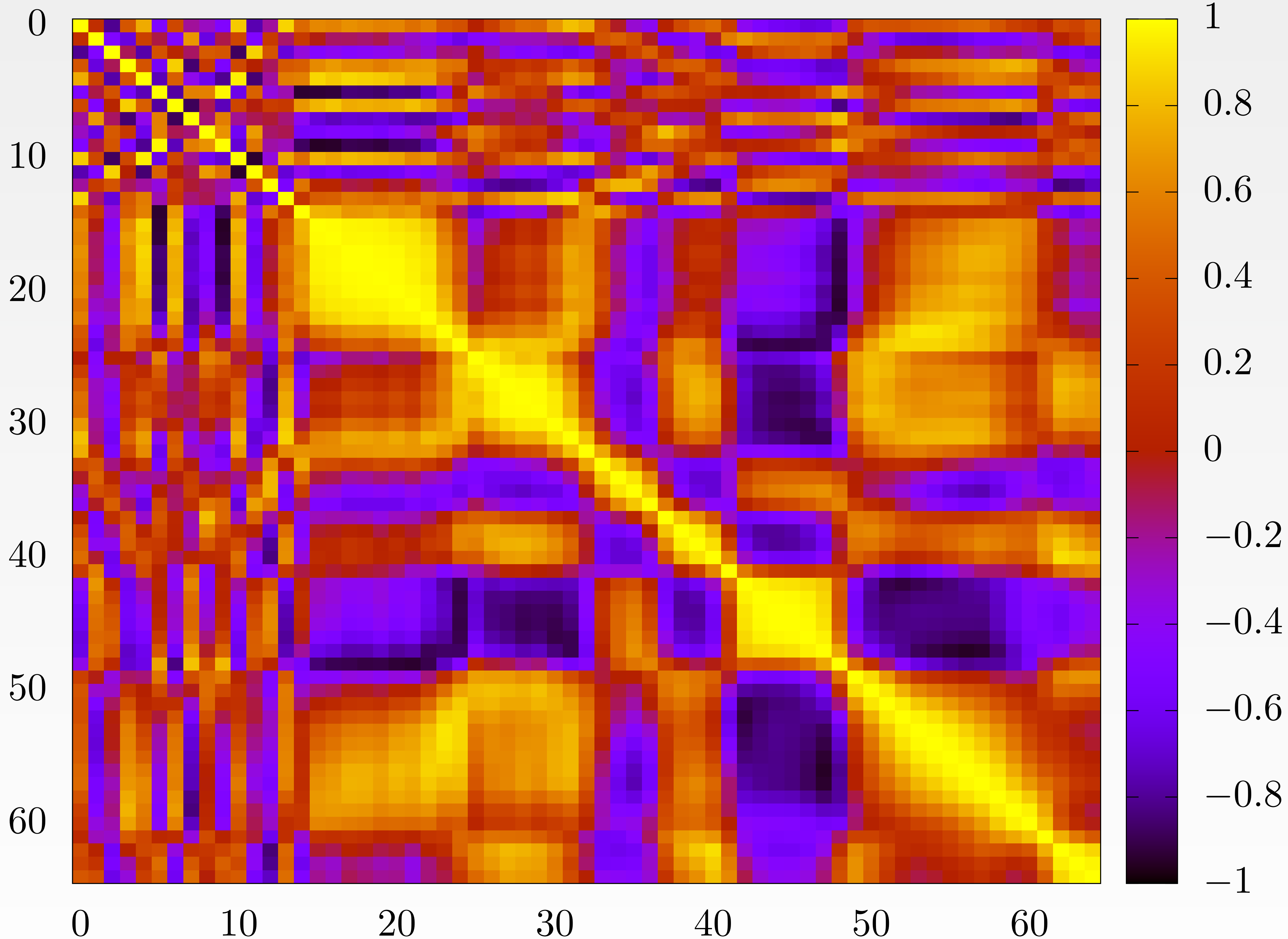
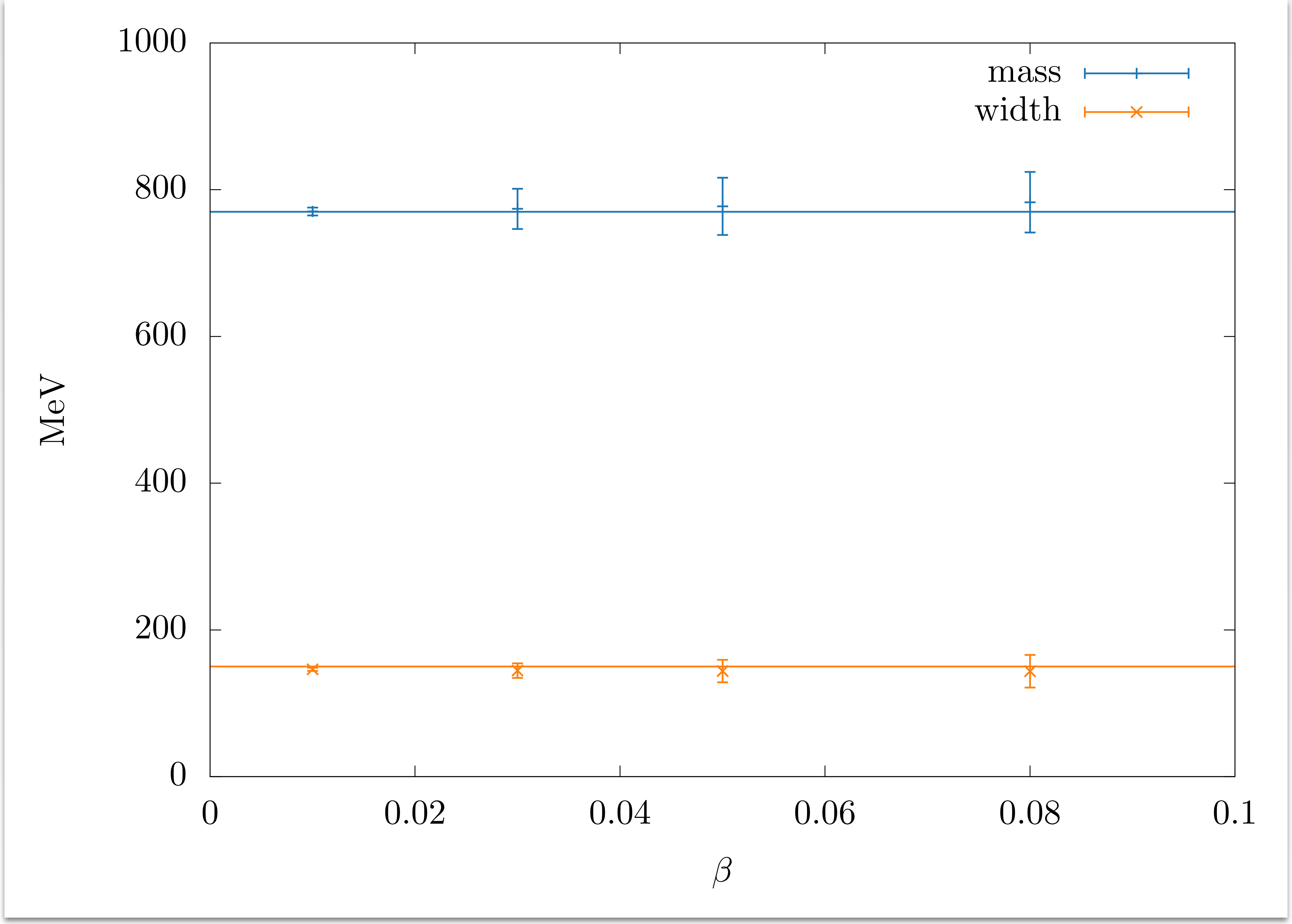


