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H_2-Optimal Estimation of Linear Delayed and PDE Systems

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***Abstract*— The H_2 norm is a commonly used performance metric in the design of estimators. However, H_2-optimal estimation of most PDEs is complicated by the lack of state-space and transfer function representations. To address this problem, we re-characterize the H_2-norm in terms of a map from initial condition to output. We then leverage the Partial Integral Equation (PIE) state-space representation of systems of linear PDEs coupled with ODEs to recast this characterization of H_2 norm as a convex optimization problem defined in terms of Linear Partial Integral (LPI) inequalities. We then parameterize a class of PIE-based observers and solve the associated H_2-optimal estimation problem. The resulting observers are validated using numerical simulation.**

I. INTRODUCTION

Partial Differential Equations (PDEs) are used to describe the evolution of some process whose state is distributed over a spatial domain. Examples of such processes include fluid flow [1], vibroacoustics [2], chemical reaction networks [3], and time-delay systems [4], where the corresponding distributed states are velocity profile, displacement, species concentration, and history. For such systems, it is often desirable to be able to track the evolution of the system using sensor measurements – either for the purpose of feedback control [5], [6] or for monitoring and fault detection [7], [8].

Unlike Ordinary Differential Equations (ODEs) and other such lumped-parameter systems, however, direct measurement of the system state of a PDE requires an uncountable number of sensors – a practical impossibility. Consequently, there has been significant interest in the development of observers wherein by tracking a finite set of measurements, we may infer real-time estimates of the entire distributed state. Furthermore, for PDEs, the need to integrate boundary conditions and the distributed states precludes the existence of a convenient and universal state-space representation. This means that most efforts to design estimators for such systems are ad hoc – requiring significant modification for even minor changes in the model [9]. As a result, most approaches to the estimation of the PDE state entail a reduction of the PDE state to finite dimensions, either through early-lumping [10], [11], by reducing the distributed states to finite-dimensions, or late-lumping [12], [13], which enforces synthesis conditions on a finite number of test functions.

Recently, efforts have been made to synthesize observers for PDE systems without lumping through the use of a more convenient state-space representation of PDEs. This method integrates the PDE evolution equation with the boundary conditions by defining the state as the highest spatial derivative of the distributed state and parameterizing the evolution of this state by means of integral operators with polynomial kernels. This method has the advantage that such operators form an algebra, which can be represented using matrices and optimized using Linear Matrix Inequalities (LMIs). The representation of a PDE using such operators is referred to as a Partial Integral Equation (PIE), and methods for the construction of PIE representations of a broad class of PDEs are well-established [14]–[17].

Observer designs for PDE systems that admit a PIE representation have previously been presented in [18], [19] and for time-delay systems in [20]. These results parameterize the observer dynamics using PIEs and pose conditions for stability and performance bounds as the solution of a convex optimization problem expressed in terms of Partial Integral (PI) operator variables and Linear Partial integral Inequalities (LPIs), which can be enforced using recently developed Matlab toolboxes such as [21]. These results ensure stability and bound the L_2-gain of a tracking error to disturbances such as sensor noise. The problem with minimization of L_2-gain, however, is that disturbances such as sensor noise are not typically characterized in terms of energy, but rather in terms of frequency content and power spectral density – implying that the H_2 norm is a more suitable performance metric in design of observers (e.g. LQG and Kalman filters).

The goal of this paper, then, is to formulate and solve the problem of H_2-optimal observer synthesis. Unlike H_∞-optimal observer synthesis, wherein a proxy for H_∞ performance is L_2-gain, the main technical difficulty for H_2-optimal estimation is the identification of a time-domain proxy for H_2 performance. To address this difficulty, we rely on an initial condition to output L_2-gain characterization of the H_2 metric as proposed in [22]. This allows us to extend classical LMIs for H_2-performance to LPI-type conditions to performance bounds on the error dynamics of the PIE-based observer.

This paper is structured as follows. First, Section II defines PI operators, PIEs, and LPIs. Section III introduces a time-domain characterization of the H_2 norm and formulates H_2-optimal observer synthesis problem. Section IV gives an LPI characterization of the H_2-norm of a PIE and Section V extends this result to give an LPI condition for computing H_2-optimal observer gains. Section VI gives a procedure to find observer gains from the LPI solution and Section VII presents numerical examples for observer validation.

Notation: $L_2^p[a, b]$ is the space of *Lesbegue* square-

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integrable $\mathbb{R}^p$-valued functions on spatial domain $s \in [a,b]$, endowed with the standard inner product. $\mathbb{R}L_2^{m,p}[a,b]$ denotes the Hilbert space $\mathbb{R}^m \times L_2^p[a,b]$. Occasionally, we omit domain and simply write L_2^p or $\mathbb{R}L_2^{m,p}$. We use the bold font, (e.g. $\mathbf{x}$) to indicate scalar or vector-valued functions of a spatial variable. For Hilbert spaces X, Y, $\mathcal{L}(X,Y)$ denotes the set of bounded linear operators from X to Y with $\mathcal{L}(X) := \mathcal{L}(X,X)$. We use the calligraphic font (e.g. $\mathcal{A}$) to represent such bounded linear operators.

II. State Space and Convex Optimization: PIs, PIEs, and LPIs

In this section, we introduce the algebra of Partial Integral (PI) operators, the class of systems modelled using Partial Integral Equations (PIEs), and the class of convex optimization problems defined in terms of Linear PI (LPI) Inequality constraints.

