Three-body scattering amplitudes

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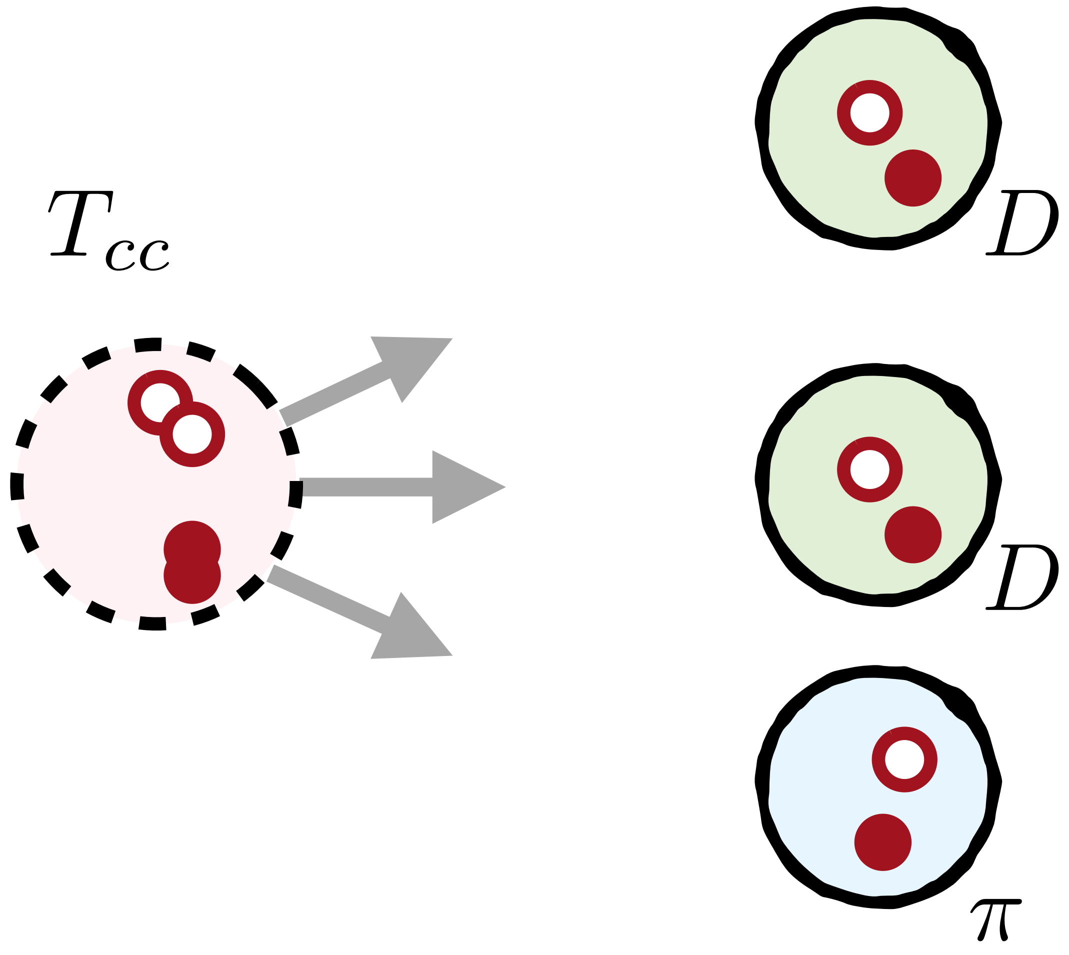
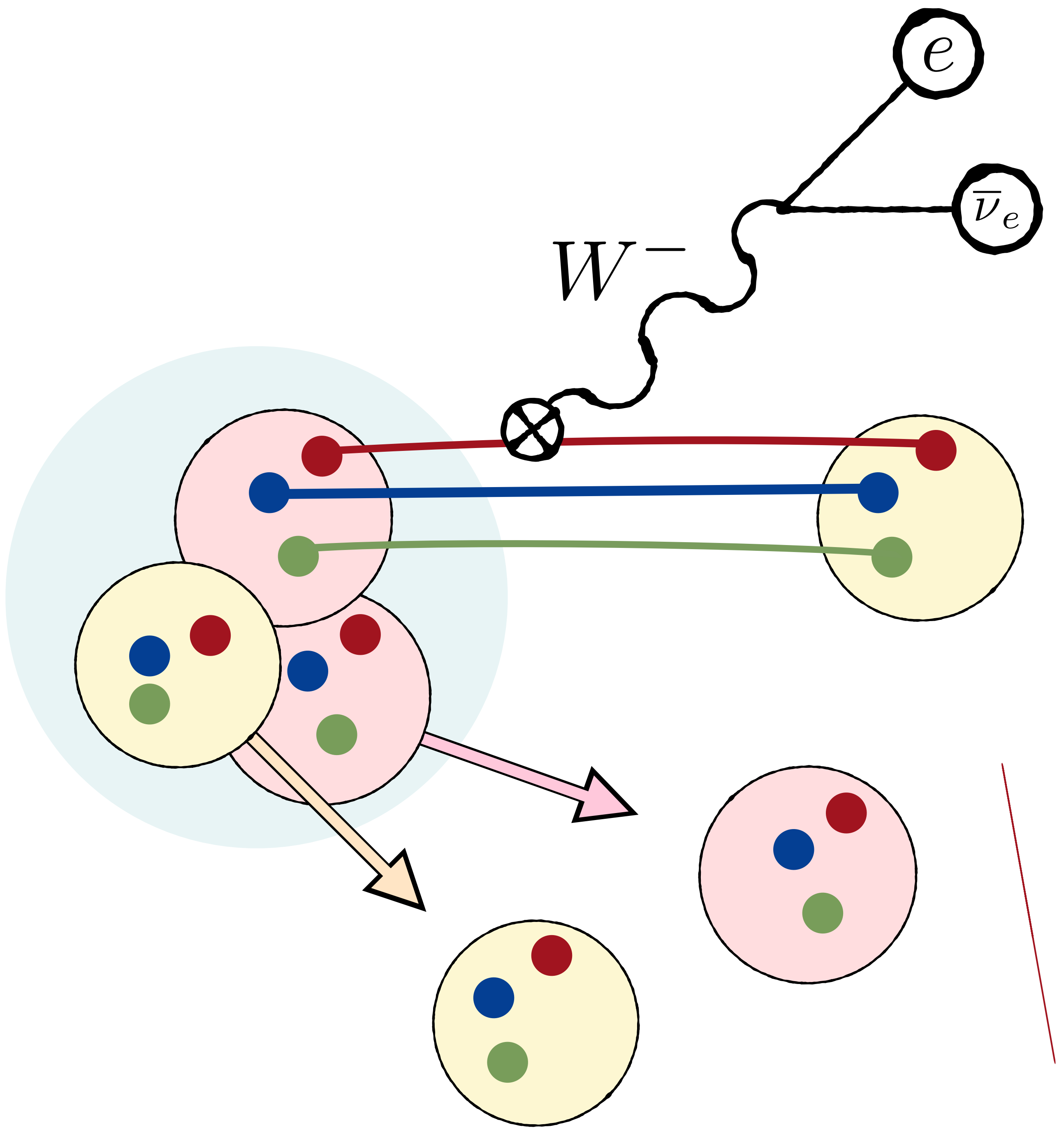
big picture