Prediction correlations

?!?



Summary of prediction validations



Summary & perspectives

Summary

- Neural networks are an **attractive approach to inverse problems** for data analysis.
- Neural networks allow to **infer non-linear generalisations** to linear algorithms.
- Spectral reconstruction: possible for unsmeared density using **deep** neural networks.
- **Seems to survive an prediction test** using realistic synthetic Monte-Carlo data.

Perspectives

- Direct comparison to Backus-Gilbert techniques.
- Study of generalisation
  (multi-peaks or even more arbitrary)
- Other approaches *e.g.* classifiers?
- Test on real large-volume lattice data.

Monument Valley 25/08/2019

Thank you!

Infinite-volume 2-pion contribution

- Amplitude and spectral measure separation

$$A(\omega_\alpha^2) = \sum_\mu |\langle 0| J_\mu(0) |\alpha\rangle|^2 \qquad \mu(s) = \int \mathrm{d}\alpha\, \delta(\omega_\alpha^2 - s)$$

$$\rho(s) = \frac{1}{3s} A(s)\mu(s)$$

- 2-pion contribution

$$A_{\pi\pi}(s) = \frac{2}{\sqrt{s}}(s - 4M_\pi^2)|F_\pi(s)|^2 \quad \mu_{\pi\pi}(s) = \frac{1}{32\pi^2}(s - 4M_\pi^2)^{\frac{1}{2}}$$

$$\rho_{\pi\pi}(s) = \frac{1}{48\pi^3}\left(1 - \frac{4M_\pi^2}{s}\right)^{\frac{3}{2}} |F_\pi(s)|^2$$