A. The Algebra of Partial Integral Operators

We begin by defining the algebra of partial integral operators which will be used to parameterize partial integral equations in Subsection II-B.

Definition 1. *We say* $\mathcal{P} = \mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right] \in \mathbf{\Pi}_4 \subset \mathcal{L}(\mathbb{R}L_2^{m_1,n_1}, \mathbb{R}L_2^{m_2,n_2})$ *if there exists a matrix* P *and polynomials* $Q_1, Q_2, R_0, R_1,$ *and* R_2 *such that*

$$\left(\mathcal{P}\begin{bmatrix} x \\ \mathbf{x}\end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_a^b Q_1(\theta)\mathbf{x}(\theta)d\theta \\ Q_2(s)x + \mathcal{R}\mathbf{x}(s)\end{bmatrix},$$

$$(\mathcal{R}\mathbf{x})(s) = R_0(s)\mathbf{x}(s) + \int_a^s R_1(s,\theta)\mathbf{x}(\theta)d\theta + \int_s^b R_2(s,\theta)\mathbf{x}(\theta)d\theta.$$

We refer to $\mathbf{\Pi}_4$ as the set of 4-PI operators. If $m_1 = m_2$ and $n_1 = n_2$, this set of PI operators is closed under composition, addition, and adjoint; explicit formulae for these operations can be obtained in terms of the polynomial matrices used to parameterize them [15].

As in Defn. 1, the notation $\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]$ is used to indicate the 4-PI operator associated with the matrix P and polynomial parameters Q_i, R_j. The associated dimensions (m_1, n_1, m_2, n_2) are inherited from the dimensions of the constant matrix $P \in \mathbb{R}^{m_2 \times m_1}$ and polynomial matrices $Q_1(s) \in \mathbb{R}^{m_2 \times n_1}$, $Q_2(s) \in \mathbb{R}^{n_2 \times m_1}$, and $R_0(s), R_1(s,\theta), R_2(s,\theta) \in \mathbb{R}^{n_2 \times n_1}$. In the case where a dimension is zero, we use $\emptyset$ in place of the associated parameter with zero dimension.

B. Partial Integral Equations

It has been shown in, e.g. [15], that a large class of PDE coupled with ODEs, with sensed and regulated outputs, $y(t) \in \mathbb{R}^{n_y}, z(t) \in \mathbb{R}^{n_z}$, and in-domain disturbances, $w(t) \in \mathbb{R}^{n_w}$, may be equivalently represented using a partial integral equation (PIE) of the form

$$\begin{aligned} \partial_t(\mathcal{T}\mathbf{x}(t)) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t), \qquad \mathbf{x}(0) \in \mathbb{R}L_2, \\ z(t) &= \mathcal{C}_1\mathbf{x}(t), \qquad y(t) = \mathcal{C}_2\mathbf{x}(t) + \mathcal{D}_{21}w(t), \end{aligned} \tag{1}$$

where the parameters $\mathcal{A}, \mathcal{B}_1, \mathcal{C}_2$, etc., are all 4-PI operators and where the solution of the PIE, $\mathbf{x}(t) \in \mathbb{R}L_2^{m,n}[a,b]$ yields a solution to the PDE as $\mathcal{T}\mathbf{x}(t)$. The PIE state, $\mathbf{x}(t)$, combines the ODE state with a spatial derivative of the PDE state and admits no boundary conditions or continuity constraints.

The solution of this class of PIE is formally defined as follows, where $x \in L_{2e}^p[0,\infty)$ means $x(t) \in \mathbb{R}^p$ and $\int_0^T \|x(t)\|^2\, dt$ is finite for all $T \geq 0$.

Definition 2 (PIE solution)**.** *Given PI operators* $\mathcal{T}$, $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{D}_{21}$ *we say* $\{\mathbf{x}, z, y\}$ *is a solution to the PIE system for given initial condition* $\mathbf{x}(0) \in \mathbb{R}L_2^{m,n}[a,b]$ *and input* $w \in L_{2e}^{n_w}[0,\infty)$*, if* $\mathcal{T}\mathbf{x}(t)$ *is Frechét differentiable for all* $t \in [0,\infty)$*, and if* $\mathbf{x}(t) \in \mathbb{R}L_2^{m,n}[a,b]$*,* $z \in L_{2e}^{n_z}[0,\infty)$*, and* $y \in L_{2e}^{n_y}[0,\infty)$ *satisfy Eq.* (1) *for all* $t \in [0,\infty)$*.*

C. Linear PI Operator Inequalities

As described in Subsection II-A, 4-PI operators of the form given in Defn. 1 constitute a composition algebra of bounded linear operators and are parameterized by polynomial matrices, which in turn can be parameterized by the coefficients of those polynomials. In this paper, we reformulate the problem of H_2-optional estimator synthesis as an optimization problem where the decision variables are themselves PI operators and are subject to inequality constraints which are affine in those decision variables – See, e.g. Eqn. (12) in Thm. 8. Optimization problems in this form may be solved by using matrices to parameterize the coefficients of the polynomials that define the PI operator variables. Inequalities are enforced by using positive matrices to parameterize positive PI operators, as described in [14], and implemented in the PIETOOLS Matlab toolbox [21].

