Finite volume matrix elements of two-body states with one current insertion

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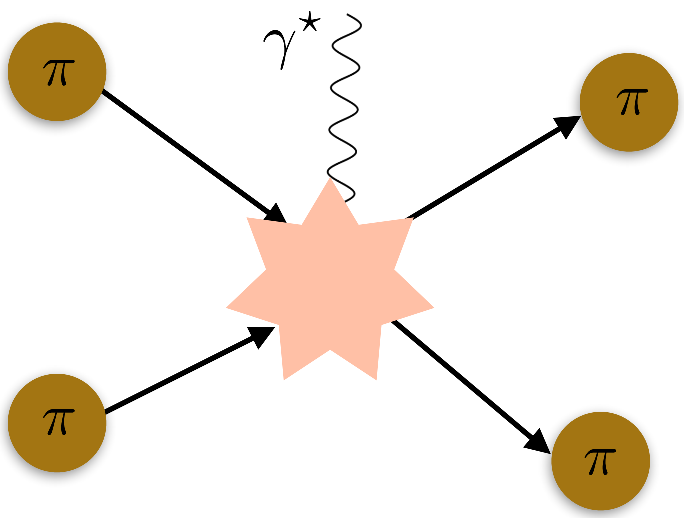
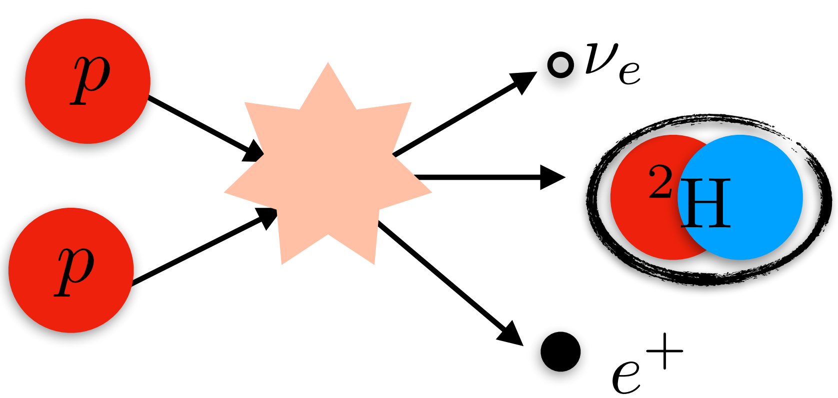
Few-body electroweak processes



- Can be used to access resonance
  - Obtain structural information about resonances (form factors)



- Start of the main chain reaction that powers the Sun

Recently in LQCD (Savage et al. 2015)

Plenty of other reactions relevant for nuclear physics:

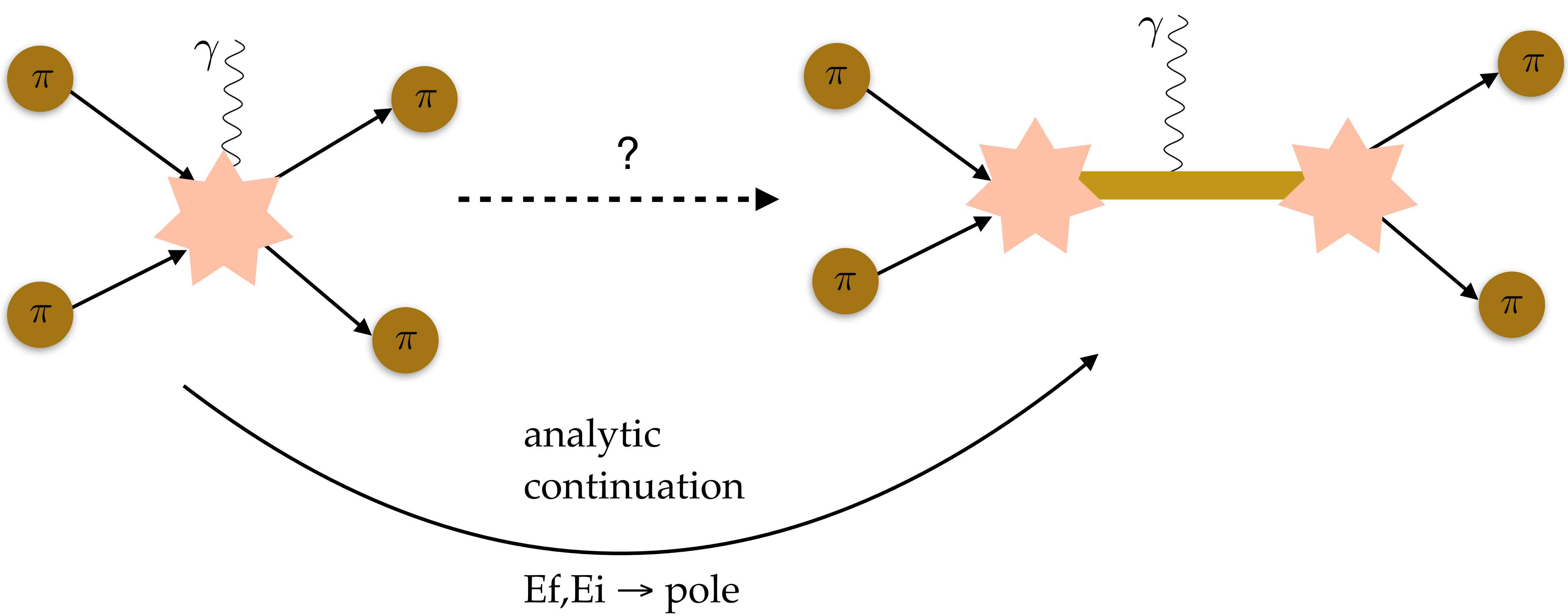
- $n + p \to d + \gamma$
- Neutrino-deuteron low energy scattering

$\cdots$

Form factor of a resonance

- Resonance are not asymptotic states → Form factor?

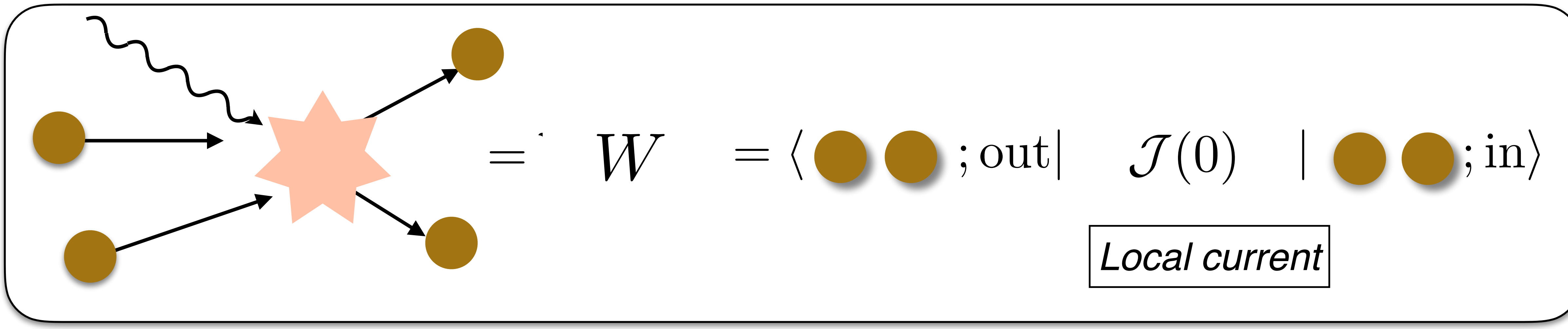
(Standard LSZ theorem cannot be used)