Finite-volume 2-pion contribution

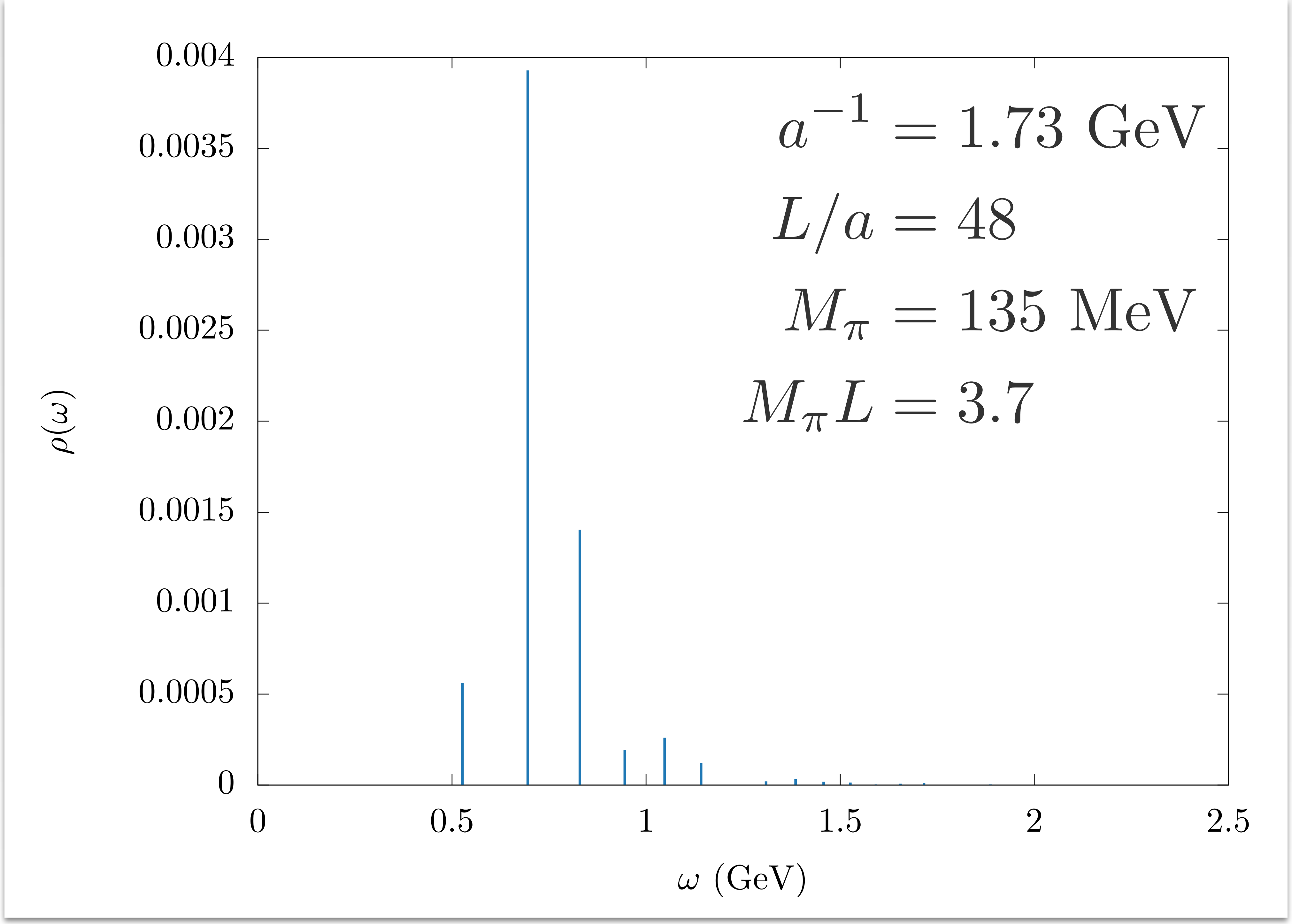
- Two assumptions:
  1) use free pions quantisation
  2) use infinite-volume amplitudes

Only accurate if $(M_\pi L)^{-3}$ **can be neglected**
(typically $M_\pi L \gtrsim 10$)

$$\mu_{\pi\pi,\mathrm{FV}}(s) = \frac{1}{L^3}\sum_{n=0}^{+\infty} r_3(n)\delta[s - 4\omega_\pi(\tfrac{4\pi^2}{L^2}n)^2]$$

$$\rho_{\pi\pi,\mathrm{FV}}(s) = \frac{2}{3L^3\sqrt{s}}\left(1 - \frac{4M_\pi^2}{s}\right)|F_\pi(s)|^2\sum_{n=0}^{+\infty} r_3(n)\delta[s - 4\omega_\pi(\tfrac{4\pi^2}{L^2}n)^2]$$