III. Problem Formulation

In this section, we introduce a suitable time-domain characterization of the H_2 norm and use this characterization to define the problems of H_2 norm bounding and H_2-optimal estimation for systems that admit a PIE representation.

A. The H_2 norm of a PIE

For this subsection, we restrict our consideration to the characterization of the H_2 norm of a system represented by a PIE of the form

$$\begin{aligned} \partial_t(\mathcal{T}\mathbf{x}(t)) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t), \\ z(t) &= \mathcal{C}_1\mathbf{x}(t), \qquad \mathbf{x}(0) = 0, \end{aligned} \tag{2}$$

where $\mathbf{x}(t) \in \mathbb{R}L_2^{m,n}[a,b]$ is the state, $w(t) \in \mathbb{R}^{n_w}$ is a disturbance, and $z(t) \in \mathbb{R}^{nz}$ is the output. Specifically, in Definition. 3, we define the H_2 norm of this system as L_2-gain of initial condition to output of an auxiliary system with no disturbance. While non-standard, we will see that this characterization of H_2 performance is equivalent in a certain sense to the standard definition of H_2 norm.

Definition 3. *Consider solutions of the auxiliary PIE*

$$\begin{aligned} \partial_t(\mathcal{T}\mathbf{x}(t)) &= \mathcal{A}\mathbf{x}(t), \\ z(t) &= \mathcal{C}_1\mathbf{x}(t), \qquad \mathcal{T}\mathbf{x}(0) = \mathcal{B}_1 x_0. \end{aligned} \tag{3}$$

We define the H_2 norm of System (2) (denoted G) as

$$\|G\|_{H_2} := \sup_{\substack{z,\mathbf{x} \text{ satisfy } (3)\\ \|x_0\|=1}} \|z\|_{L_2}.$$

To see the relationship between the definition of H_2 norm in Definition 3 and the standard definition, recall the usual state-space representation of an ODE.

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \qquad \forall t \in [0, \infty). \tag{4}$$

Corollary 4. *Suppose A is Hurwitz and $\hat{G}(s) = C(sI - A)^{-1}B$ with $B \in \mathbb{R}^{n_x \times n_w}$. Consider solutions of the auxiliary ODE*

$$\dot{x}(t) = Ax(t),$$
$$z(t) = Cx(t), \qquad x(0) = Bx_0, \tag{5}$$

Then

$$\sup_{\substack{z,x \text{ satisfies } (5)\\ \|x_0\|=1}} \|z\|_{L_2} \le \|G\|_{H_2} \le \sqrt{n_w} \sup_{\substack{z,x \text{ satisfies } (5)\\ \|x_0\|=1}} \|z\|_{L_2}.$$

Proof. Suppose $\{x, z\}$ satisfy (5) with initial condition $x(0) = Bx_0$. Then $x(t) = e^{At}Bx_0$ and hence if $\|x_0\| = 1$, we have

$$\begin{aligned}
\|z\|_{L_2}^2 &= \int_0^\infty x(\tau)^T C^T C x(\tau) d\tau \\
&= \int_0^\infty x_0^T B^T e^{A^T\tau} C^T C e^{A\tau} B x_0 d\tau \\
&\le \bar{\sigma}\left(\int_0^\infty B^T e^{A^T\tau} C^T C e^{A\tau} B d\tau\right) \\
&\le \text{trace}\left(\int_0^\infty B^T e^{A^T\tau} C^T C e^{A\tau} B d\tau\right) = \|G\|_{H_2}^2.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|G\|_{H_2}^2 &= \text{trace}\left(\int_0^\infty B^T e^{A^T\tau} C^T C e^{A\tau} B d\tau\right) \\
&\le n_w \bar{\sigma}\left(\int_0^\infty B^T e^{A^T\tau} C^T C e^{A\tau} B d\tau\right) \\
&= n_w \sup_{\|x_0\|=1} \int_0^\infty x_0^T B^T e^{A^T\tau} C^T C e^{A\tau} B x_0 d\tau \\
&= n_w \sup_{\|x_0\|=1} \|z\|_{L_2}^2.
\end{aligned}$$

□

Clearly, if the PIE has a single input, the proposed definition of H_2 norm coincides with the typical definition. Alternatively, in the case of multiple inputs, our time-domain characterization of H_2 norm would coincide with an alternative definition of H_2 norm given by

$$\|\hat{G}\|_{H_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\sigma}\left(G^*(i\omega)G(i\omega)d\omega\right).$$

Having defined the H_2-norm, we proceed to formulate the H_2-optimal estimator synthesis problem.

B. H_2-Optimal Estimators

Our goal is to design observers for the class of coupled ODE-PDE system which admit a PIE representation of form

$$\partial_t(\mathcal{T}\mathbf{x}(t)) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t), \qquad \mathbf{x}(0) = 0,$$
$$z(t) = \mathcal{C}_1\mathbf{x}(t), \qquad y(t) = \mathcal{C}_2\mathbf{x}(t) + \mathcal{D}_{21}w(t), \tag{6}$$

where recall the state of the original PDE is obtained from the solution of the PIE as $\mathcal{T}\mathbf{x}(t)$. The signal $y(t)$ are measurements of the PDE and $z(t)$ represents those parts of the state by which we will measure the performance of our estimator. Our estimator dynamics are then assumed to have the Luenberger observer structure