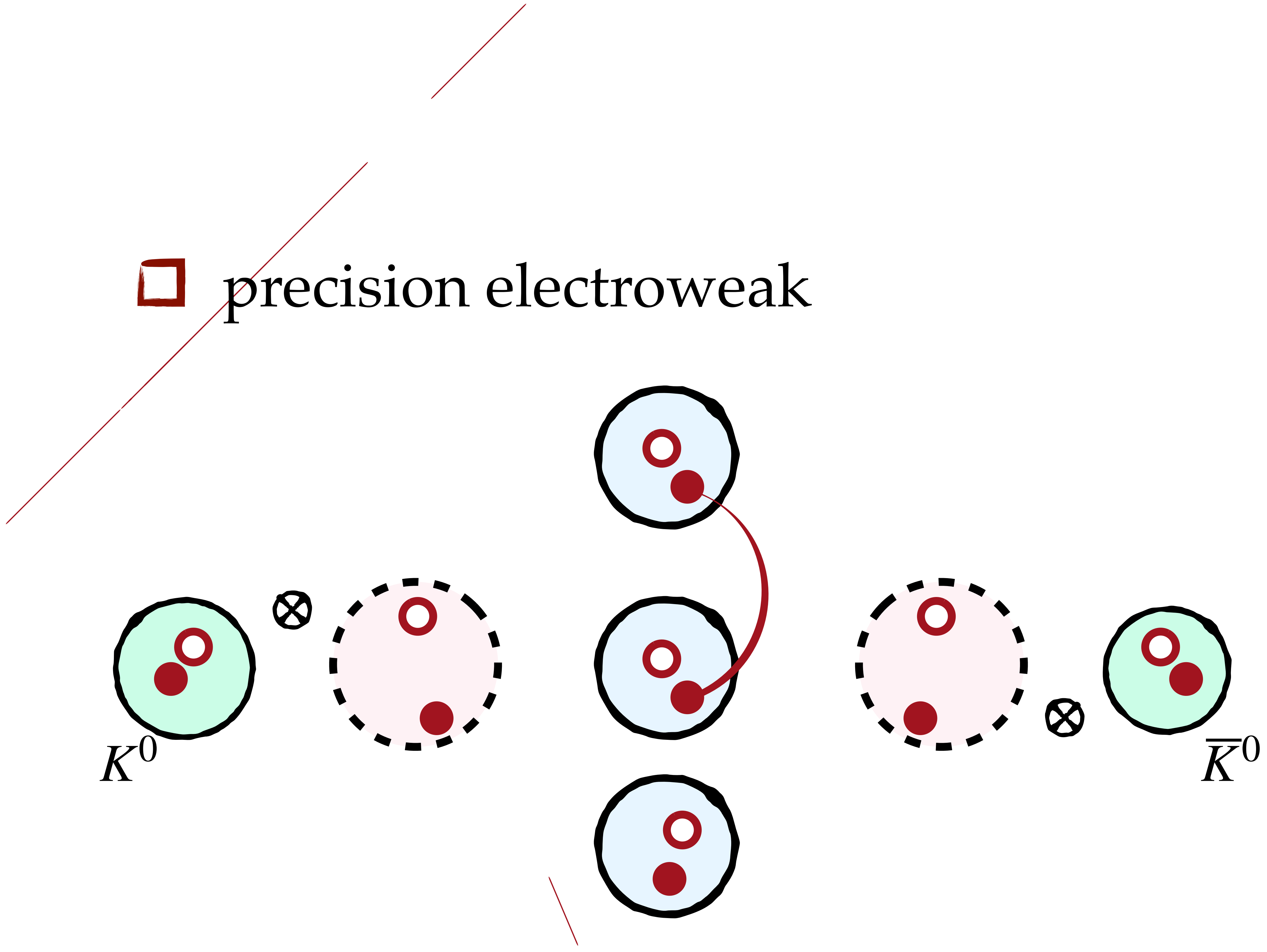
- hadron spectroscopy



- nuclear structure / neutrino physics

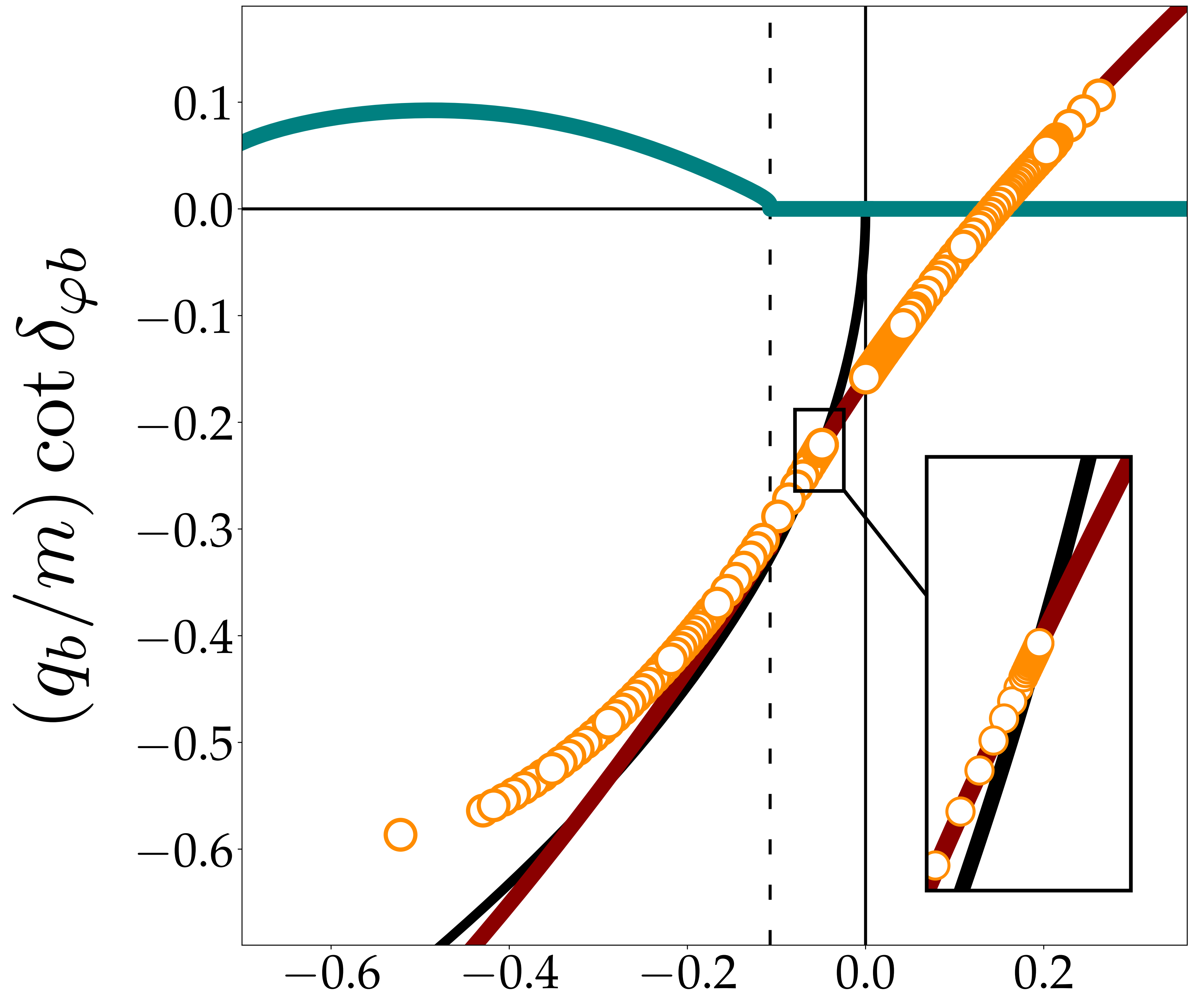


- precision electroweak



outline

- [ ] an example of the full procedure being done
- [ ] integral equations
- [ ] angular momentum projection