Finite-volume spectrum



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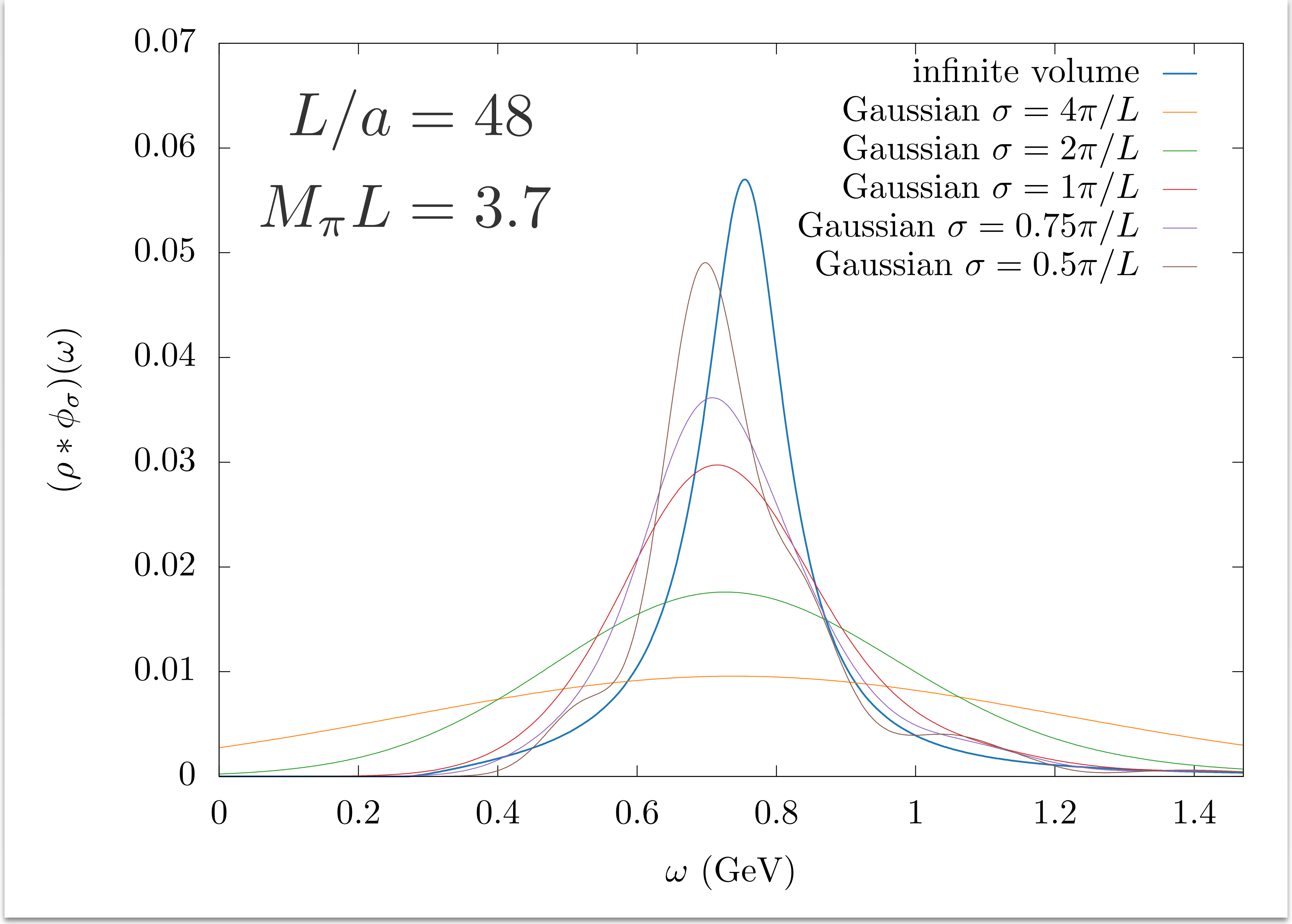

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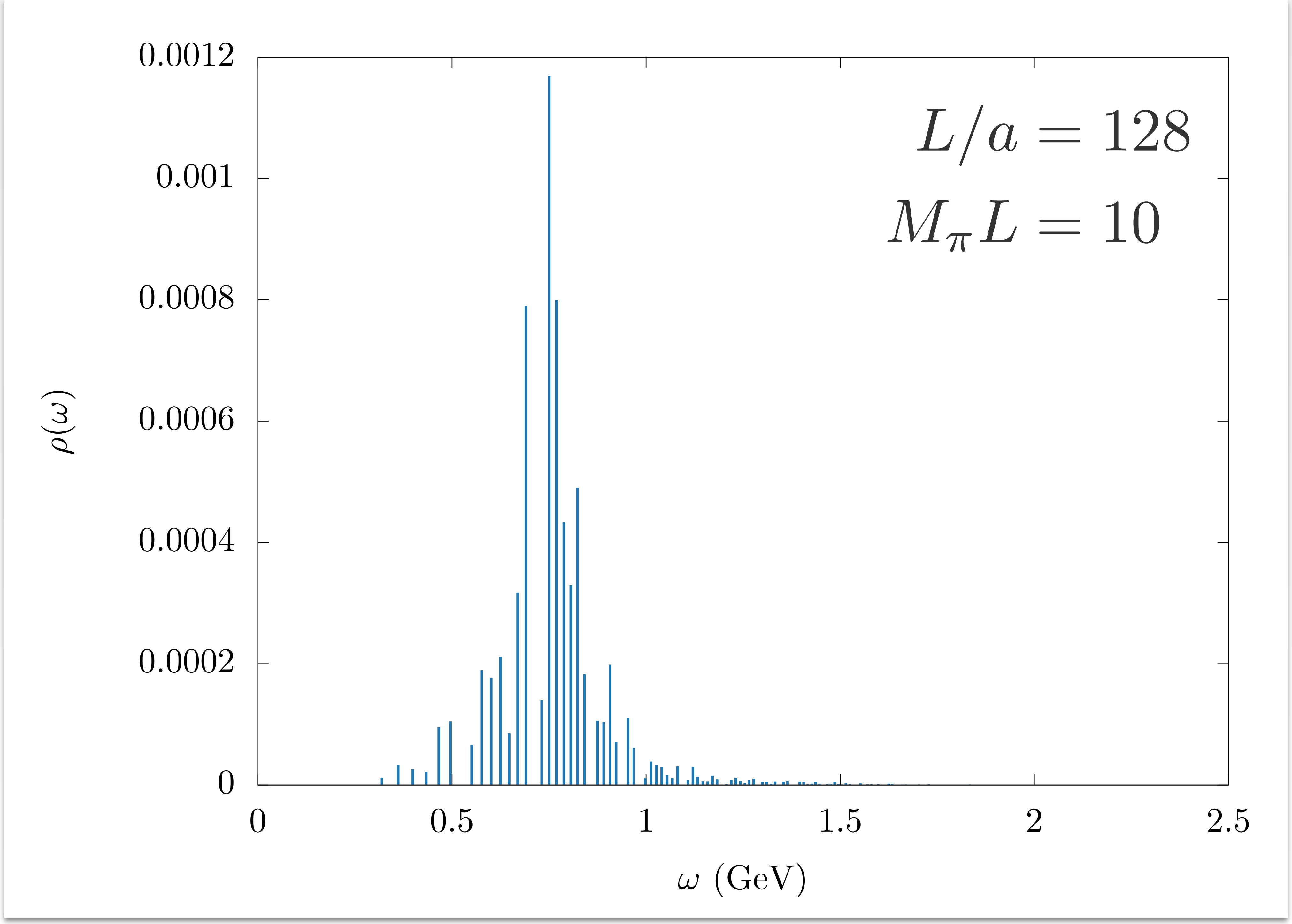