Looking at the scattering amplitude we can access

Two-to-two scattering with one current insertion

- Scalar particles

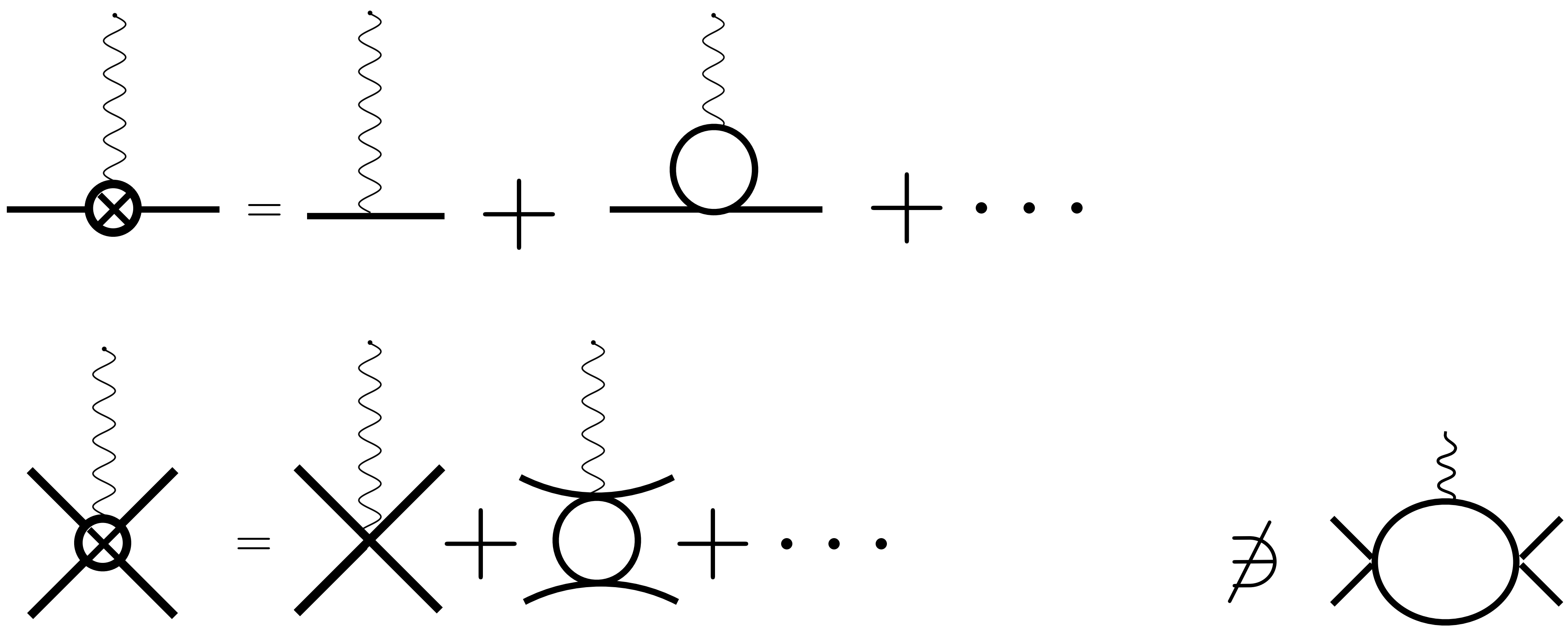


- How can we use all orders “perturbation” theory to represent this quantity?

Briceño & Hansen (2016)

All orders "perturbation" theory

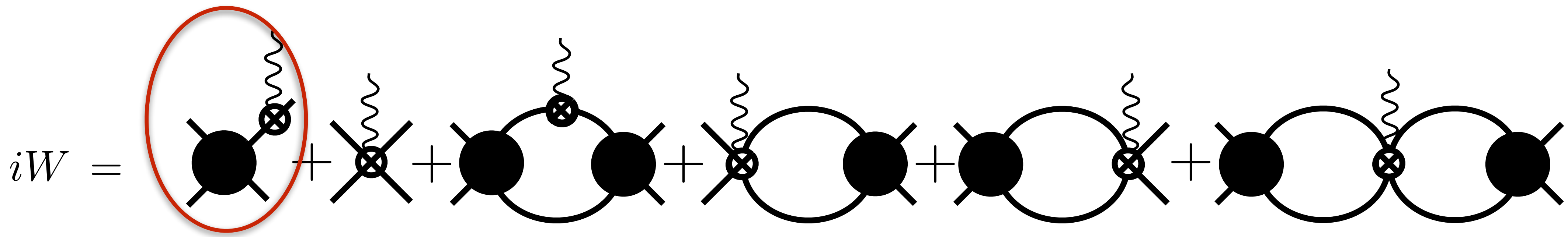
- Define the following kernels (all Feynman diagrams but s-channel)



- **In the kinematic window of interest (below multi particle thresholds) there are no singularities at finite values of the intermediate momenta**

All orders “perturbation” theory

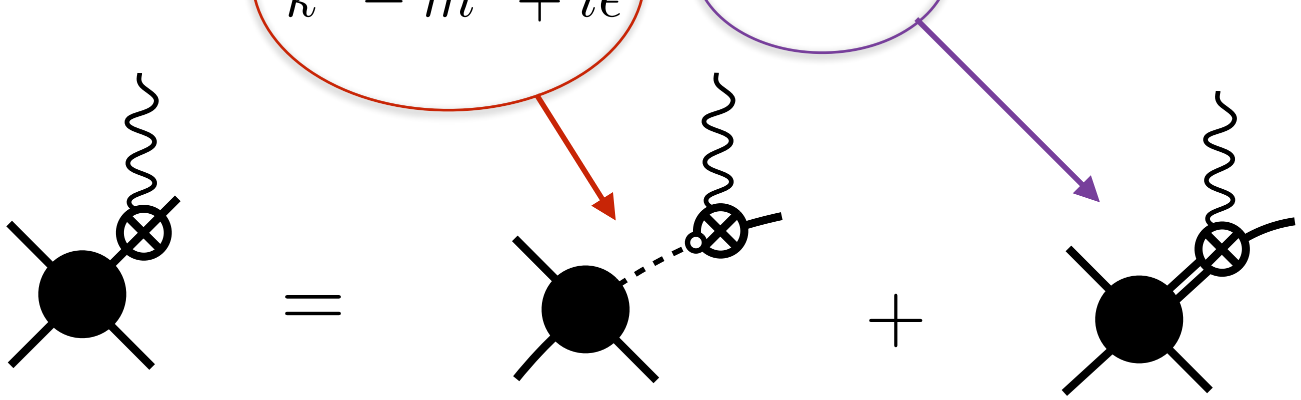
- Using the kernels of the previous slide we can define a skeleton expansion for the matrix element