$$\partial_t(\mathcal{T}\tilde{\mathbf{x}}(t)) = \mathcal{A}\tilde{\mathbf{x}}(t) + \mathcal{L}\left(\mathcal{C}_2\tilde{\mathbf{x}}(t) - y(t)\right), \quad \tilde{\mathbf{x}}(0) = 0, \tag{7}$$

which mirror the dynamics of the observed system, but without the disturbance, which is unknown. The term, $\mathcal{C}_2\tilde{\mathbf{x}}(t) - y(t)$, reflects the difference between the predicted and measured output from the PDE. This term is weighted by the observer gain, $\mathcal{L} : \mathbb{R}^{n_y} \to \mathbb{R}L_2^{m,n}$ which is taken to be a PI operator. By combining the observer in Eqn. (7) with the measured output of a PDE, real-time estimates of the PDE state can be obtained as $\mathcal{T}\tilde{\mathbf{x}}(t)$ and used in conjunction with state-feedback controllers or fault detection algorithms.

The H_2-optimal estimation problem, then, is to choose $\mathcal{L}$ which minimizes the H_2-norm of the map from disturbance w to error in the regulated output, which we define as $e_z(t) = \mathcal{C}_1\tilde{\mathbf{x}}(t) - z(t)$. This map can likewise be represented as a PIE with state $\mathbf{e}(t) = \tilde{\mathbf{x}}(t) - \mathbf{x}(t)$, where $\tilde{\mathbf{x}}$ satisfies Eqn. (7) and $\mathbf{x}$ satisfies Eqn. (6) so that

$$\partial_t\left(\mathcal{T}\mathbf{e}(t)\right) = (\mathcal{A} + \mathcal{L}\mathcal{C}_2)\mathbf{e}(t) - \left(\mathcal{B}_1 + \mathcal{L}\mathcal{D}_{21}\right)w(t),$$
$$e_z(t) = \mathcal{C}_1\mathbf{e}(t), \qquad \mathbf{e}(0) = 0. \tag{8}$$

We see that System (8) is of the form in Eqn. (2) with $\mathcal{A} \mapsto \mathcal{A} + \mathcal{L}\mathcal{C}_2$, $\mathcal{B}_1 \mapsto -\left(\mathcal{B}_1 + \mathcal{L}\mathcal{D}_{21}\right)$ and $\mathcal{C}_1 \mapsto \mathcal{C}_1$. Thus we can formulate the H_2-optimal synthesis problem using the auxiliary PIE from Defn. 3

$$\partial_t\left(\mathcal{T}\mathbf{e}(t)\right) = (\mathcal{A} + \mathcal{L}\mathcal{C}_2)\mathbf{e}(t),$$
$$e_z(t) = \mathcal{C}_1\mathbf{e}(t), \quad \mathcal{T}\mathbf{e}(0) = -\left(\mathcal{B}_1 + \mathcal{L}\mathcal{D}_{21}\right)x_0. \tag{9}$$

as

$$\min_{\mathcal{L}\in\mathbf{\Pi}_4} \sup_{\substack{z,\mathbf{e} \text{ satisfy } (9)\\ \|x_0\|=1}} \|e_z\|_{L_2}. \tag{10}$$

In Section V, we will reformulate the H_2-optimal estimation problem as an LPI. First, however, we need to address the problem of computing H_2-norm of a PIE using LPIs.

IV. An LPI for the H_2 Norm

In this section, we show how to use LPIs to compute the H_2 norm of a PIE. We begin by reformulating the following result from [22].

Theorem 5. *Suppose (2) is defined by $\mathcal{T}, \mathcal{A}, \mathcal{B}_1, \mathcal{C}_1 \in \mathbf{\Pi}_4$. If there exists some $\epsilon > 0$ and $\mathcal{P} \succcurlyeq \epsilon I$ such that:*

$$trace(\mathcal{B}_1^*\mathcal{P}\mathcal{B}_1) \le \gamma^2,$$
$$\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{C}_1^*\mathcal{C}_1 \preccurlyeq -\epsilon I, \tag{11}$$

then $\|G\|_{H_2} \le \gamma$.

We now use an extension of the Schur complement to obtain an LPI for bounding the H_2 norm which will be used for estimator design in Section V. This reformulation, however, requires us to define vertical and horizontal concatenation of $\mathbf{\Pi}_4$ operators such that the concatenated operator is in $\mathbf{\Pi}_4$ (See Lemmas 39 and 40 from [15]). This definition separately concatenates the real and distributed portions of the operator so that if, e.g. $\mathcal{P} \in \mathcal{L}(\mathbb{R}L_2^{n,m})$ and $\mathcal{Q} \in \mathcal{L}(\mathbb{R}L_2^{p,q})$, then

$$\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \in \mathcal{L}(\mathbb{R}^{n+p} \times L_2^{m+q}).$$

In proof of the following lemma, we do not re-order rows and columns. However, the result holds for the standard definition of concatenation since inequalities are preserved under symmetric reordering of rows and columns.

Lemma 6 (Schur Complement)**.** *Suppose $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathbf{\Pi}_4$. Then the following are equivalent.*

1) $\begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{Q}^* & \mathcal{R} \end{bmatrix} \succcurlyeq \epsilon I$ for some $\epsilon > 0$.

2) $\mathcal{R} - \mathcal{Q}^*\mathcal{P}^{-1}\mathcal{Q} \succcurlyeq \epsilon I$ and $\mathcal{P} \succcurlyeq \epsilon I$ for some $\epsilon > 0$.