- [ ] toy model calculations
- [ ] consistency checks and the breakdown of Lüscher
- [ ] Efimov physics



Jackura

Dawid

Islam

Romero-López, Sharpe, Blanton, RB, & Hansen (2019)
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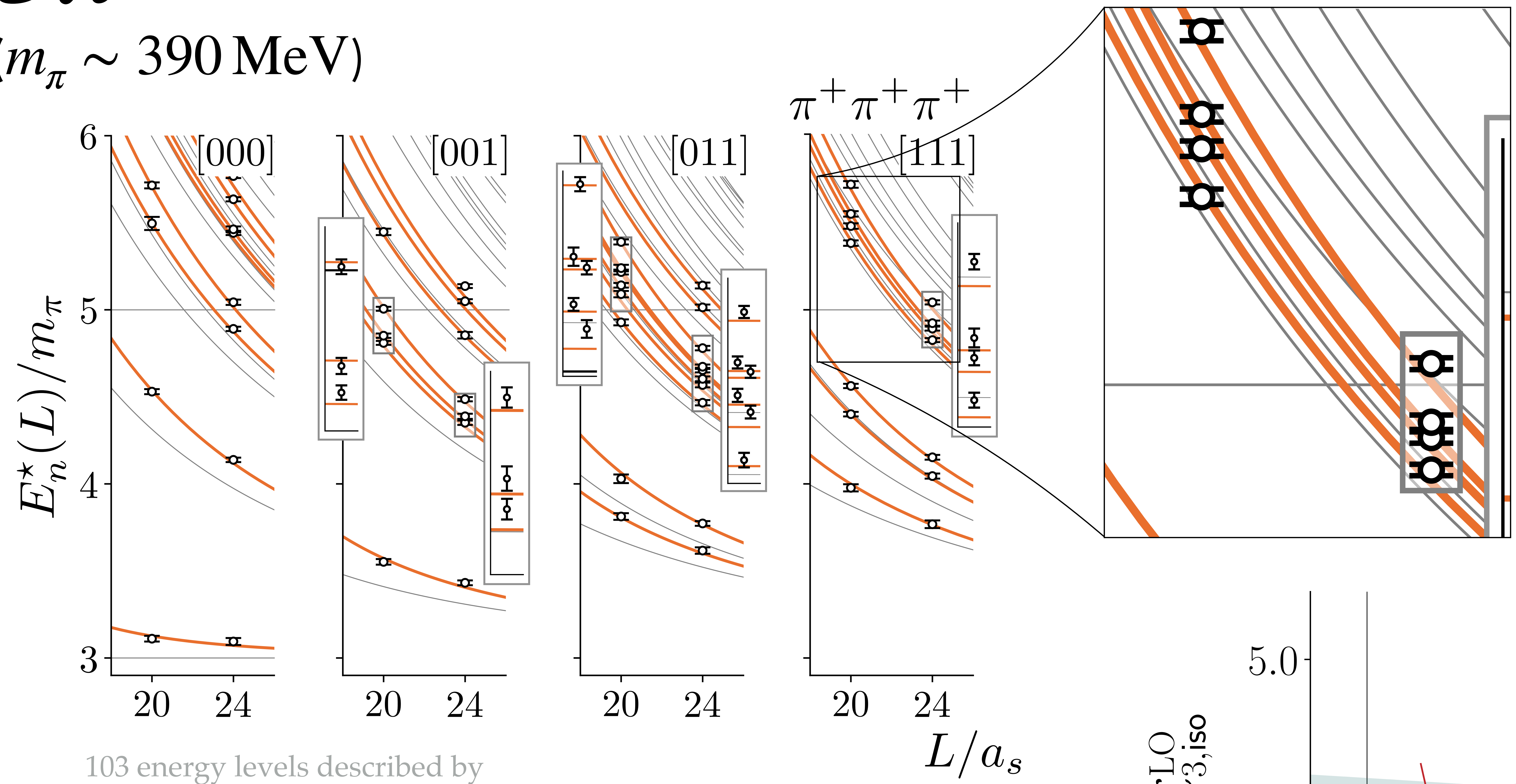
Three-hadron systems