Can have kinematic singularities

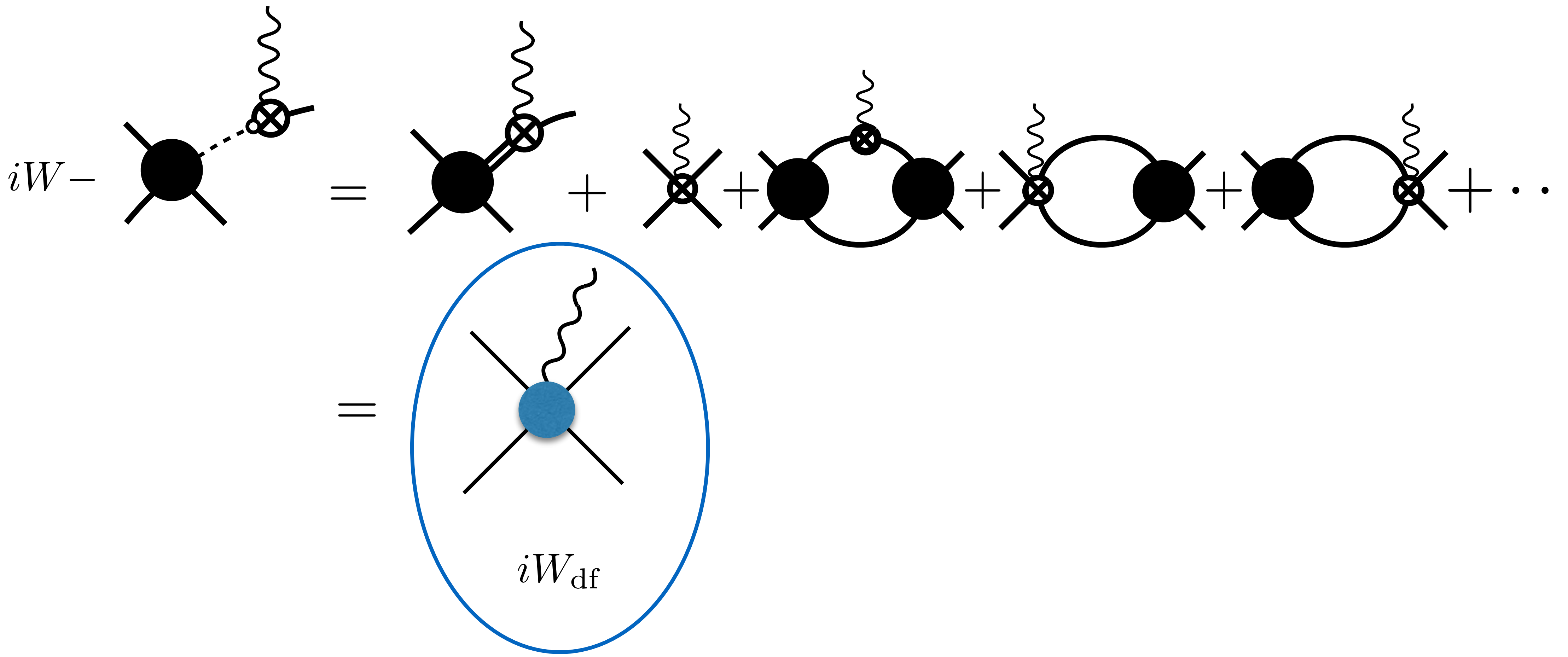
- For fully dressed propagators

$$i\Delta = \frac{i}{k^2 - m^2 + i\epsilon} + \text{smooth}$$



- Briceño & Hansen (2016)
- AB, Briceño, Hansen, Ortega (2018)

Two-to-two scattering with current

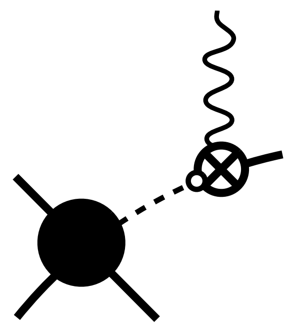


- The object related to the FV matrix element is *not* the full scattering amplitude
- We can recover the full scattering amplitude adding back diagrams with kinematic singularities

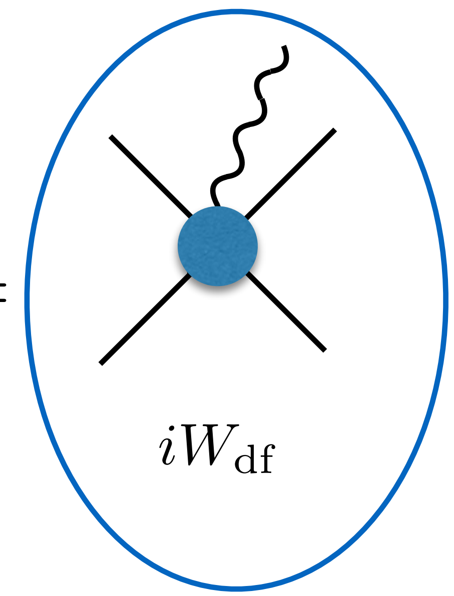
Two-to-two scattering with current

For resonances and bound states

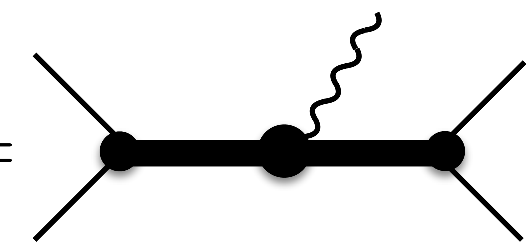
$$\lim_{\substack{E_i^{\rm cm},E_f^{\rm cm}\to E_R \\ (E_B)}} iW -$$