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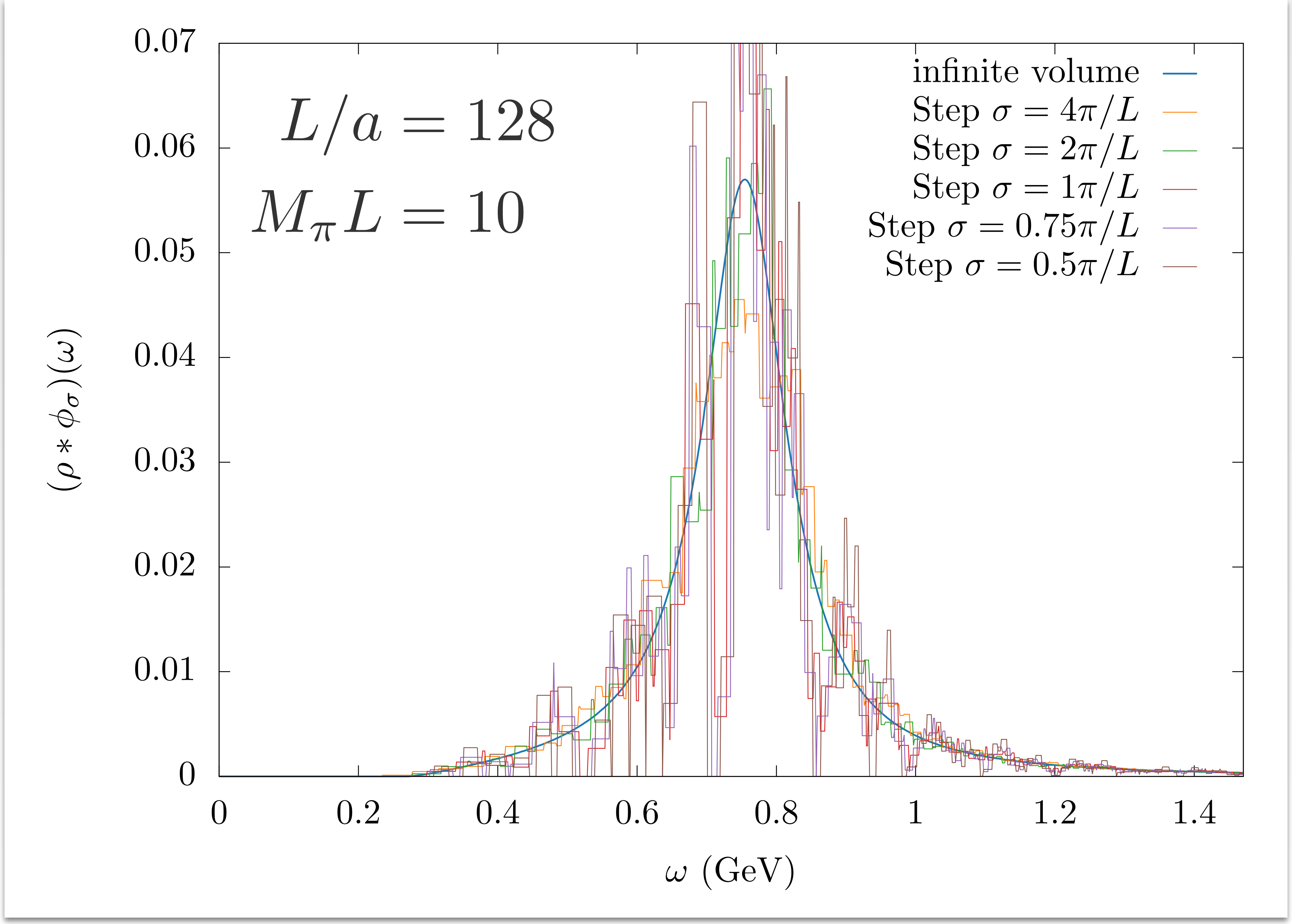

