Small-World-Synchronized Computing Networks

Regular lattices are commonly used to study physical systems with short-range interactions. New studies motivated by the natural/artificial systems have been focusing on coupled multi-component systems where the interaction between the components is facilitated by a complex network. The basic question here is how the collective behavior of the system is influenced by this possibly complex interaction topology. We study a synchronization problem in Small-World (SW) networks with "local" relaxation in a noisy environment. This study is directly applicable to synchronization in certain parallel simulation schemes implemented on a network of computers such as parallel discrete-event simulation systems (PDES) [1]. It also addresses generic and universal characteristics of SW-synchronized systems with such dynamics.

As the number of available processing elements (PE) on parallel architectures increases to tens of thousands or grid-computing networks proliferate over the Internet, fundamental questions of synchronizability and, in turn, the scalability of the underlying algorithms must be addressed. For scalability the time horizon formed by the local simulated times of the processing elements in PDES should have a bounded spread (width) as the number of PEs, N_{PE}, goes to infinity. In this study we examine different network communication topologies and show a possible way to construct scalable parallel algorithms for systems with *asynchronous* dynamics and short-range interactions on regular lattices.

If the spread of the simulated time horizon is very large, significant amount of memory should be reserved to store the data temporarily until the processors with low simulated time values reach the common level. In order to achieve a scalable algorithm we consider the parallel simulation itself as a complex interacting system where the specific synchronization rules correspond to the "microscopic dynamics". Our approach exploits a mapping between non-equilibrium surface growth and the evolution of simulated time horizon [2] so that we can use the tools and framework of statistical mechanics. We focus on the steady-state simulated time landscapes of the synchronization schemes and this work has also relevance to criticality on SW networks.

Since the discrete events in PDES are not synchronized by a global clock, the processing elements have to synchronize themselves by communicating with others. One of the first approaches to this problem for self-initiating processes is the original scheme where there are only nearest-neighbor interactions mimicking [2] the interaction topology of the underlying physical system. This basic model associated each component or site with one PE (worst-case scenario) under periodic boundary conditions. In this original scheme, at each time step only those PEs whose local simulated time are smaller than the local simulated times of their nearest neighbors are incremented by an exponentially distributed random amount so that the discrete events exhibit Poisson asynchrony. If the time of any PE is not smaller than its neighbors' time then no update occurs, i.e., PE idles. Using a mapping between simulated times and surface site heights in the coarse-grained description, it was shown that [2] the simulated time horizon of the original scheme is governed by the Kardar-Parisi-Zhang (KPZ) equation.

When analyzing the statistical and morphological properties of the stochastic landscape of the simulated times, it is convenient to study the height-height correlation or its Fourier transform, the height-height structure factor. The equal-time height-height structure factor $S(k,t)$ is defined as $S(k,t)N\delta_{k,-k'} = \langle\tilde{\tau}_k(t)\tilde{\tau}_{k'}(t)\rangle$ where $\tilde{\tau}_k = \sum_{j=1}^{N} e^{-ikj}\tau_j$ is the Fourier transform of the simulated times with the wave number $k{=}2\pi n/N$, $n{=}0,1,2,...,N-1$ and $\delta_{k,-k'}$ is the Kronecker delta. Structure factor essentially contains all the "physics" needed to describe the scaling behavior of the time surface. Here we focus on the steady-state properties ($t{\to}\infty$) of the time horizon where the structure factor becomes independent of time, $\lim_{t\to\infty} S(k,t){=}S(k)$. In the long time limit in one dimension, for a KPZ/EW surface one has $S(k){=}\frac{D}{2[1-\cos(k)]}$ where D is the constant factor in the second moment of the noise, i.e. $D{=}\frac{\langle\eta(x,t)\eta(x\prime,t\prime)}{2\delta(x-x\prime)\delta(t-t\prime)}$. The measured steady-state structure factor for 1D, obtained by simulating the original scheme based on the exact rules for the evolution of the local simulated times confirms the coarse-grained prediction for small k values, $S(k){\sim}k^{-2}$. For 2D, the structure factor as a function of the magnitude of the wave vector for the original scheme also exhibits a power-law behavior confirmed by using restricted solid-on-solid simulations as $S(|\vec{k}|){\sim}|\vec{k}|^{-2.78}$ for small values of k. The relation between the steady-state structure factor and the steady-state average width is $\langle w^2\rangle{=}\frac{1}{N^d}\sum_{k\neq 0} S(k)$ and one can easily show that in the original scheme $\langle w^2\rangle{\sim}N$ for 1D and $\langle w^2\rangle{\sim}N^{0.78}$ for 2D. Hence in the original scheme for both 1D and 2D, the width diverges with the system size.