Proof. In this proof, there is no rearrangement of rows or columns. Now, mirroring the standard proof of the Schur complement, suppose that 1) is true. Then, we have

$$\langle \mathbf{x}, \mathcal{P}\mathbf{x} \rangle = \left\langle \begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix}, \begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{Q}^* & \mathcal{R} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix} \right\rangle \geq \epsilon \left\| \mathbf{x} \right\|^2,$$

which implies that $\mathcal{P}$ is invertible. Now note that

$$\begin{bmatrix} \mathcal{P} & 0 \\ 0 & \mathcal{R} - \mathcal{Q}^*\mathcal{P}^{-1}\mathcal{Q} \end{bmatrix} = \begin{bmatrix} I & -\mathcal{P}^{-1}\mathcal{Q} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{Q}^* & \mathcal{R} \end{bmatrix} \begin{bmatrix} I & -\mathcal{P}^{-1}\mathcal{Q} \\ 0 & I \end{bmatrix},$$

and hence

$$\begin{aligned} &\langle \mathbf{x}, (\mathcal{R} - \mathcal{Q}^*\mathcal{P}^{-1}\mathcal{Q})\mathbf{x} \rangle = \left\langle \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} \mathcal{P} & 0 \\ 0 & \mathcal{R} - \mathcal{Q}^*\mathcal{P}^{-1}\mathcal{Q} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} -\mathcal{P}^{-1}\mathcal{Q}\mathbf{x} \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{Q}^* & \mathcal{R} \end{bmatrix} \begin{bmatrix} -\mathcal{P}^{-1}\mathcal{Q}\mathbf{x} \\ \mathbf{x} \end{bmatrix} \right\rangle \\ &\geq \epsilon \left\| \begin{bmatrix} -\mathcal{P}^{-1}\mathcal{Q}\mathbf{x} \\ \mathbf{x} \end{bmatrix} \right\|^2 \geq \epsilon \left\| \mathbf{x} \right\|^2. \end{aligned}$$

For the converse, suppose 2) is true. Then

$$\begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{Q}^* & \mathcal{R} \end{bmatrix} = \begin{bmatrix} I & \mathcal{P}^{-1}\mathcal{Q} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \mathcal{P} & 0 \\ 0 & \mathcal{R} - \mathcal{Q}^*\mathcal{P}^{-1}\mathcal{Q} \end{bmatrix} \begin{bmatrix} I & \mathcal{P}^{-1}\mathcal{Q} \\ 0 & I \end{bmatrix},$$

which implies

$$\left\langle \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{Q}^* & \mathcal{R} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\rangle \geq \epsilon \left\| \begin{bmatrix} I & \mathcal{P}^{-1}\mathcal{Q} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|^2.$$

Now, define $\left\| \begin{bmatrix} I & \mathcal{P}^{-1}\mathcal{Q} \\ 0 & I \end{bmatrix}^{-1} \right\|_{\mathcal{L}(\mathbb{R}L_2)} = \delta$. Then

$$\left\| \begin{bmatrix} I & \mathcal{P}^{-1}\mathcal{Q} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|^2 \geq \delta \left\| \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|^2,$$

and hence

$$\left\langle \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{Q}^* & \mathcal{R} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\rangle \geq \epsilon\delta \left\| \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|^2,$$

as desired. □

Theorem 7. *Suppose $\mathcal{T}, \mathcal{A}, \mathcal{B}_1, \mathcal{C}_1 \in \mathbf{\Pi}_4$. Suppose there exists some matrix W, $\epsilon > 0$, and a 4-PI operator $P \succcurlyeq \epsilon I$ such that:*

$$\begin{bmatrix} -\gamma I & \mathcal{C}_1 \\ \mathcal{C}_1^* & \mathcal{T}^* P \mathcal{A} + \mathcal{A}^* P \mathcal{T} \end{bmatrix} \preccurlyeq -\epsilon I \tag{12}$$

$$\begin{bmatrix} W & \mathcal{B}_1^*\mathcal{P} \\ \mathcal{P}\mathcal{B}_1 & \mathcal{P} \end{bmatrix} \succcurlyeq \epsilon I \tag{13}$$

$$trace(W) \leq \gamma. \tag{14}$$

Then $\sup_{\substack{z,\mathbf{x} \text{ satisfy } (3) \\ \|x_0\|=1}} \|z\|_{L_2} \leq \gamma.$

Proof. Suppose $\gamma, \mathcal{P}, \mathcal{Z}$ are as stated above. Then, Inequality (12) combined with Lemma 6 implies

$$\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} + \frac{1}{\gamma}\mathcal{C}^*\mathcal{C} \preccurlyeq -\epsilon I.$$

Likewise, Inequality (13) combined with Lemma 6 implies

$$W - \mathcal{B}^*\mathcal{P}\mathcal{P}^{-1}\mathcal{P}\mathcal{B} = W - \mathcal{B}^*\mathcal{P}\mathcal{B} \succ 0.$$

Now W and $\mathcal{B}^*\mathcal{P}\mathcal{B}$ are matrices and hence $\mathrm{trace}(\mathcal{B}^*\mathcal{P}\mathcal{B}) < \mathrm{trace}\, W \leq \gamma$. Define $\hat{\mathcal{P}} = \gamma\mathcal{P}$ so that $\mathcal{P} = \frac{1}{\gamma}\hat{\mathcal{P}}$ and hence

$$\mathcal{A}^*\hat{\mathcal{P}}\mathcal{T} + \mathcal{T}^*\hat{\mathcal{P}}\mathcal{A} + \mathcal{C}^*\mathcal{C} \preccurlyeq -\gamma\epsilon I, \qquad \mathrm{trace}(\mathcal{B}^*\hat{\mathcal{P}}\mathcal{B}) \leq \gamma^2,$$

which implies the conditions of Thm 5 are satisfied. □

In the next section, this result is used to design observers which minimize a bound on the H_2 norm of the error dynamics.