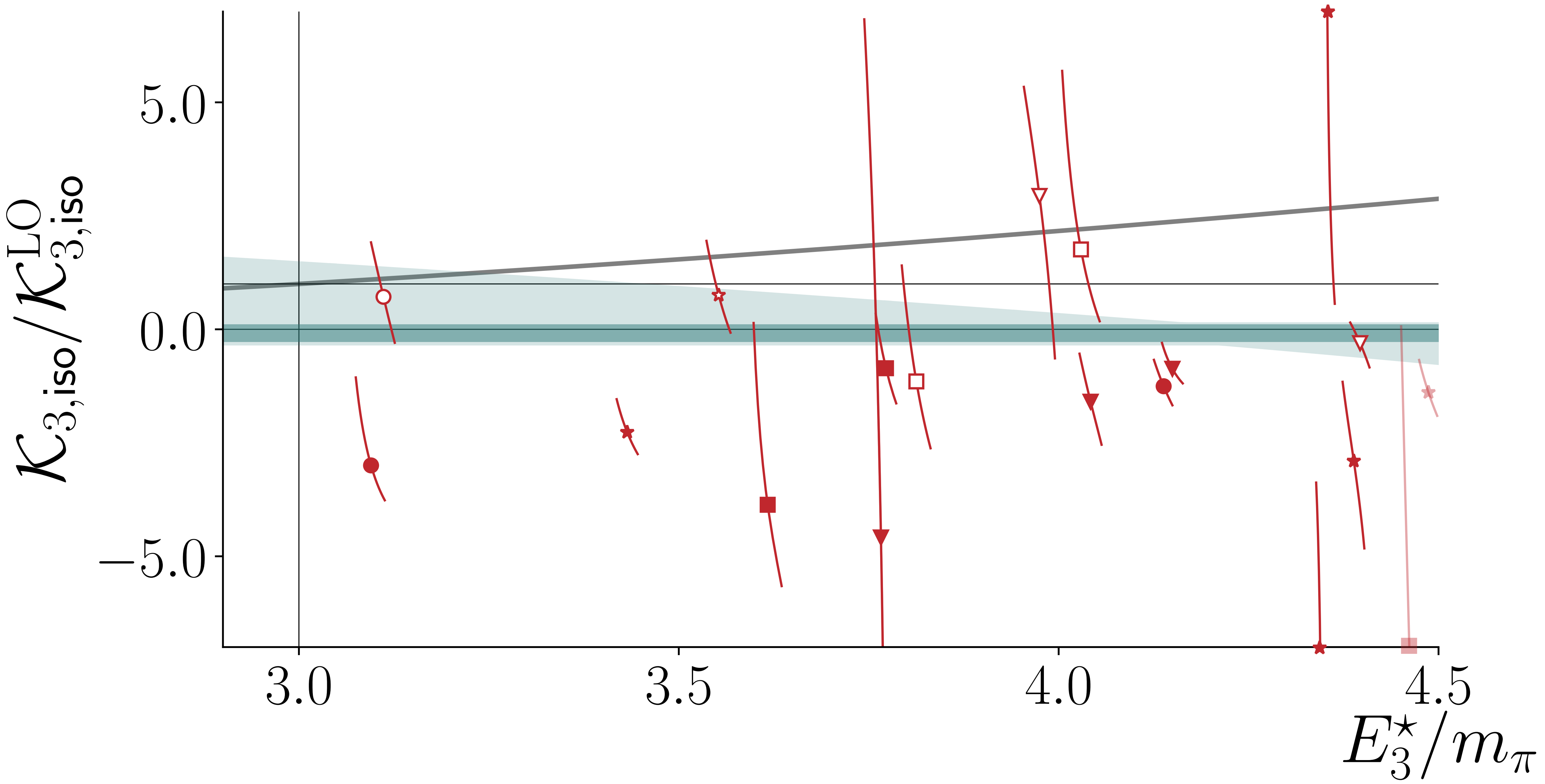
$3\pi^+$

$(m_\pi \sim 390\,\mathrm{MeV})$



103 energy levels described by three numbers: m_π, $a_{\pi\pi}$, $\mathcal{K}_{3,\mathrm{iso}}$