$$- \cdots =$$



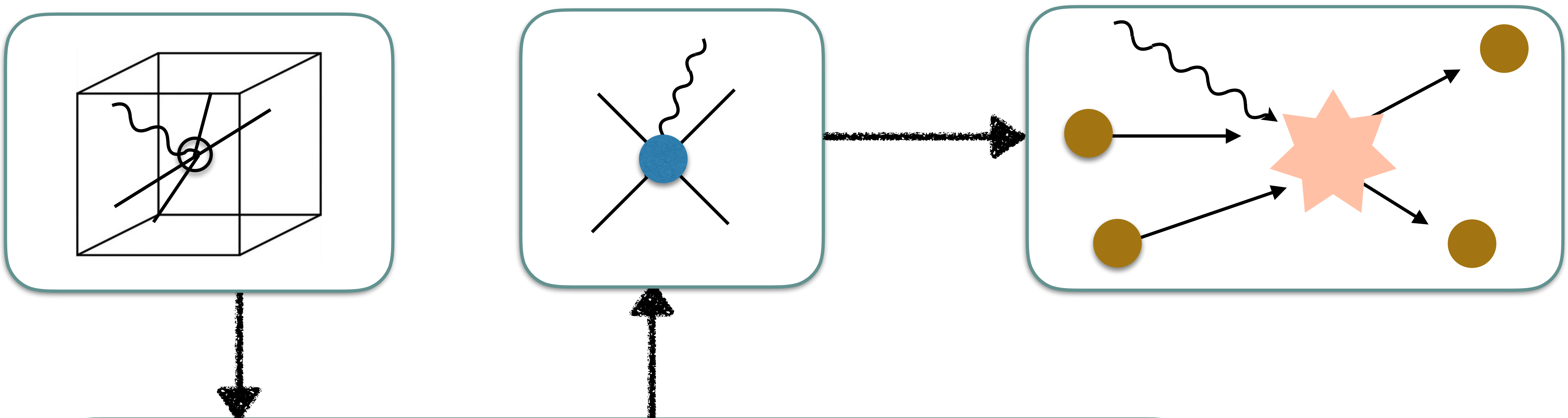
$$=$$



Form factors directly related to $W_{\rm df}$

$2 + \mathcal{J} \to 2$

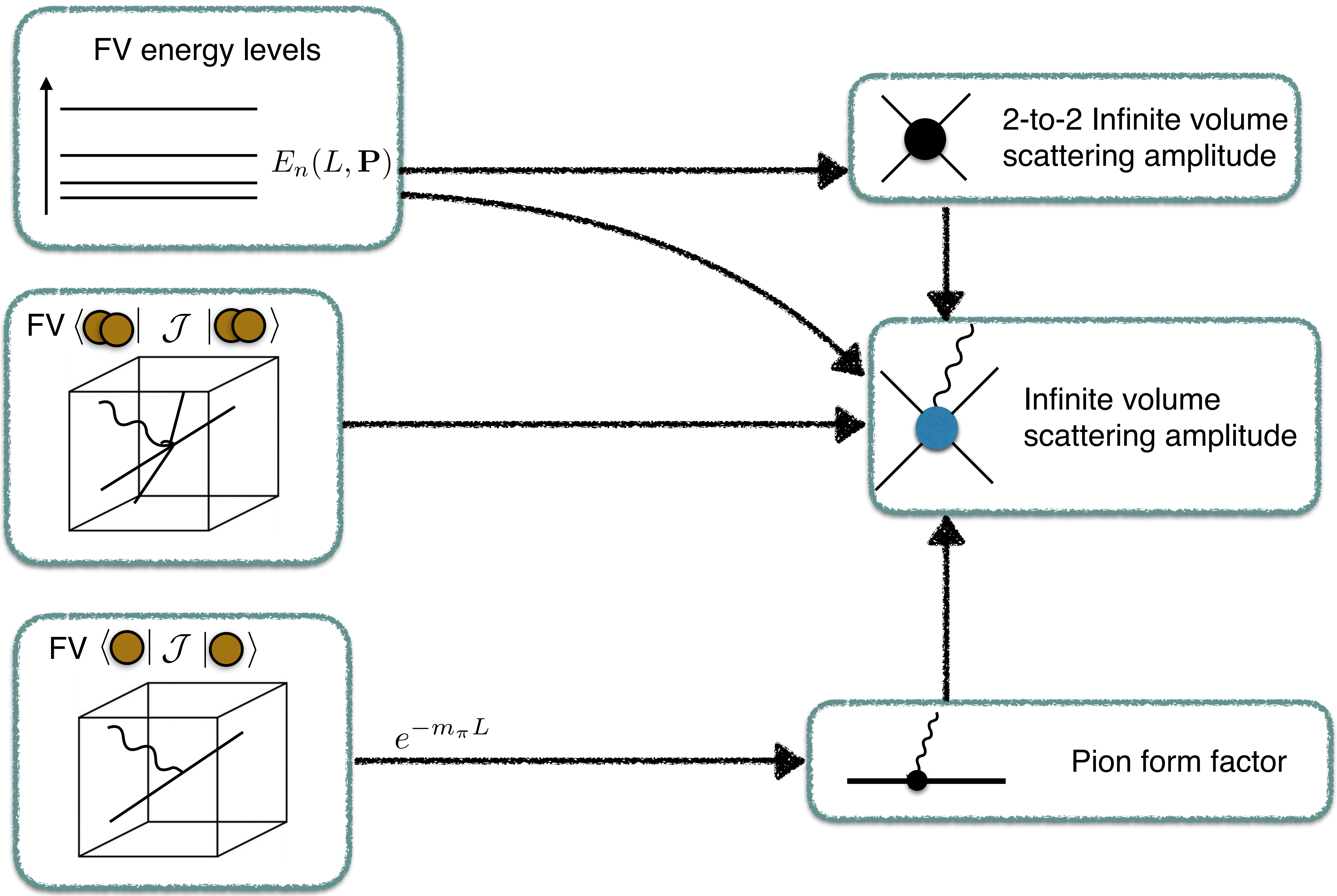
FV matrix elements to infinite volume electroweak amplitudes



$$|\langle 2|\mathcal{J}|2\rangle|_L^2 = \frac{1}{L^6}\mathrm{Tr}\left[R(E_L, L)W_{L,\mathrm{df}}R(E_L, L)W_{L,\mathrm{df}}\right]$$

- Briceño & Hansen (2016)
- AB, Briceño, Hansen, Ortega (2019)

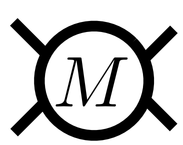
Roadmap


FV energy levels
$E_n(L, \mathbf{P})$
2-to-2 Infinite volume scattering amplitude
FV $\langle \bigcirc\!\bigcirc | \mathcal{J} | \bigcirc\!\bigcirc \rangle$
Infinite volume scattering amplitude
FV $\langle \bigcirc | \mathcal{J} | \bigcirc \rangle$
$e^{-m_\pi L}$
Pion form factor

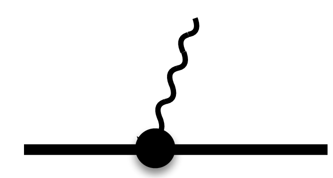
$2+\mathcal{J}\to 2$

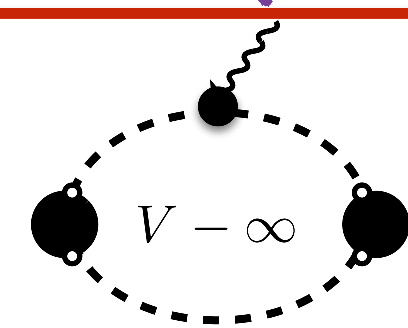
$$\langle 2|\mathcal{J}|2\rangle|_L^2 = \frac{1}{L^6}\mathrm{Tr}\left[R(E_L,L)W_{L,\mathrm{df}}R(E_L,L)W_{L,\mathrm{df}}\right]$$

$$W_{L,\mathrm{df}} = W_{\mathrm{df}} + MG(L,w)M$$

$M =$ 

2→ 2 scatt. Amplitude
Calculated at FV energies

$w =$ 



$$G = \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right](\cdots)$$

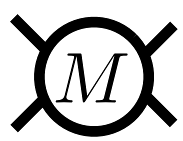
Crucial ingredient that we studied

$2+\mathcal{J}\rightarrow 2$

Special case

$$\langle 2|\mathcal{J}|2\rangle|_L^2 = \frac{1}{L^6}\mathrm{Tr}\left[R(E_L,L)W_{L,\mathrm{df}}R(E_L,L)W_{L,\mathrm{df}}\right]$$

$$W_{L,\mathrm{df}} = W_{\mathrm{df}} + MG(L,w)M$$

$M =$ 

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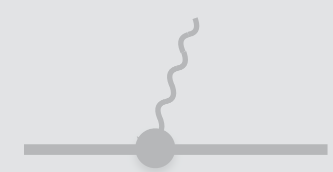
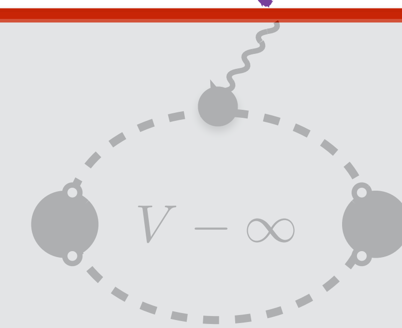
$w =$ 

