Oscillating elastic defects

A fundamental question in elasticity theory is to understand the patterns of deformation and strains created by stress, defects or inhomogeneities in a material. In a celebrated 1957 paper [1], Eshelby derived an exact solution for the strain fields created by an inhomogeneous elliptic inclusion in an isotropic elastic medium. This result is a cornerstone of the theory of inhomogeneous elastic media, nowadays routinely used in engineering sciences. From a physicist’s point of view, it demonstrates that a local perturbation induces long range strain fields, slowly decaying as \(1/r^d\) in \(d\) dimensions (\(d \geq 2\), and \(r\) being the distance from the perturbation). These long-range and anisotropic interactions between strain fields induce, for instance, pattern formation in martensitic materials [3].

Eshelby’s work considered static inhomogeneities only; in a number of applications in physics and material science, it is actually important to consider the dynamics of the strain fields created by an elastic inclusion. The inhomogeneities themselves can be oscillating: for instance, polarons tunneling between two positions in high \(T_c\) superconductors can be viewed as locally oscillating defects. Other applications include the design of micro electromechanical devices (MEMS). Although the dynamical generalizations of the Eshelby problem have been recently investigated in the context of engineering sciences [2], no effort has been made in that direction from a physicist’s point of view. In this work [4], we consider the dynamical Eshelby problem for localized oscillating defects, and focus on the general and qualitative features of the resulting strain fields. In particular, we ask the following specific questions: how do the strain fields depend on the oscillating frequency? How do defects interact with each other through the strain fields they create?

One defect case

We consider a two-dimensional elastic medium, and we impose the presence of an oscillating defect, represented by a locally oscillating deviatoric strain \(e_3\) (\(e_3\) essentially measures locally how much a square is deformed into a rectangle): \(e_3(\vec{r} = \vec{0}) = e_0 \sin \omega_0 t\). At the level of linear elasticity, the equations can be solved exactly, and expressions are obtained for \(e_3\), as well as for the dilatation strain \(e_1\) and the shear strain \(e_2\). Fig. 1 shows such strain profiles for different frequencies \(\omega_0\), and Fig. 2 shows a surface plot of the \(e_3\) field.

This study allows us to demonstrate two important results:

- The strain fields created by the defect decay at large distances as \(1/r^2\), as in the static case.

Figure 1: Profile along \((Ox)\) of the deviatoric strain \(e_3\), for different frequencies \(\omega_0 = 1\) (blue), \(\omega_0 = 10\) (red) and \(\omega_0 = 100\) (black). The higher the frequency \(\omega_0\), the more localized is the perturbation.

Figure 2: Surface plot of the deviatoric strain \(e_3\), for \(\omega_0 = 1\). Notice the anisotropy of the strain field.
• The deformation induced by the defect is progressively more localized as \( \omega_0 \) is increased; this has an important consequence for the interaction between defects: it can be suppressed by tuning the frequency. This point is developed in the next section.

**Two defects case**

Next, we consider the case of two defects, to probe the interaction between them. We suppose that the two defects, located at \( r = 0 \) and \( r = r_0^* \), oscillate at the same frequency \( \omega_0 \), and have a phase difference \( \varphi \). Again, at the level of linear elasticity, the equations can be solved exactly, giving access to the strain profiles. An example is given in Fig. 3.

![Figure 3: Profiles of the deviatoric strain \( e_3 \) with two defects, for \( \omega_0 = 1 \) (blue), \( \omega_0 = 10 \) (red) and \( \omega_0 = 100 \) (black). The phase difference between the two defects is \( \varphi = \pi \).](image)

From a given configuration of the strain fields, it is easy to calculate the associated elastic energy; this energy obviously oscillates with frequency \( \omega_0 \). To probe the interaction between the two defects, we average this energy over one period, and obtain the average energy \( U(\varphi, r^*, \omega_0) \) as a function of the phase difference between the defects \( \varphi \), the distance between them \( r_0^* \) and their common frequency \( \omega_0 \). We focus on a situation where the defects are pinned, and interact through their relative phase; the relevant quantity is then \( U(\varphi) \), at fixed \( r_0^* \) and \( \omega_0 \). These energies are well described by the “XY-like” function:

\[
U(\varphi, r^*, \omega_0) = C (r^*, \omega_0) + J (r^*, \omega_0) \cos \varphi. \tag{1}
\]

A positive \( J \) corresponds to a ferromagnetic, or phase locking, interaction, whereas a negative \( J \) leads to an antiferromagnetic interaction.

Fig. 4 shows some representative energy profiles, and emphasizes the effects of varying the frequency or the distance:

- As expected, \( J \) goes to zero for large distances and/or large frequencies.
- More dramatically, \( J \) changes sign: at low enough \( \omega_0 \) and short enough distances, the interaction is ferromagnetic; and antiferromagnetic otherwise.

This last property may have important consequences for the collective behavior of an assembly of defects: one can imagine inducing phase transitions between phase locked and disordered states by tuning the defect oscillation frequency.

**Conclusion**

Summarizing, we have extended the classical study by Eshelby of static defects in the 50s to dynamical oscillating defects. As in the static case, local perturbation of the strain fields propagates to large distances and decays away only as \( 1/r^2 \), \( r \) being the distance from the perturbation. However, the strain profile strongly depends on the oscillation frequency: the higher the frequency, the more localized the strain field. This has important consequences for the interaction between several such oscillating defects, opening the door to collective effects controlled by the frequency. This may also have applications for the interactions between polarons in high \( T_c \) superconductors, and for the design of micromechanical devices.

**References**


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