$$F_{3,\mathrm{iso}}^{-1}(P,L) + \mathcal{K}_{3,\mathrm{iso}}(P^2) = 0$$



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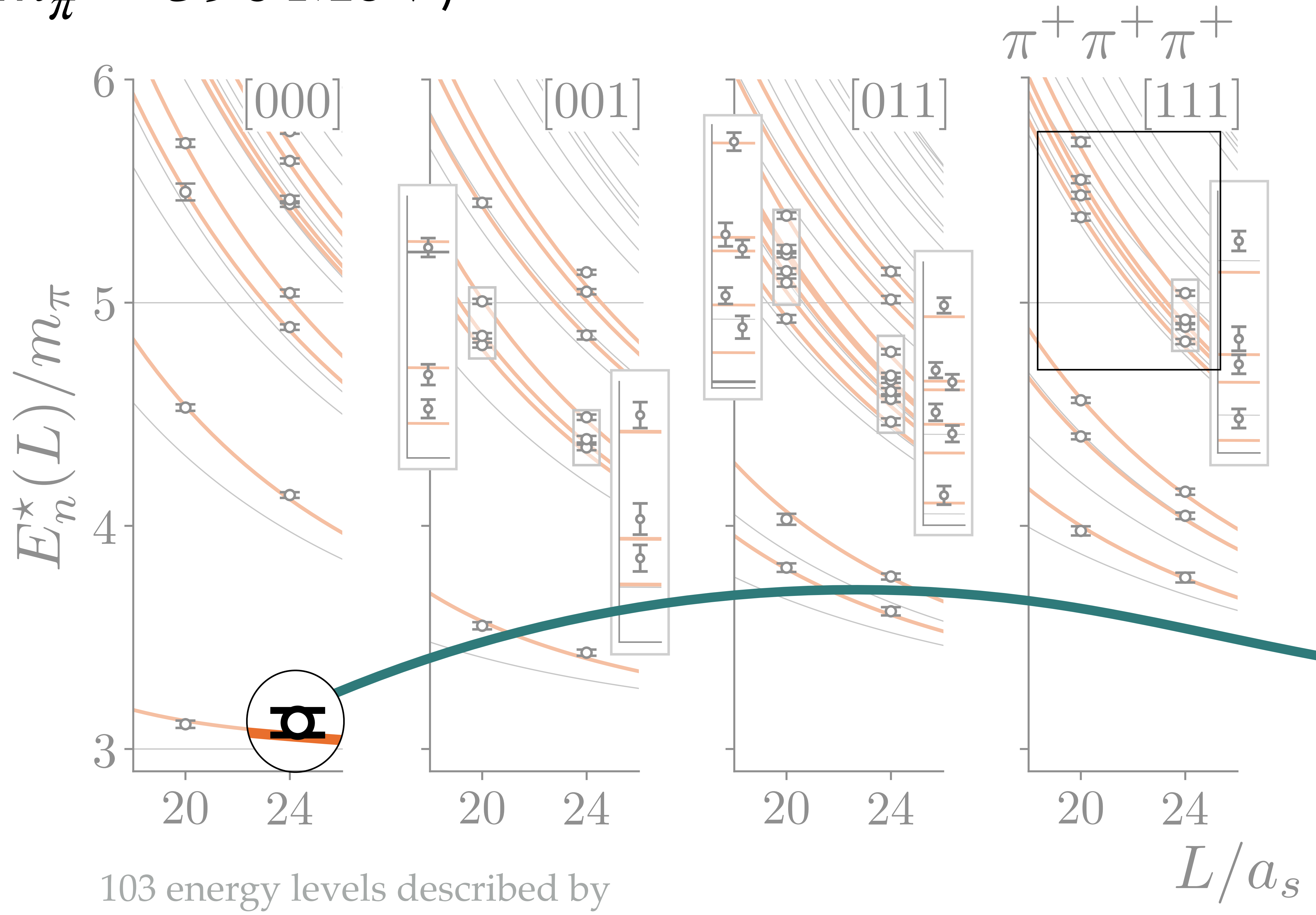

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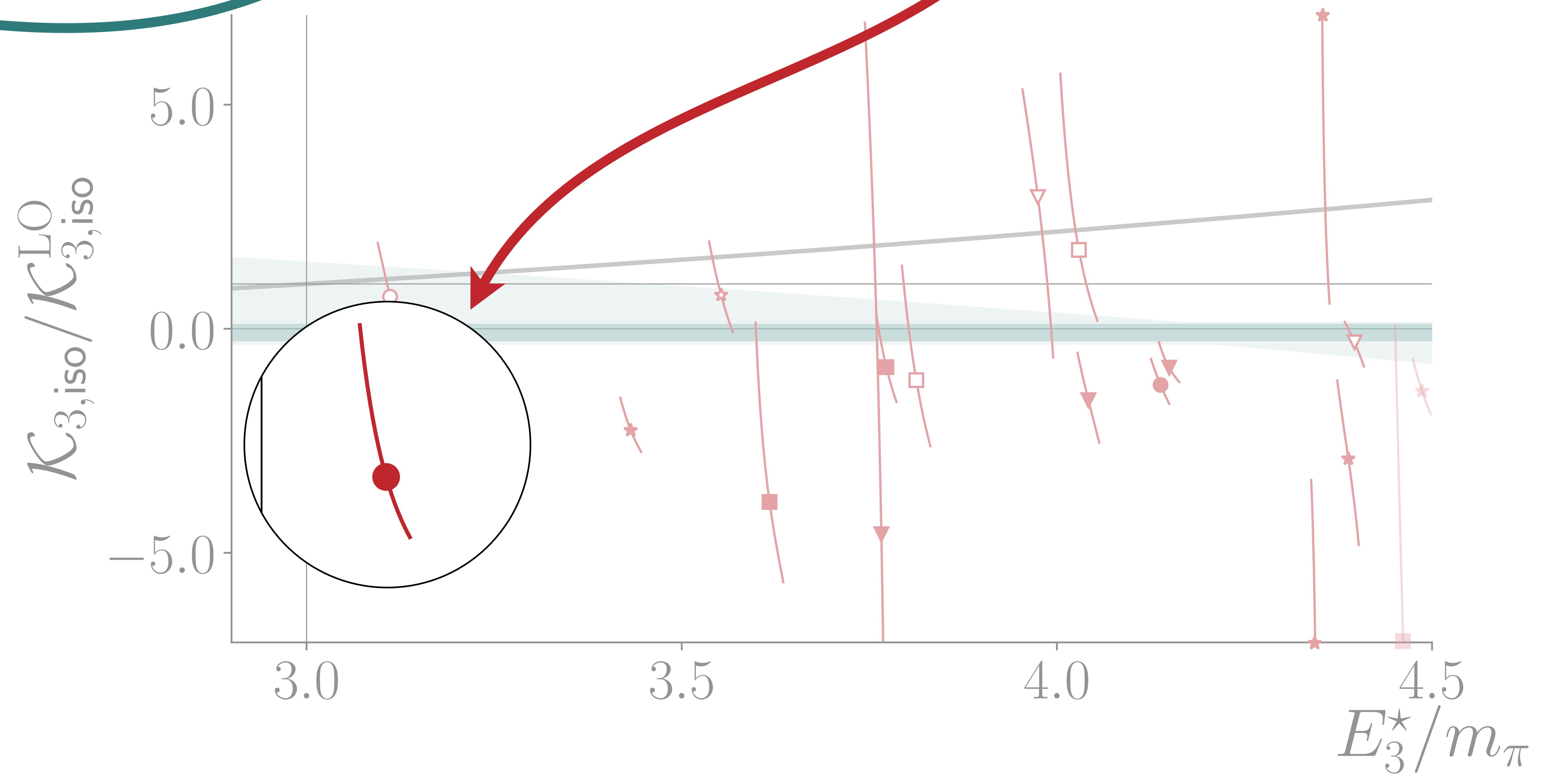