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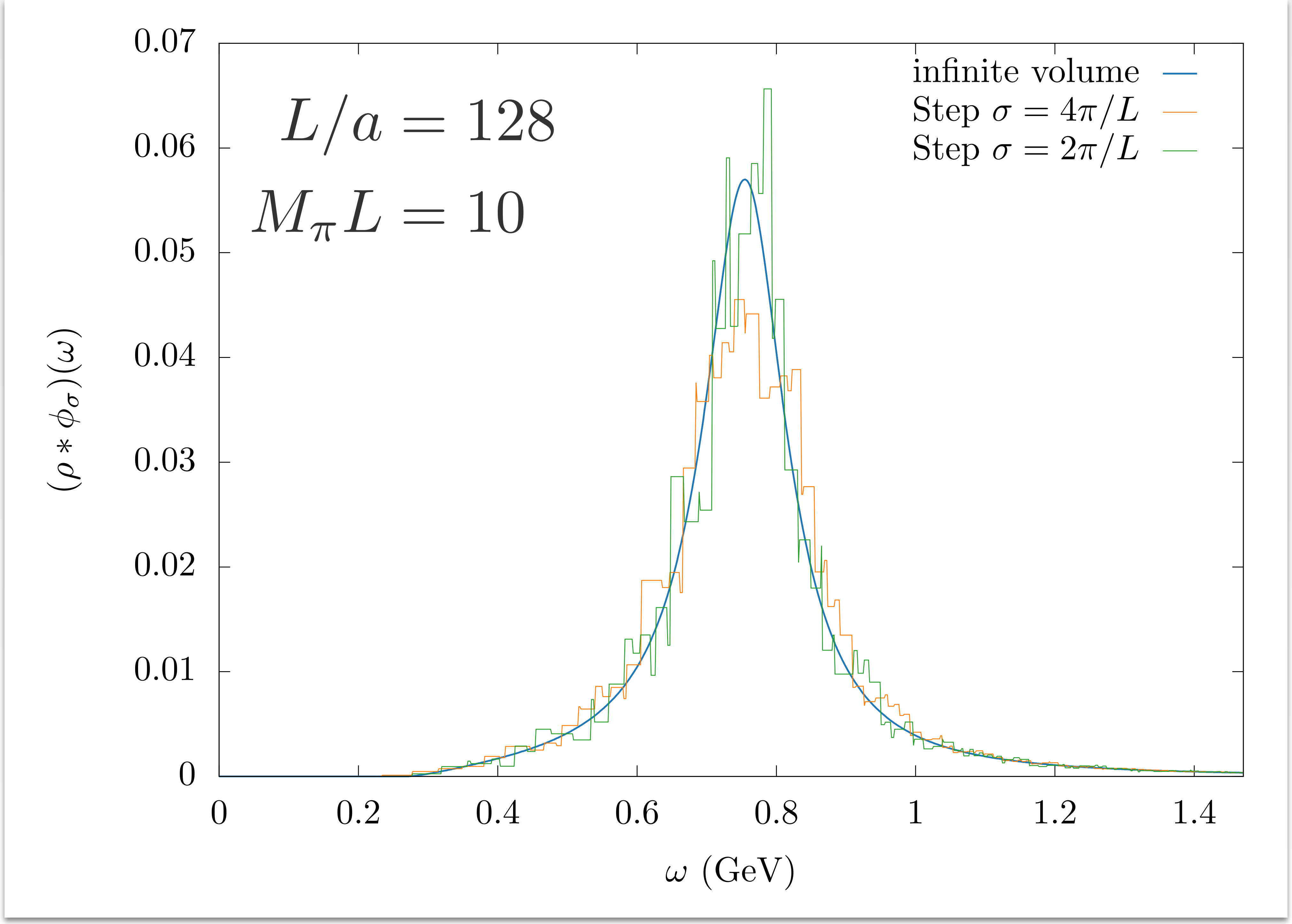