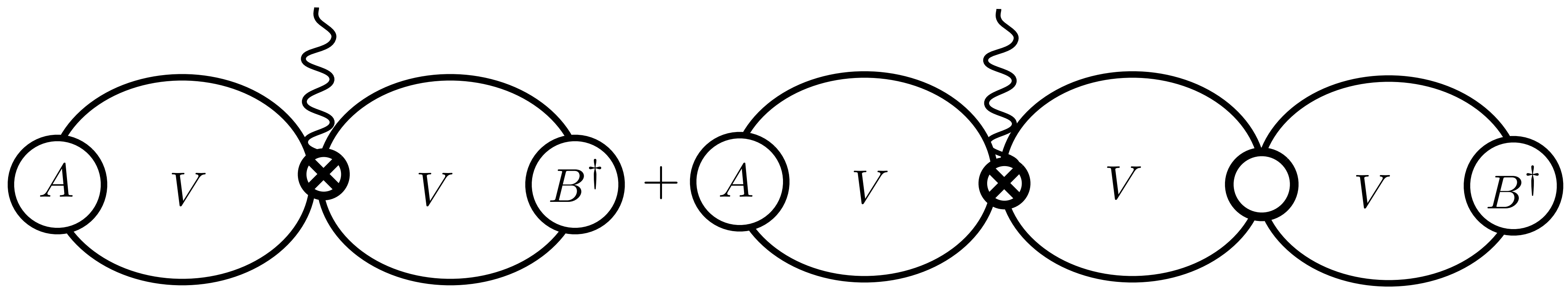


$$G = \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right](\cdots)$$

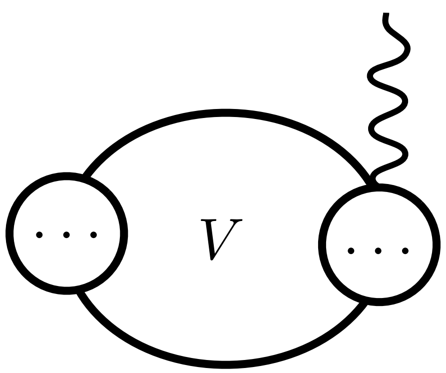
Crucial ingredient that we studied

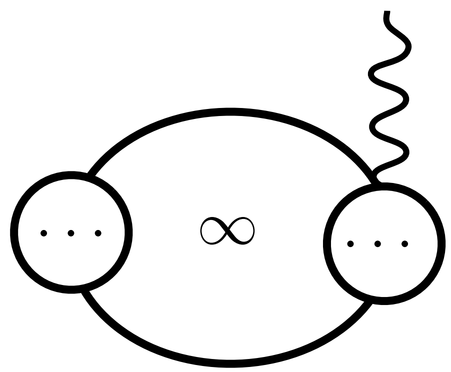
Finite volume correlator

Special case

$C_L^{3\,pt.}(P_i, P_f) =$ 

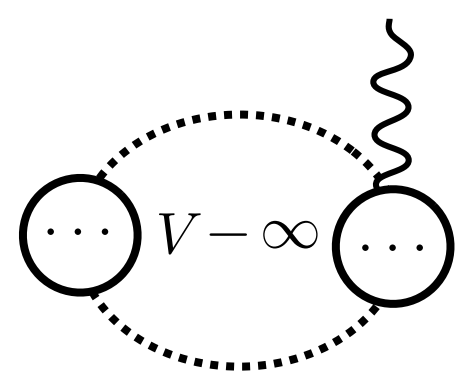
$+ \cdots$



$-$ 

$$= \left[\frac{1}{L^3} \sum_{\mathbf{k}} - \int d\mathbf{k} \right] \int dk^0 \frac{(\cdots)\,(\cdots)}{(k^2 - m^2 + i\epsilon)[(P-k)^2 - m^2 + i\epsilon]}$$

$$= \left[\frac{1}{L^3} \sum_{\mathbf{k}} - \int d\mathbf{k} \right] \frac{(\cdots)\,(\cdots)|_{k^0=\omega_k}}{(2\omega_k)^2 (E - 2\omega_k + i\epsilon)} =$$



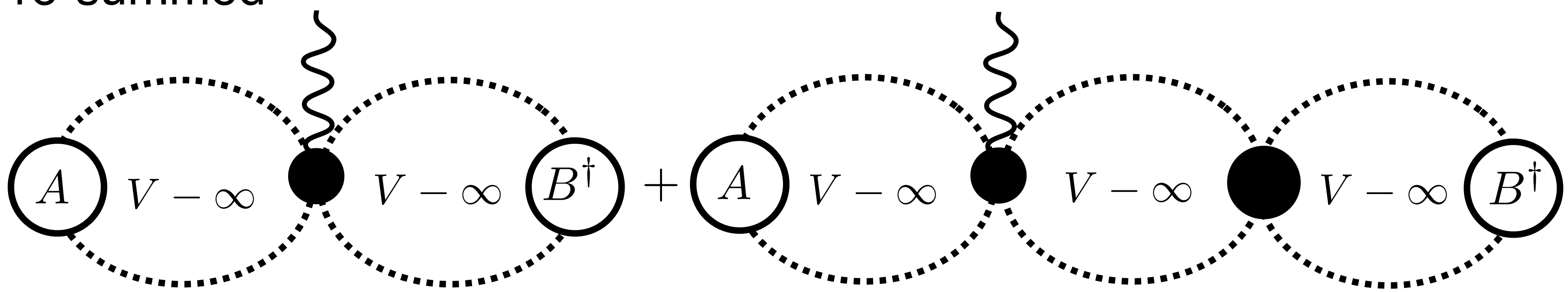
$= (\cdots)|_{\text{on}}\, iF\, (\cdots)|_{\text{on}}$

Finite volume correlator

Special case

- Series below can be re-summed

$C_L^{3\,pt.}(P_i, P_f) =$



$$+\cdots$$

$$= \quad A(P_i)\frac{1}{F^{-1}(P_i, L) + M(P_i)} W(P_i, P_f) \frac{1}{F^{-1}(P_f, L) + M(P_f)} B^\dagger(P_f)$$

$$+\cdots$$

- Fourier transform of the correlator

$$C_L^{3\,pt.}(t_i, t_f, \mathbf{P}_i, \mathbf{P}_f) = L^3 \int \frac{dP_i^0}{2\pi} \int \frac{dP_f^0}{(2\pi)} e^{iP_i^0 t_i + iP_f^0 t_f} C_L^{3\,pt.}(P_i, P_f)$$

$$= L^3 \sum_{n_i, n_f} e^{-(E_{n_i} t_i + E_{n_f} t_f)} A(E_{n_i}, \mathbf{P}_i) R(E_{n_i}, \mathbf{P}_i) W(P_i, P_f) R(E_{n_f}, \mathbf{P}_f) B^\dagger(E_{n_f}, \mathbf{P}_f)$$

Finite volume correlator

Special case

- From previous slide

$$C_L^{3\,pt.}(t_i,t_f,\mathbf{P}_i,\mathbf{P}_f) = L^3 \sum_{n_i,n_f} e^{-(E_{n_i}t_i+E_{n_f}t_f)} A(E_{n_i},\mathbf{P}_i) R(E_{n_i},\mathbf{P}_i) W(P_i,P_f) R(E_{n_f},\mathbf{P}_f) B^\dagger(E_{n_f},\mathbf{P}_f)$$