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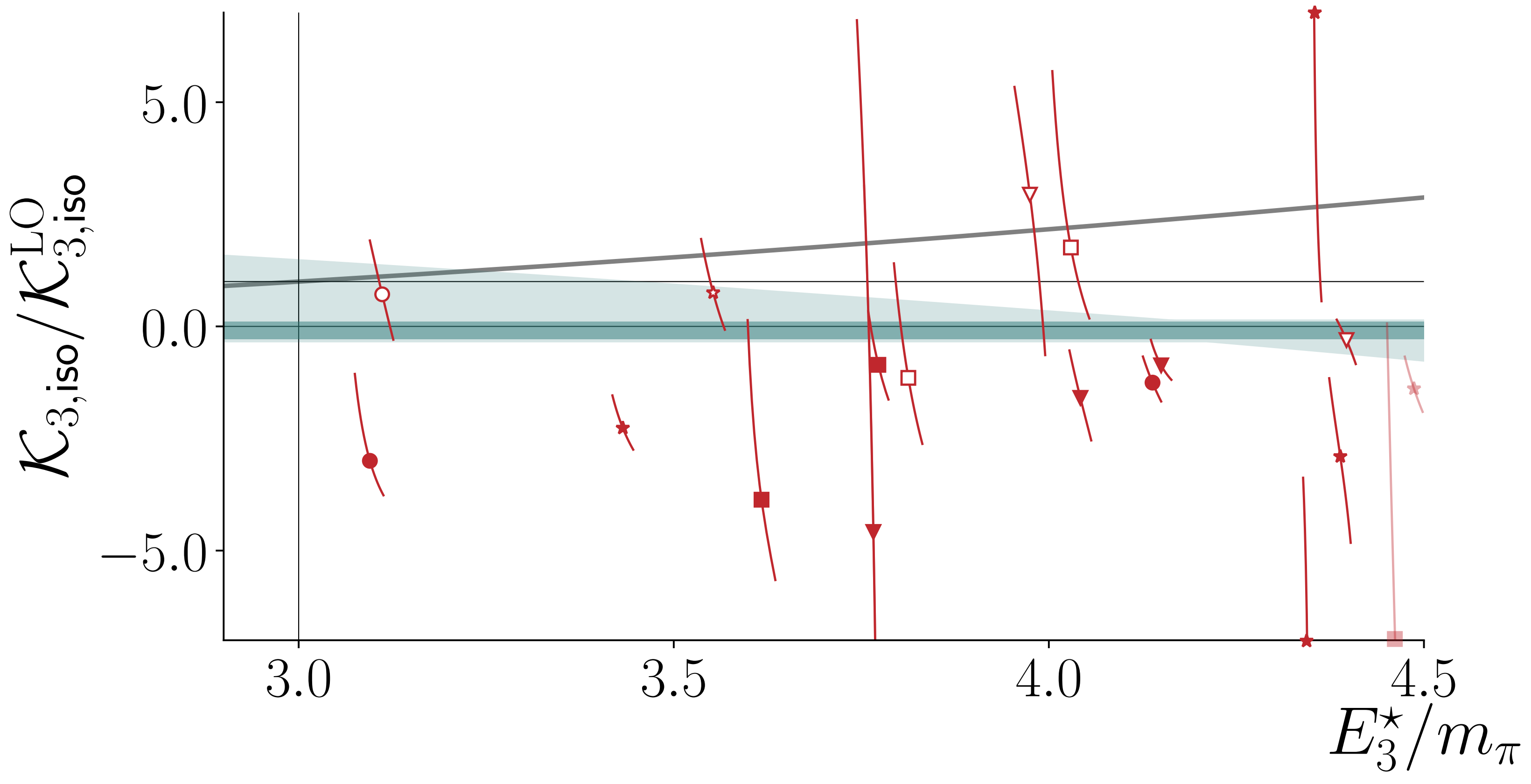