V. An LPI for H_2-optimal Estimator

In this section, we consider the problem of designing the estimator gain $\mathcal{L} \in \mathbf{\Pi}_4$ which minimizes a bound on the H_2 norm of the error dynamics defined in Subsection III-B.

Theorem 8. *Suppose there exist $\epsilon > 0$, matrix W, and 4-PI operators $\mathcal{P} \succcurlyeq \epsilon I$ and $\mathcal{Z}$, such that*

$$\begin{bmatrix} -\gamma I & \mathcal{C}_1 \\ \mathcal{C}_1^* & \mathcal{T}^* P \mathcal{A} + \mathcal{A}^* P \mathcal{T} + \mathcal{T}^*\mathcal{Z}\mathcal{C}_2 + \mathcal{C}_2^*\mathcal{Z}^*\mathcal{T} \end{bmatrix} \preccurlyeq -\epsilon I,$$

$$\begin{bmatrix} W & -(\mathcal{B}_1^*\mathcal{P} + D_{21}^T\mathcal{Z}^*) \\ -(\mathcal{P}\mathcal{B}_1 + \mathcal{Z}D_{21}) & \mathcal{P} \end{bmatrix} \succcurlyeq \epsilon I,$$

$$\mathrm{trace}(W) \leq \gamma.$$

Then, if $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$, the H_2-norm of the system in Eq. (8) is upper bounded by γ.

Proof. Let $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$. Then

$$\begin{aligned} &\begin{bmatrix} -\gamma I & \mathcal{C}_1 \\ \mathcal{C}_1^* & \mathcal{T}^*\mathcal{P}\left(\mathcal{A} + \mathcal{L}\mathcal{C}_2\right) + \left(\mathcal{A} + \mathcal{L}\mathcal{C}_2\right)^* P\mathcal{T} \end{bmatrix} \\ &= \begin{bmatrix} -\gamma I & \mathcal{C}_1 \\ \mathcal{C}_1^* & \mathcal{T}^*\mathcal{P}\left(\mathcal{A} + \mathcal{P}^{-1}\mathcal{Z}\mathcal{C}_2\right) + \left(\mathcal{A} + \mathcal{P}^{-1}\mathcal{Z}\mathcal{C}_2\right)^* P\mathcal{T} \end{bmatrix} \\ &= \begin{bmatrix} -\gamma I & \mathcal{C}_1 \\ \mathcal{C}_1^* & \mathcal{T}^* P \mathcal{A} + \mathcal{A}^* P \mathcal{T} + \mathcal{T}^*\mathcal{Z}\mathcal{C}_2 + \mathcal{C}_2^*\mathcal{Z}^*\mathcal{T} \end{bmatrix} \preccurlyeq -\epsilon I, \end{aligned}$$

and

$$\begin{aligned} &\begin{bmatrix} W & -\left(\mathcal{B}_1 + \mathcal{L}D_{21}\right)^* \mathcal{P} \\ -\mathcal{P}\left(\mathcal{B}_1 + \mathcal{L}D_{21}\right) & \mathcal{P} \end{bmatrix} \\ &= \begin{bmatrix} W & -(\mathcal{B}_1^*\mathcal{P} + D_{21}^T\mathcal{Z}^*) \\ -(\mathcal{P}\mathcal{B}_1 + \mathcal{Z}D_{21}) & \mathcal{P} \end{bmatrix} \succcurlyeq \epsilon I. \end{aligned}$$

Since $\mathrm{trace}(W) \leq \gamma$, from Theorem 7, we conclude that γ is an upper bound on the H_2-norm of the PIE system defined by $\{\mathcal{T}, (\mathcal{A} + \mathcal{L}\mathcal{C}_2), -(\mathcal{B}_1 + \mathcal{L}D_{21}), \mathcal{C}_1\}$ as in Eq. 8. □

VI. Estimator Gain Reconstruction

In this section, we suppose that we have obtained $\mathcal{P}, \mathcal{Z}$ which satisfy Thm. 8. Our next step is to construct the observer gain $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$ and use this gain in combination with the PIE estimator in (7) to track the state of a PDE.