- From standard spectral decomposition of the matrix element

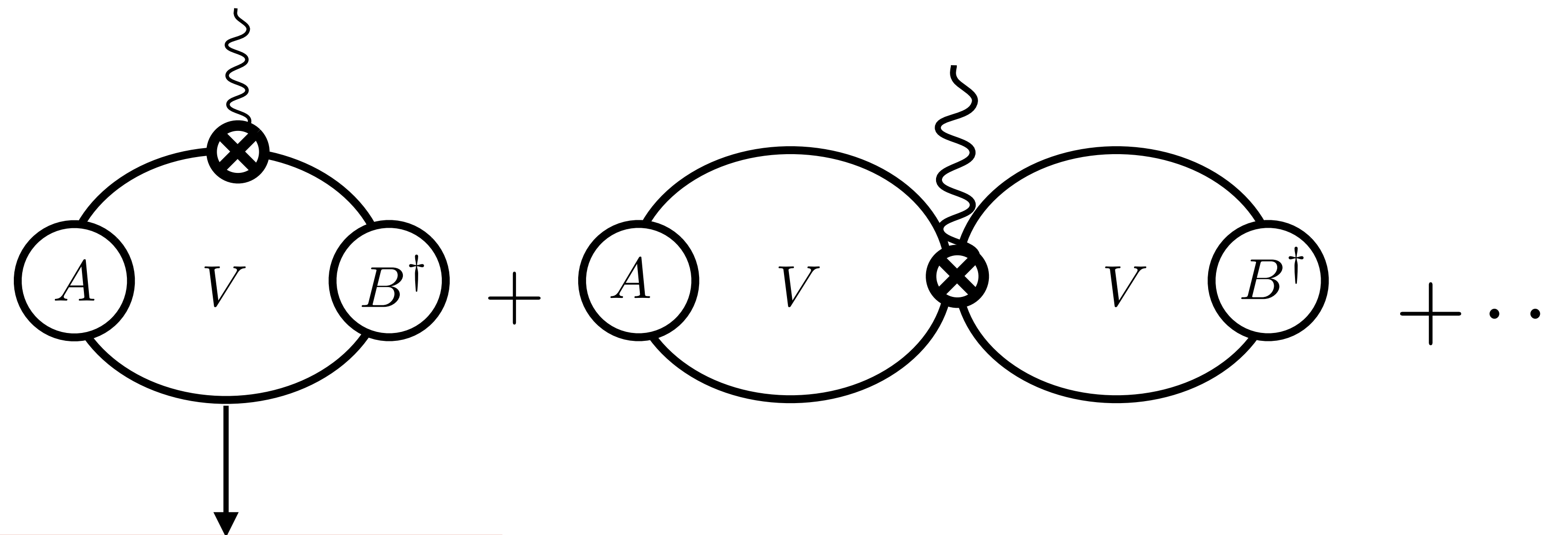
$$C_L^{3\,pt.}(t_i,t_f,\mathbf{P}_i,\mathbf{P}_f) = L^9 \sum_{n_i,n_f} e^{-(E_{n_i}t_i+E_{n_f}t_f)} \left[\langle 0|A(0)|E_{n_i},\mathbf{P}_i,L\rangle\right]_L \left[\langle E_{n_i},\mathbf{P}_i,L|\mathcal{J}(0)|E_{n_f},\mathbf{P}_f,L\rangle\right]_L \left[\langle E_{n_f},\mathbf{P}_f,L|B^\dagger(0)|0\rangle\right]_L$$

- Matching the two expressions we obtain the wanted relation (and few more passages..)

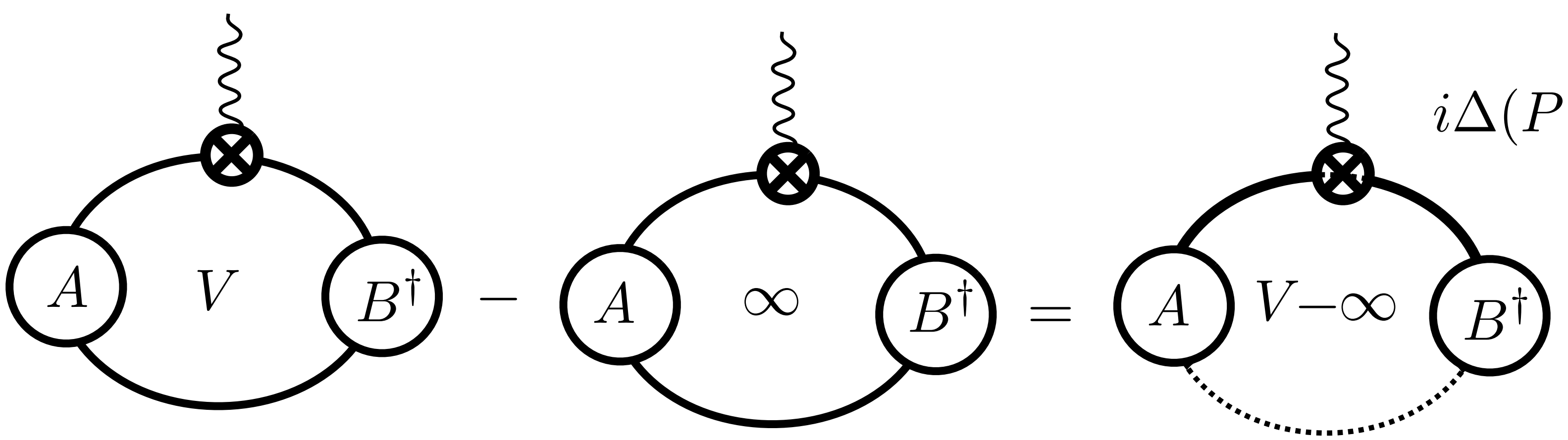
$$\left[\langle E_{n_i},\mathbf{P}_i,L|\mathcal{J}(0)|E_{n_f},\mathbf{P}_f,L\rangle\right]_L^2 = \frac{1}{L^6}\mathrm{Tr}\left[W(P_i,P_f)R(E_{n_f},\mathbf{P}_f)W(P_f,P_i)R(E_{n_i},\mathbf{P}_i)\right]$$

Finite volume correlator

$$C_L^{3\,pt.}(P_i, P_f) =$$



New topology not present before



$$i\Delta(P-k) = \frac{i}{(P-k)^2 - m^2 + i\epsilon} + \text{smooth}$$

$$= \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right] \int dk^0 \frac{iA(P_f,k)\Delta(P_f-k)\ w^{off}(P_f-k, P_i-k)\ \Delta(P_i-k) iB^\dagger(P_i,k)}{(k^2 - m^2 + i\epsilon)}$$

$$= \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \frac{1}{2\omega_{k2}} iA(P_f,k)\Delta(P_f-k)\ w^{off}(P_f-k, P_i-k)\ \Delta(P_i-k) iB^\dagger(P_i,k)\Big|_{k^0=\omega_{k2}}$$

+exp suppressed terms

Finite-infinite volume bubble with one current insertion

$$\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \frac{1}{2\omega_{k2}} iA(P_f,k)\Delta(P_f-k)\ w^{off}(P_f-k,P_i-k)\ \Delta(P_i-k) iB^\dagger(P_i,k)\Big|_{k^0=\omega_{k2}}$$

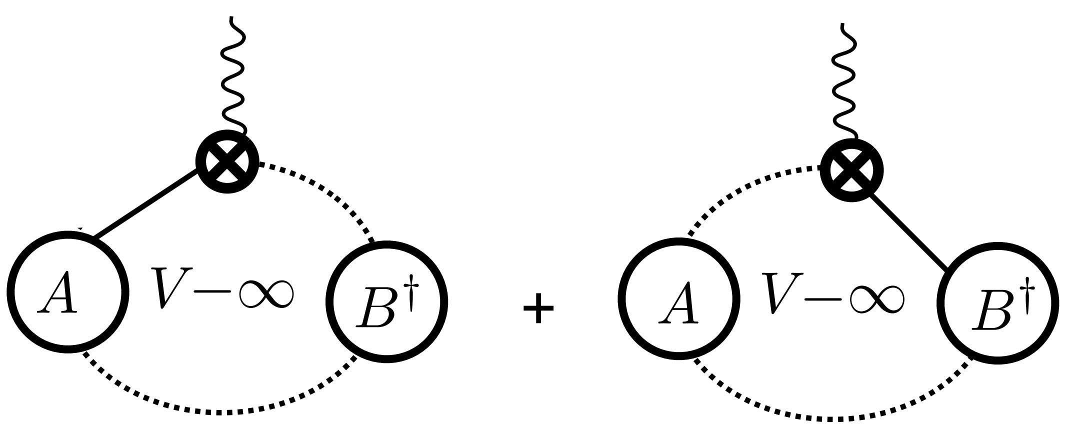
$$i\Delta(P-k) = \frac{i}{(P-k)^2-m^2+i\epsilon} + \text{smooth}$$