The divergent width for very large systems discussed, is the result of the divergent lateral correlation length ξ of the simulated time surface reaching the system-size N in the steady-state. To de-correlate

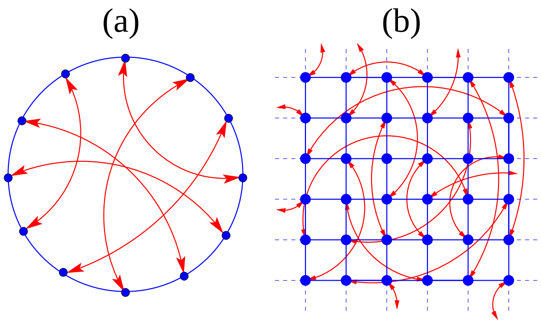

Figure 1: SW synchronization network in (a) 1D and (2) 2D. Red arrows show the random SW links in both.

the simulated time horizon we modify the virtual communication topology of the PEs. The resulting communication network must include the original short-range (nearest-neighbor) connections to faithfully simulate the dynamics of the underlying system. We add one random link with strength p (the probability to include the random link as well in the simulated time comparison) for every node as shown in Fig. 1. Note that the occasional extra checking (at every $1/p$ parallel steps on average) of the simulated time of the random link is *not* needed for the faithfulness of the simulation. It is merely introduced to control the width of the time horizon.

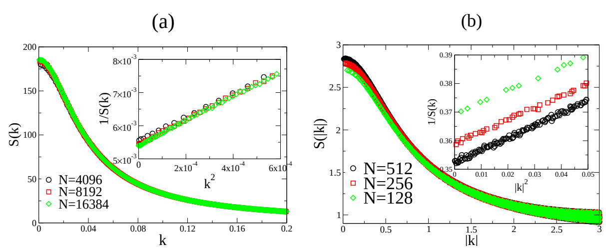

Figure 2: (a) Structure factor for the SW synchronization scheme in 1D for p=0.1. (b) Structure factor as a function of $|\vec{k}|$ for SW synchronization scheme in 2D for p=1. The insets show the linear behavior of the structure factor when plotted on rescaled axis, $1/S(k)$ vs. k^2 for small values of k.

The random links in the system create a relaxation term in the stochastic growth equation and this implies that $\lim_{k\to 0} S(k) < \infty$, that is, there are no large amplitude long-wavelength modes in the surface. Consequently, the width is also finite. We see that the simulated time surface of the SW network is macroscopically smooth when compared to the KPZ surface resulted from a short-range (original scheme) network. Considering only the linear terms in KPZ equation, we obtain $S(k) \propto \frac{1}{\gamma + k^2}$ where γ is a monotonically increasing function of probability parameter p and $\gamma(p)$=0 as can be seen in Fig. 2. In this approximation, the lateral correlation length of the surface fluctuations is $\xi \sim \gamma^{-1/2}$, that is, it is finite for all $p \neq 0$.

This kind of behavior in structure factor implies that the width of the simulated time surface saturates to an asymptotic value for a nonzero value of p as the number of PEs goes to infinity [Fig. 3]. The generalization about adding random links to a higher-dimensional underlying regular lattice is intuitively clear. Because one- and two-dimensional cases with random links are effectively governed by the mean-field equation, in higher dimensions it will be even more so (i.e., the critical dimension of the model with SW links is less than one). Since we have a finite correlation length and consequently a finite width in 1D and 2D, we can argue that in higher dimensions the SW networks should behave in the same way.

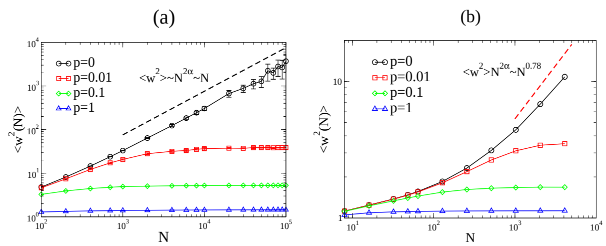

Figure 3: The average steady-state width as a function of system-size for different p values in (a) 1D SW (b) 2D SW synchronization network. Note that in 2D, $N = N_{PE}^{1/d} = N_{PE}^{1/2}$ is the linear system size.

In conclusion, the simulated time horizon for the SW-synchronized PDES scheme becomes *macroscopically* smooth and essentially exhibits mean-field like characteristics. The random links, on top of a regular lattice, generate an effective *mass* for the propagator of the simulated time horizon (in a field theory sense) corresponding to a finite correlation length [3]. There is growing evidence that systems without inherent frustration exhibit (strict or anomalous) mean-field-like behavior when the original short-range interaction topology is modified to a SW network.

References

[1] G. Korniss, M.A. Novotny, H. Guclu, Z. Toroczkai, and P.A. Rikvold, *Science* **299**, 677 (2003).

[2] G. Korniss, Z. Toroczkai, M.A. Novotny, and P.A. Rikvold, *Phys. Rev. Lett.* **84**, 1351 (2000).

[3] H. Guclu, G. Korniss, M.A. Novotny, Z. Toroczkai, and Z. Rácz, To be published in *Phys. Rev. E.*

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