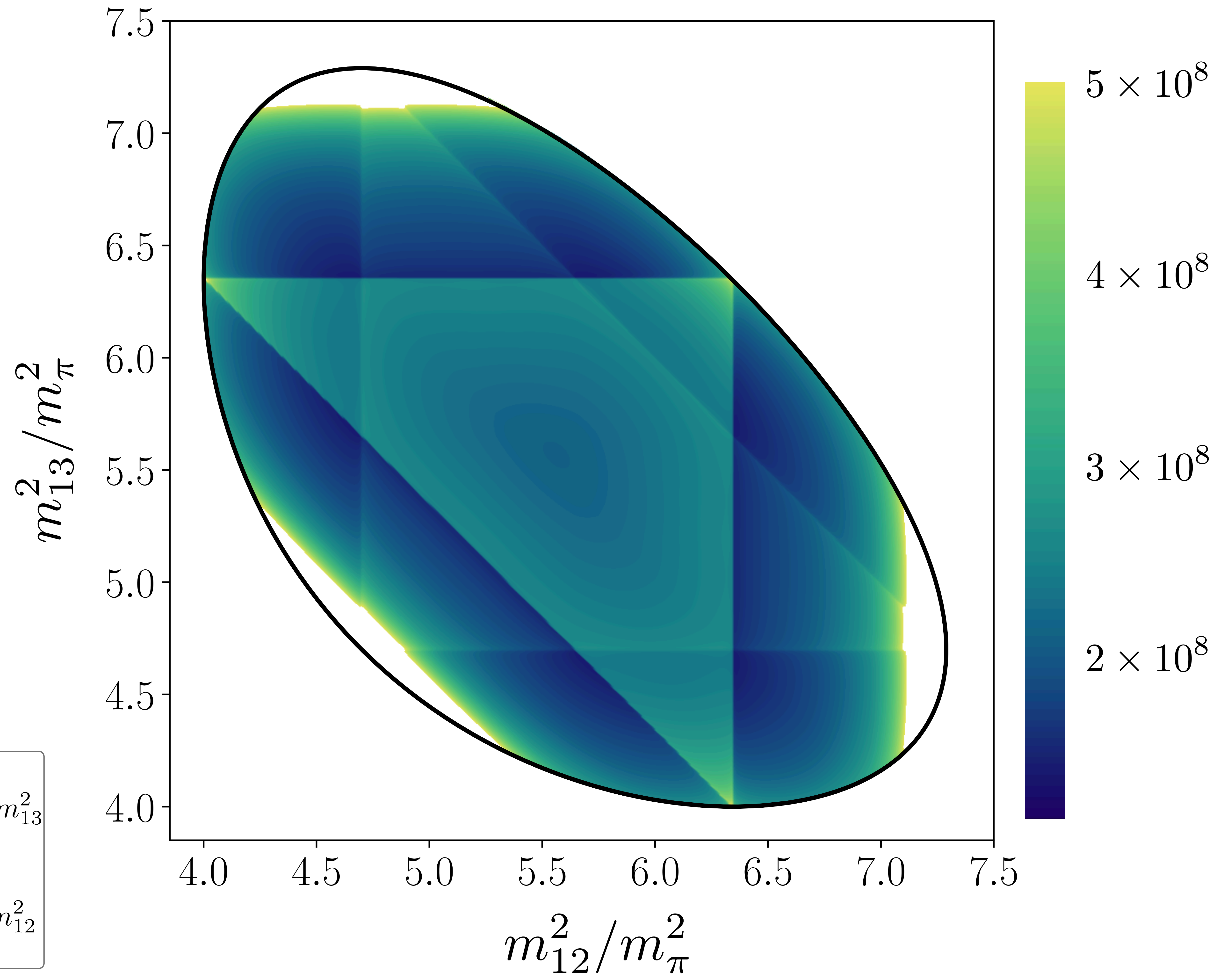
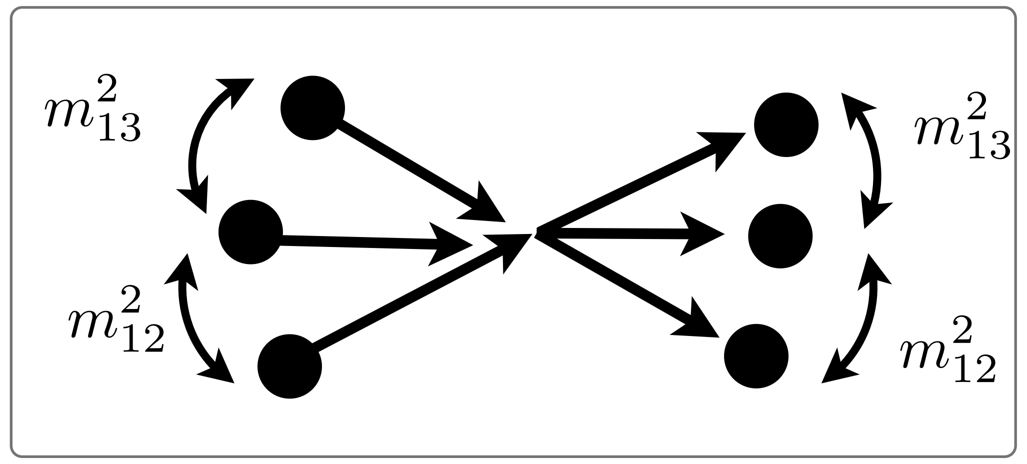