Separate remaining parts in off and on shell terms

$2^6 = 64$ Terms, can be divided into 3 classes

0 free propagator → Smooth contributions

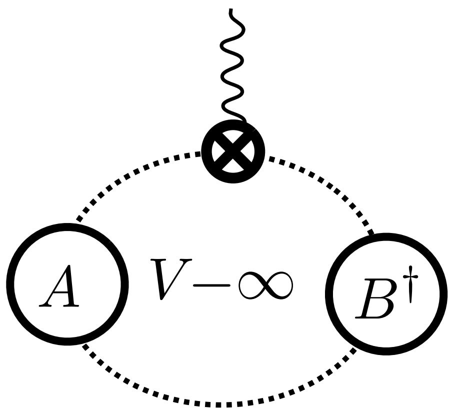
1 free propagator →



They will give the F function

Finite-infinite volume bubble with one current insertion

2 free propagators:



Maximally singular term

Numerator all on shell

$$= \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right] \int dk^0 \frac{A(P_i)\, w(P_i, Pf, k)\ B^\dagger(P_f)}{(k^2 - m^2 + i\epsilon)[(P_i - k)^2 - m^2 + i\epsilon][(P_f - k)^2 - m^2 + i\epsilon]}$$

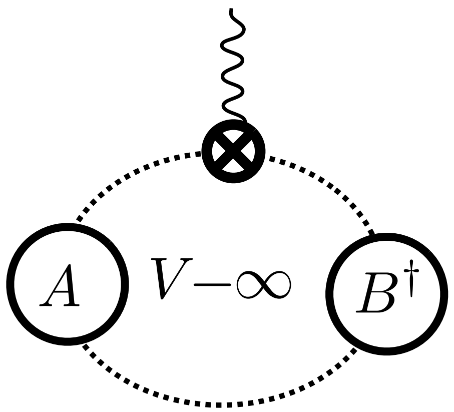
$$= \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right] \frac{A(P_i)\, w(P_i, Pf, k)\ B^\dagger(P_f)}{2\omega_k[(P_i - k)^2 - m^2 + i\epsilon][(P_f - k)^2 - m^2 + i\epsilon]}\Bigg|_{k_0=\omega_k}$$

$+$exp suppressed terms

Inserting the above contributions in the previously showed skeleton expansion (and with *proper* rearranging) we obtain the final equation

$$|\langle 2|\mathcal{J}|2\rangle|^2_L = \frac{1}{L^6}\mathrm{Tr}\left[R(E_L, L)W_{L,\mathrm{df}}R(E_L, L)W_{L,\mathrm{df}}\right]$$

New kinematic function



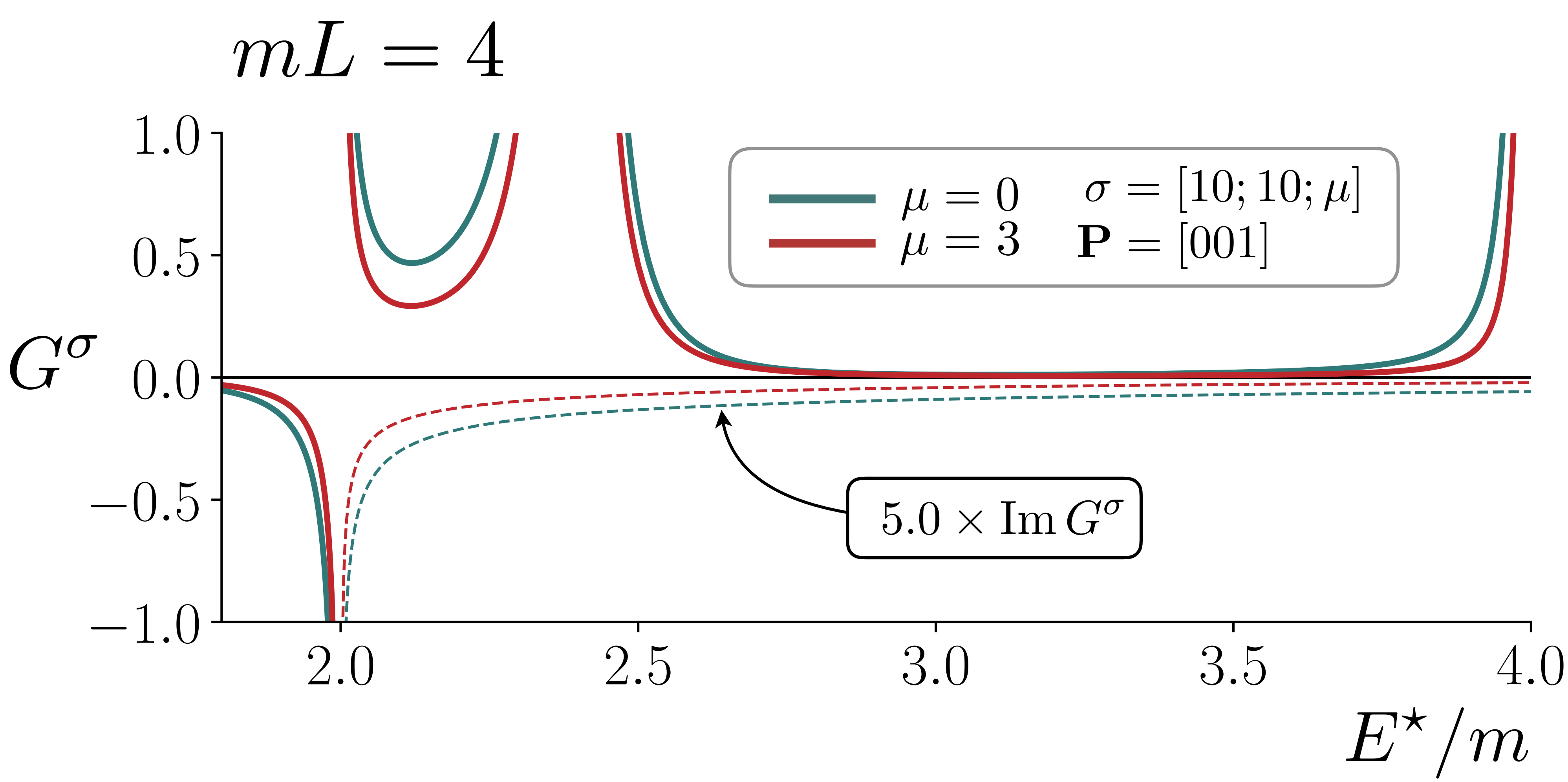
$$= \quad G(P_i, P_f, L)$$

$$= \left(\frac{1}{L^3} \sum_{\mathbf{k}} - \int_{\mathbf{k}} \right) \mathcal{N}(P_i, P_f, \mathbf{k}) \frac{1}{2\omega_{\mathbf{k}}} \frac{1}{(P_f - k)^2 - m^2 + i\epsilon} \frac{1}{(P_i - k)^2 - m^2 + i\epsilon} \bigg|_{k^0 = \omega_k}$$

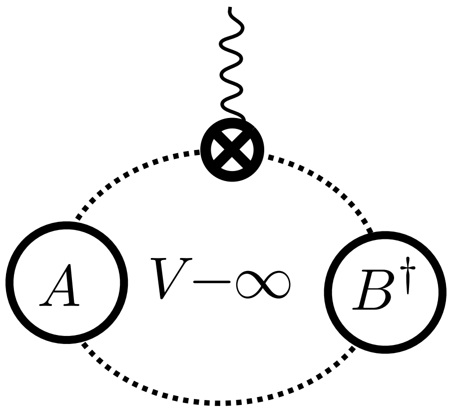
- The sum is "straightforward"
- The integral is highly not trivial (spectator particle goes on-shell)
    - integrand singularities are two surfaces in three-dimension
    - using mathematical trickery we can isolate the singularities, treat them with standard field theory techniques, and be left with a 3D smooth integral
    - Solution holds for generic momenta