First, if $\mathcal{P} \in \mathbf{\Pi}_4$ is invertible, then the inverse $\mathcal{P}^{-1}$ can be computed using, e.g. Lem. 17 in [23] and numerically approximated by a PI operator

$$\mathcal{P}^{-1} \approx \hat{\mathcal{P}} := \mathcal{\Pi}\left[\begin{array}{c|c} \hat{P} & \hat{Q} \\ \hline \hat{Q}^T & \{\hat{R}_i\} \end{array}\right].$$

Then, if $\mathcal{Z} = \mathcal{\Pi}\left[\begin{array}{c|c} Z_1 & \emptyset \\ \hline Z_2 & \{\emptyset\} \end{array}\right]$, we have, by the 4-PI composition formula [15], that $\mathcal{L} = \mathcal{\Pi}\left[\begin{array}{c|c} L_1 & \emptyset \\ \hline L_2 & \{\emptyset\} \end{array}\right]$, where

$$L_1 = \hat{P}Z_1 + \int_a^b \hat{Q}(s)Z_2(s)ds,$$
$$L_2(s) = \hat{Q}(s)^T Z_1 + \hat{R}_0(s)Z_2(s) + \int_a^s \hat{R}_1(s,\theta)Z_2(\theta)d\theta + \int_s^b \hat{R}_2(s,\theta)Z_2(\theta)d\theta.$$

L_1 represents the correction to the ODE state and L_2 represents a correction to the distributed state. In the following section, we test observers designed in this manner by numerical integration of a PIE estimator using the output from the numerical integration of the PDE it is observing.

VII. Numerical Examples

In this section, we validate the proposed algorithm for observer synthesis by constructing the H_2-optimal observer gains and numerically integrating the estimator dynamics using the output from numerical integration of the associated PDEs subject to disturbances. Our illustration uses two PDE examples: an unstable non-homogeneous reaction-diffusion equation (Example A) and an energy-preserving Euler-Bernoulli beam equation (Example B).

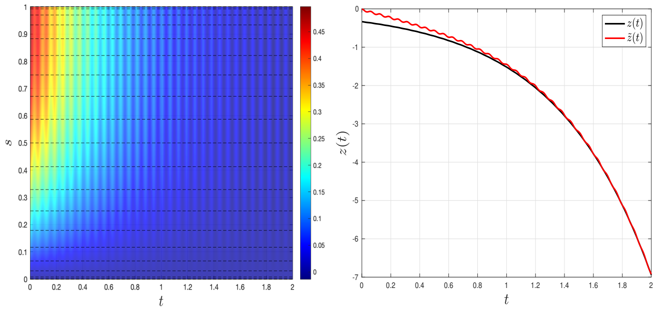
In both cases, the command-line PDE input option of PIETOOLS [21] is used to obtain the PIE representation of the PDE. Solution of the LPI in Thm. 8, operator inversion, and estimator gain reconstruction is likewise performed using PIETOOLS. Numerical integration of both the PIE estimator and PDE plant are performed using a Galerkin projection with Chebyshev bases order up to 8, and as implemented in PIESIM [24]. In each case, we plot both the evolution of the performance metric being minimized (e_z) as well as the error in the estimate of the distributed PDE state.

Example A (Unstable Reaction-Diffusion Equation)**.** *In this example, we consider the unstable, non-homogeneous reaction-diffusion PDE with both sensor and process noise where sensor measurements are taken at the boundary.*

$$\dot{\xi}(t,s) = 3\xi(t,s) + (s^2+0.2)\partial_s^2\xi(t,s) - \frac{s^2}{2}w(t),$$
$$\xi(t,0) = \partial_s\xi(t,1) = 0,$$
$$z(t) = \int_0^1 \xi(t,\theta)d\theta, \quad y(t) = \xi(t,1) + w(t). \qquad (15)$$

PIETOOLS is used to obtain the PIE representation of this PDE with PIE state, $\mathbf{x}(t) = \partial_s^2\xi(t)$*, and system parameters*

$$\mathcal{T} = \mathcal{\Pi}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{0, R_1, R_2\} \end{array}\right], \qquad \mathcal{B}_1 = \mathcal{\Pi}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline -0.5s^2 & \{\emptyset\} \end{array}\right],$$
$$\mathcal{A} = \mathcal{\Pi}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{S_0, 2R_1, 2R_2\} \end{array}\right], \qquad \mathcal{C}_1 = \mathcal{\Pi}\left[\begin{array}{c|c} \emptyset & 0.5s^2-s \\ \hline \emptyset & \{\emptyset\} \end{array}\right],$$
$$\mathcal{C}_2 = \mathcal{\Pi}\left[\begin{array}{c|c} \emptyset & -s \\ \hline \emptyset & \{\emptyset\} \end{array}\right], \qquad \mathcal{D}_{21} = 1,$$



(a) (b)

Fig. 1: Numerical estimation of an H_2-optimal estimator for an unstable reaction-diffusion equation (Eq. (15)) using measurement at the boundary along with process and sensor disturbance $w(t) = sin(100t)$ and PDE initial condition $\xi(0,s) = s^2/2 - s$ ($\mathbf{x}(0,s) = 1$). (a): Evolution of error in estimate of the PDE state $\mathcal{T}\mathbf{e}(t) = \mathcal{T}\tilde{\mathbf{x}}(t) - \xi(t)$. (b): Evolution of the regulated output $z(t)$ of both estimator and PDE.

where $R_1(s,\theta) = -\theta$*,* $R_2(s,\theta) = -s$*, and* $S_0(s) = s^2+0.2$*. In Fig. 1, we find a numerical simulation of the PDE and* H_2*-optimal estimator with a time step of* $0.002s$*,* $w(t) = sin(100t)$*, and PDE initial condition* $\xi(0,s) = s^2/2 - s$ *implying* $\mathbf{x}(0,s) = 1$*. In this simulation, we see errors in both the estimated state of the PDE and the regulated output decaying quickly despite instability in the PDE and persistent high-frequency excitation.*