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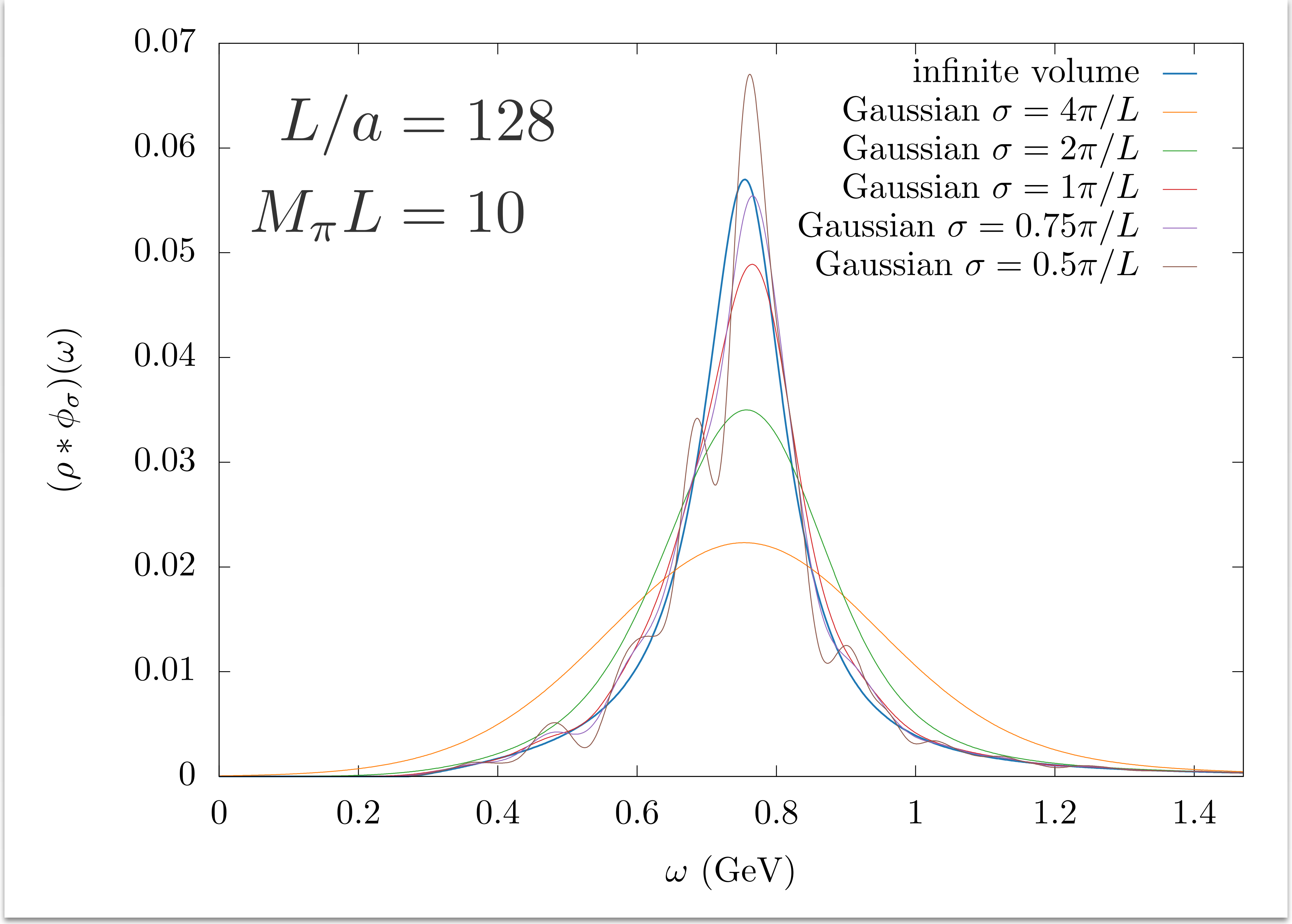

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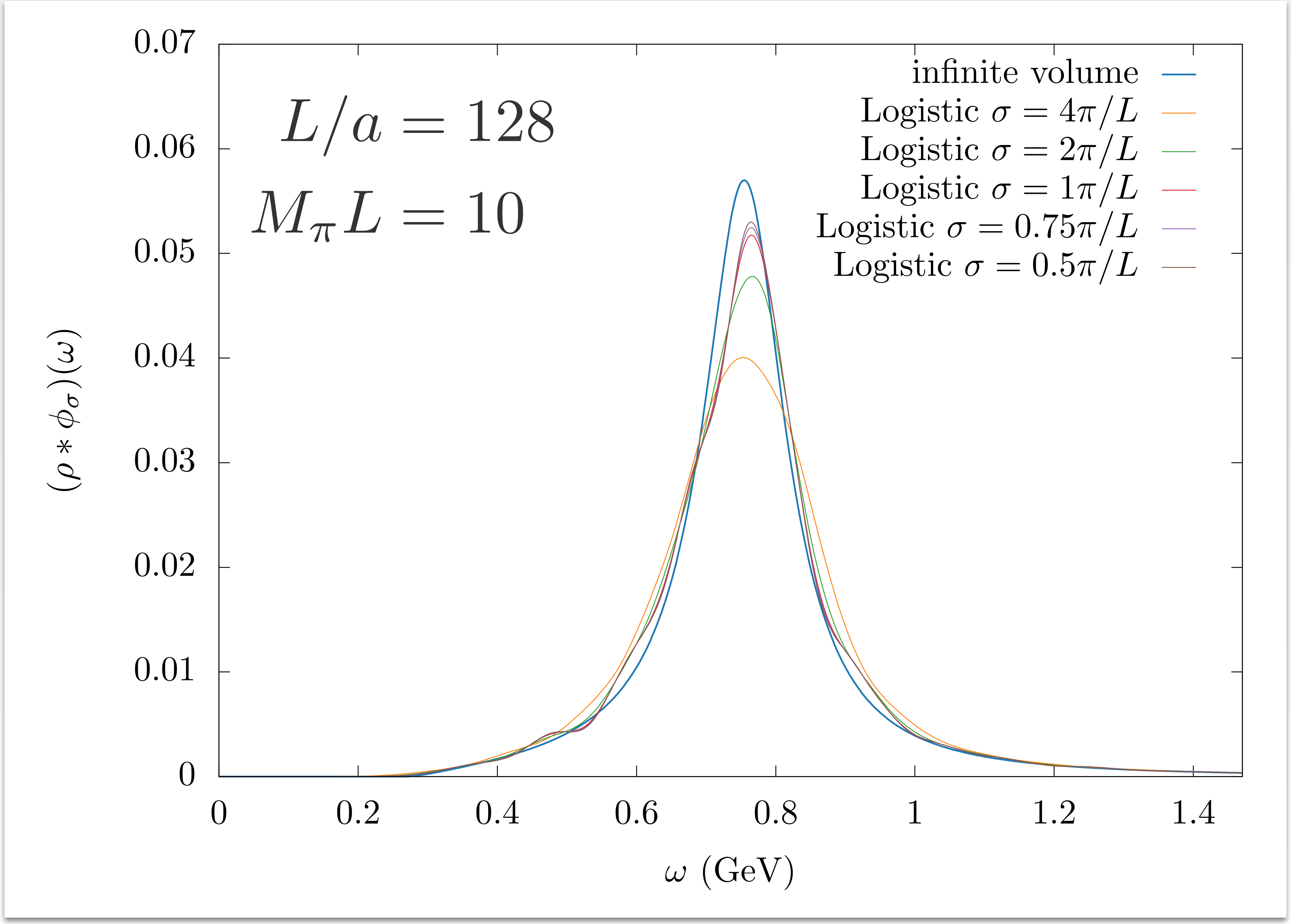