New kinematic function

For equal initial and final momenta it has been shown that the above problem has an *analytical solution*



New kinematic function



=

$$\left(\frac{1}{L^3}\sum_{\mathbf{k}} - \int_{\mathbf{k}}\right) N_{l_f,m_f;\mu;l_i,m_i}(m,\mathbf{k})\, D(m,\mathbf{k})$$

Smooth

$$\frac{1}{2\omega_{\mathbf{k}}}\frac{1}{(P_f-k)^2-m^2+i\epsilon}\frac{1}{(P_i-k)^2-m^2+i\epsilon}\bigg|_{k^0=\omega_k}$$

We focus on the case in which the smooth function is constant i.e. zero angular momenta and scalar form factor (generalization is not trivial.. explicitly written in the paper)

$$\mathcal{I} \propto \int_{\mathbf{k}} D(m,\mathbf{k})$$

More explicitly

$$\mathcal{I} \propto \int_{\mathbf{k}} D(m, \mathbf{k}) \qquad D(m, \mathbf{k}) = \frac{1}{2\omega_{\mathbf{k}}} \frac{1}{(P_f - k)^2 - m^2 + i\epsilon} \frac{1}{(P_i - k)^2 - m^2 + i\epsilon}\bigg|_{k^0 = \omega_k}$$

$$\int \frac{dk^0}{2\pi} D_c(m, k) \equiv D(m, \mathbf{k}) + D_r(m, \mathbf{k})$$

Smooth remainder

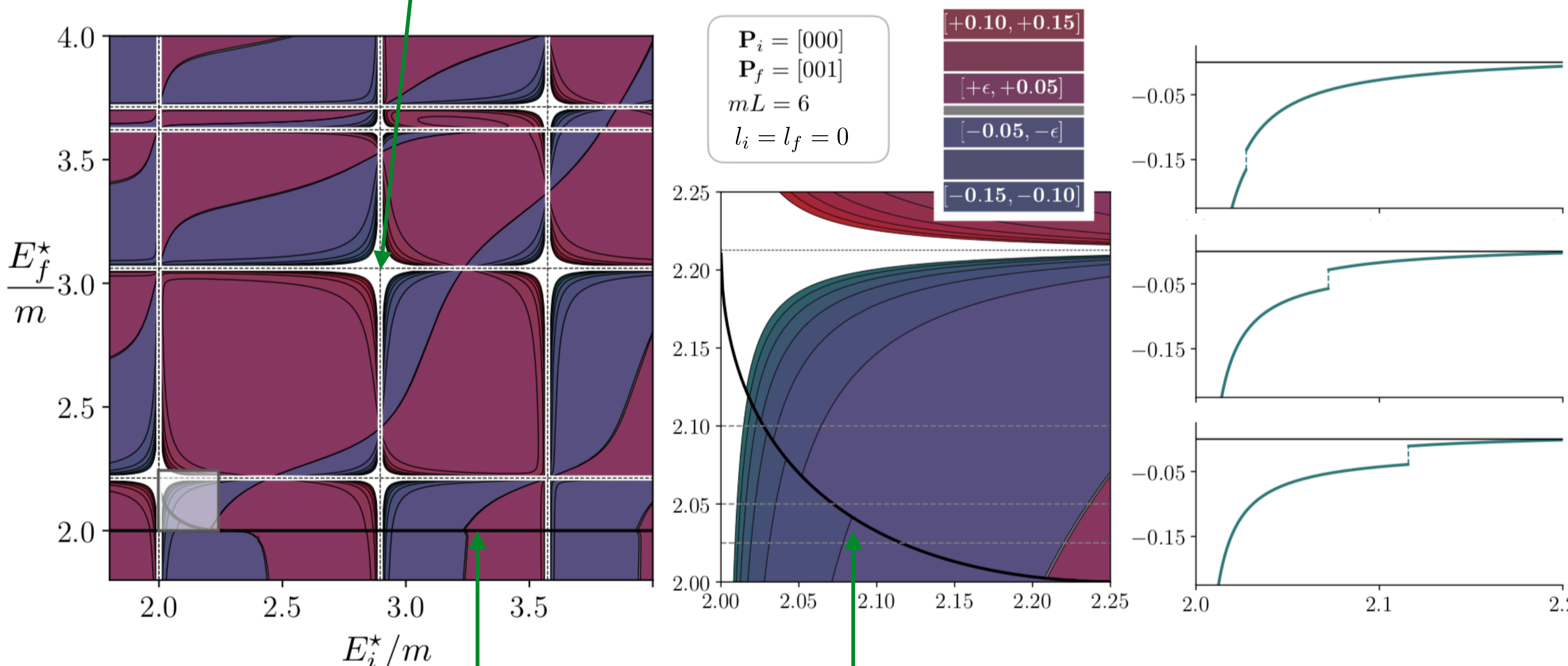
$$\frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(P_i - k)^2 - m^2 + i\epsilon} \frac{1}{(P_f - k)^2 - m^2 + i\epsilon}$$

$$I \propto \int \frac{d^4 k}{(2\pi)^4} D_c(m, \mathbf{k}) - \int_{\mathbf{k}} D_r(m, \mathbf{k})$$

Integral in 3D can be safely
Evaluated numerically

Can be reduced to a 1D integral (singularities present before will be here)

Poles at FV energies



Threshold singularities

Triangle singularities

(For some kinematic conditions emerge)

<u>*Can be properly classified using Landau's conditions*</u>

- Ward-Takahashi identity ✓

  For QED current

- Bound-State limit ✓

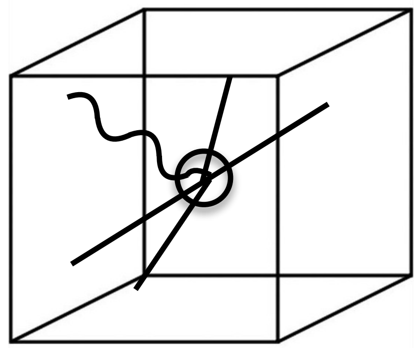
  For scalar current

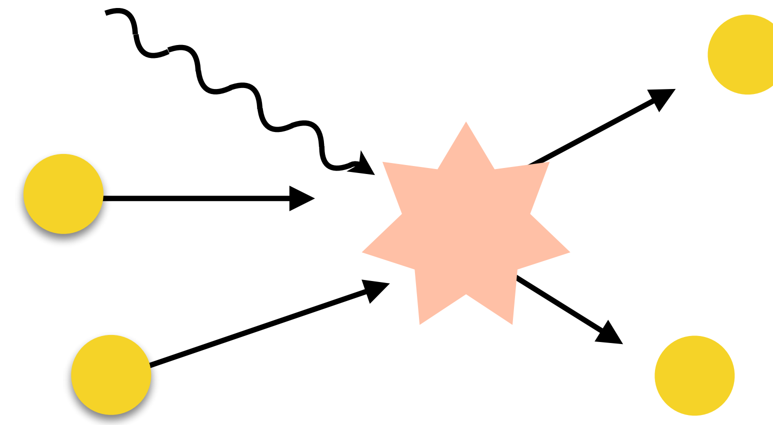
- For scalar form factors in the deuteron large corrections full formalism needed

A. Jackura
ODU

R. A. Briceño, A. Jackura, M.T. Hansen, to appear

Outlook





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- [x] Framework developed and under control
- [ ] Details of the implementation need to be fully understood.. (toy model analysis)
- [ ] Use this framework in an actual LQCD calculation

Form factors of resonances and bound states

Generalization to systems with spin
Similar study for baryons

Thank you!