Example B (Euler-Bernoulli Beam Equation)**.** *Consider a cantilevered Euler - Bernoulli beam with both sensor and process noise where the sensor measures tip velocity at the boundary.*

$$\ddot{\eta}(t,s) = -\frac{1}{10}\partial_s^4\eta(t,s) + \frac{s^2-2s}{2}w(t),$$
$$\eta(t,0) = \partial_s\eta(t,0) = \partial_s^2\eta(t,1) = \partial_s^3\eta(t,1) = 0,$$
$$z(t) = \int_0^1 \dot{\eta}(t,s)ds, \quad y(t) = \dot{\eta}(t,1) + w(t). \qquad (16)$$

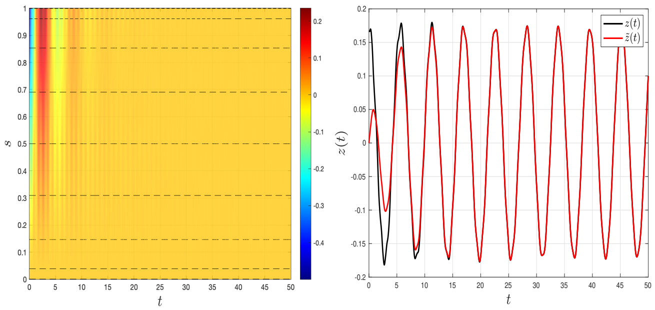
We may rewrite this equation in first-order form by defining the concatenated state $\mathbf{v}(t,s) = \big(\dot{\eta}(t,s), \partial_s^2\eta(t,s)\big)$ *[14], to obtain the coupled PDE system*

$$\dot{\mathbf{v}}(t) = \begin{bmatrix} 0 & -0.1 \\ 1 & 0 \end{bmatrix}\partial_s^2\mathbf{v}(t) + \begin{bmatrix} \frac{s^2-2s}{2} \\ 0 \end{bmatrix}w(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t),$$
$$\begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{v}(t,0) = \begin{bmatrix} 1 & 0 \end{bmatrix}\partial_s\mathbf{v}(t,0) = 0,$$
$$\begin{bmatrix} 0 & 1 \end{bmatrix}\mathbf{v}(t,1) = \begin{bmatrix} 0 & 1 \end{bmatrix}\partial_s\mathbf{v}(t,1) = 0,$$
$$z(t) = \int_0^1 \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{v}(t,s)ds, \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{v}(t,1) + w(t).$$

As in Ex. A, we find the PIE system parameters to be

$$\mathcal{T} = \mathcal{\Pi}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{R_0, R_1, R_2\} \end{array}\right], \qquad \mathcal{A} = \mathcal{\Pi}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{S_0, S_1, S_2\} \end{array}\right],$$
$$\mathcal{B}_1 = \mathcal{\Pi}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline s^2/2 & \{\emptyset\} \end{array}\right], \qquad \mathcal{C}_1 = \mathcal{\Pi}\left[\begin{array}{c|c} \emptyset & 0.5s^2-s \\ \hline \emptyset & \{\emptyset\} \end{array}\right],$$
$$\mathcal{C}_2 = \mathcal{\Pi}\left[\begin{array}{c|c} \emptyset & -s \\ \hline \emptyset & \{\emptyset\} \end{array}\right], \qquad \mathcal{D}_{21} = 1,$$

with PIE state, $\mathbf{x}(t) = \partial_s^2\mathbf{v}(t)$*, where*



(a) (b)

Fig. 2: Numerical estimation of an H_2-optimal estimator for a neutrally stable Euler-Bernoulli beam equation (Eq. (16)) using velocity measurement at the tip without disturbances and with PDE initial condition $\dot{\eta}(0,s) = s^2/2$. (a): Evolution of error in estimate of the PDE state $\tilde{\dot{\eta}}(t,\cdot) - \dot{\eta}(t,\cdot)$. (b): Evolution of the regulated output $(z(t))$ of both estimator and PDE.

$$R_0(s) = S_1(s,\theta) = S_2(s,\theta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; S_0(s) = \begin{bmatrix} 0 & -0.1 \\ 1 & 0 \end{bmatrix},$$
$$R_1(s,\theta) = \begin{bmatrix} s-\theta & 0 \\ 0 & 0 \end{bmatrix}, \; R_2(s,\theta) = \begin{bmatrix} 0 & 0 \\ 0 & \theta - s \end{bmatrix}.$$

In Fig. 2, we find a numerical simulation of the Euler-Bernoulli beam and H_2-optimal estimator with zero disturbance, PDE initial condition $\dot{\eta}(0,s) = s^2/2$ and a time step of $0.001s$. In this simulation, we see errors in both the estimated state of the PDE and the regulated output decaying quickly while the energy of the beam itself is preserved.

VIII. Conclusion

The H_2 norm is a commonly used performance metric in the estimation of linear state-space systems. However, finding observers which minimize the H_2 norm for a delayed or PDE system is complicated by the lack of an equivalent time-domain characterization of this norm. To address this problem, we have proposed an alternative initial condition to output characterization of the H_2 norm and applied this characterization to the PIE representation of the error dynamics. This approach then allowed us to pose the optimal observer synthesis problem as an LPI which can then be solved using existing software. The results were applied to estimation of the distributed state using boundary measurement subject to process and sensor noise and validated using numerical simulation of an unstable non-homogeneous reaction-diffusion equation and an energy-preserving Euler-Bernoulli beam equation.

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