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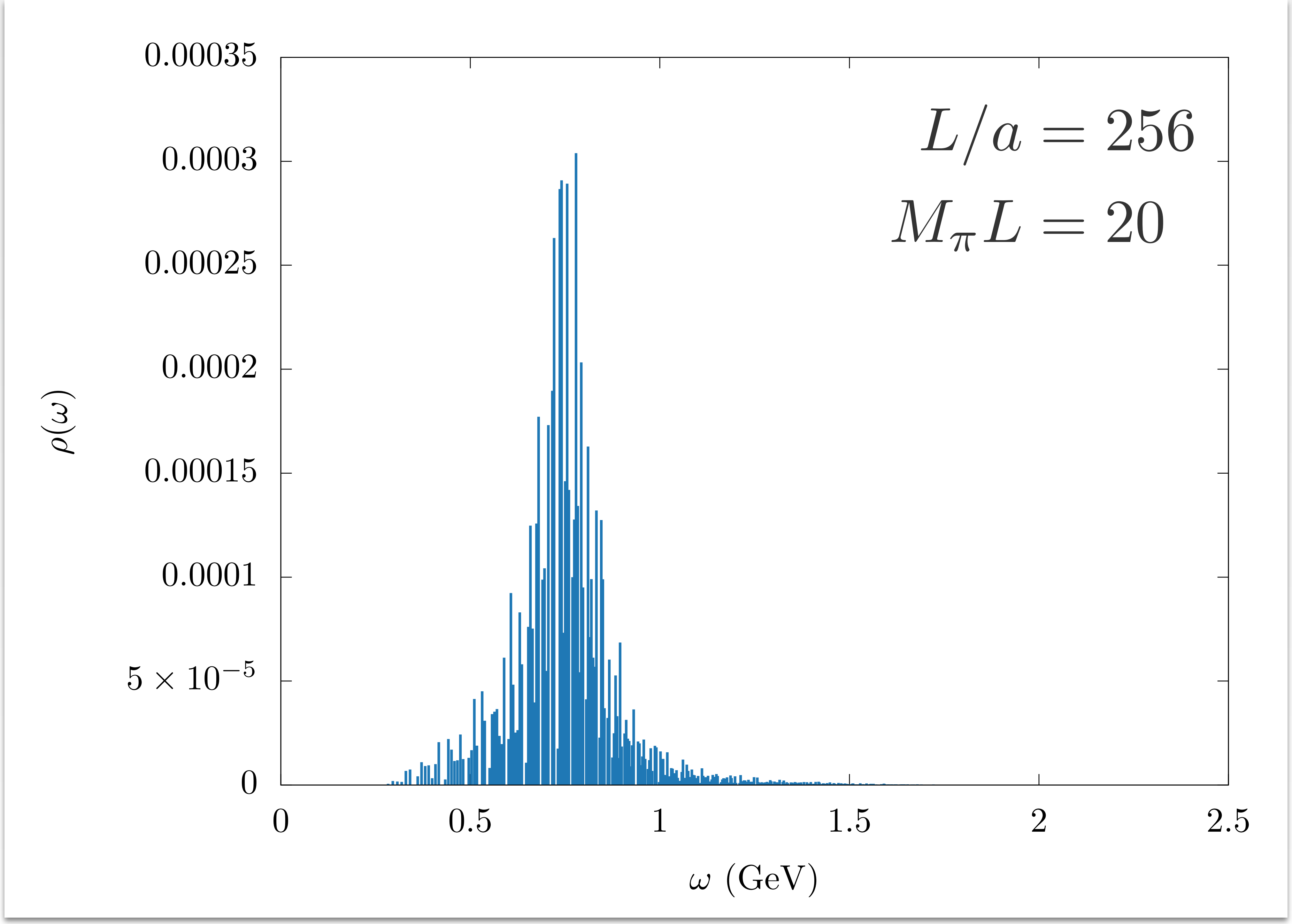

Finite-volume spectrum