$$i\mathcal{D} + i\mathcal{L}\frac{1}{\mathcal{K}_3^{-1} + F_3^\infty}\mathcal{L}$$





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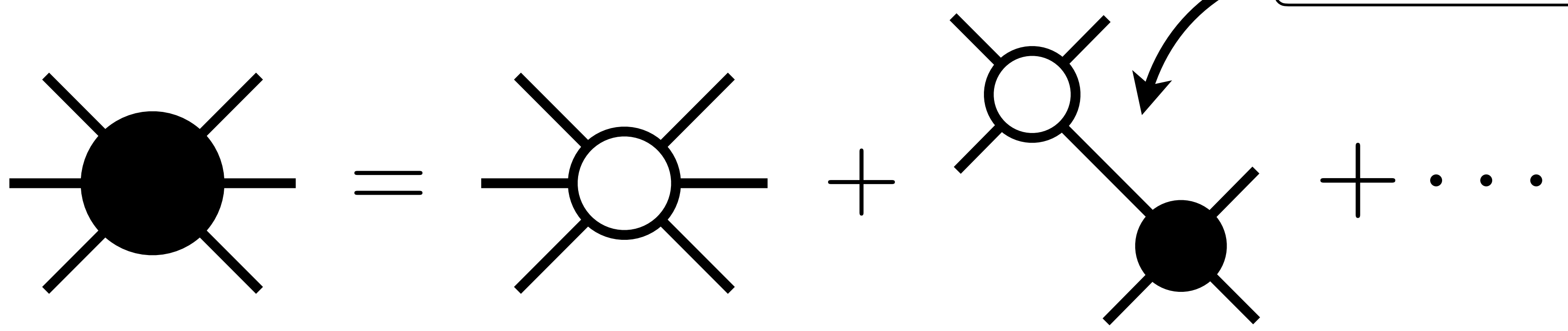
Three-hadron systems

Infinite Minkowski spacetime

$$G \sim \frac{1}{(P-p-k)^2 - m^2}$$

$$i\mathcal{M}_3 =$$



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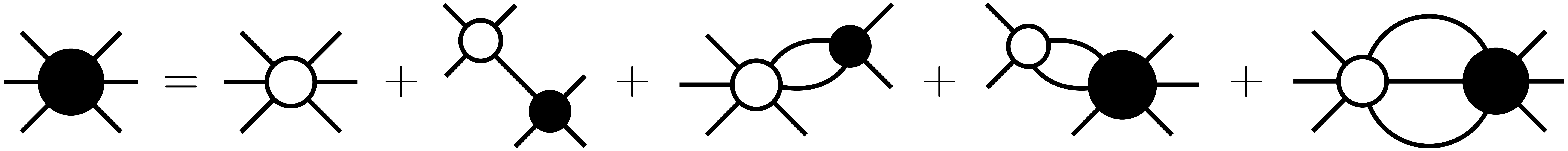


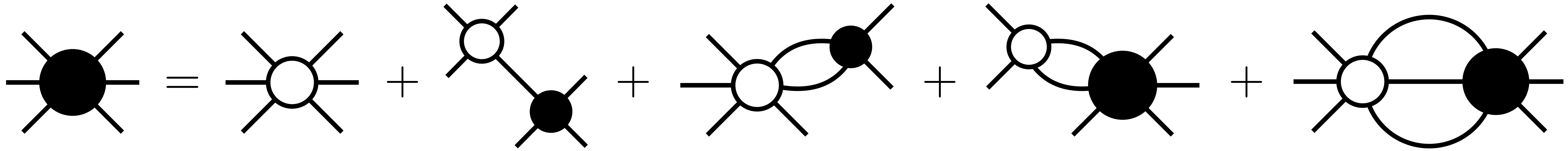
placing all legs on-shell

$$\longrightarrow i\mathcal{D} + i\mathcal{L}\frac{1}{\mathcal{K}_3^{-1} + F_3^{\infty}}\mathcal{L}$$

[isotropic approximation]

Three-hadron systems

Infinite Minkowski spacetime

$i\mathcal{M}_3 =$ 

placing all legs on-shell → $i\mathcal{D} + i\mathcal{L}\frac{1}{\mathcal{K}_3^{-1} + F_3^{\infty}}\mathcal{L}$ [isotropic approximation]

Satisfies integral equations:

$$i\mathcal{D} = i\mathcal{M}_2 iG i\mathcal{M}_2 + \int i\mathcal{M}_2 iG\, i\mathcal{D}$$

$$\mathcal{L} = \frac{1}{3} + \mathcal{M}_2\rho - \mathcal{D}\rho$$

$$F_3^{\infty} = \int \rho\mathcal{L}$$

Three-hadron systems

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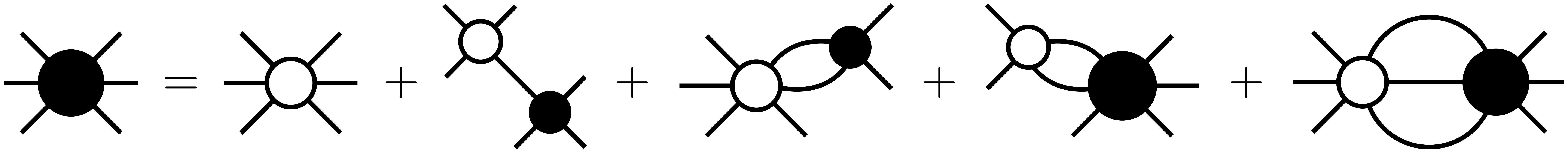

placing all legs on-shell $\longrightarrow$ $i\mathcal{D} + i\mathcal{L}\dfrac{1}{\mathcal{K}_3^{-1} + F_3^{\infty}}\mathcal{L}$ [isotropic approximation]

Satisfies integral equations:

$$i\mathcal{D} = i\mathcal{M}_2 iG i\mathcal{M}_2 + \int i\mathcal{M}_2 iG\, i\mathcal{D} \quad \Big| \quad \mathcal{L} = \frac{1}{3} + \mathcal{M}_2\rho - \mathcal{D}\rho \quad \Big| \quad F_3^{\infty} = \int \rho\mathcal{L}$$

It is convenient to introduce $\mathcal{D} = \mathcal{M}_2\, d\, \mathcal{M}_2$

where $d = -G - \int G\, \mathcal{M}_2 \mathcal{D}$

Integral equations

We need to solve:

$$d(\mathbf{p}', s, \mathbf{p}) = -G(\mathbf{p}', s, \mathbf{p}) - \int_0^{q_{\max}} \underbrace{\frac{d^3\mathbf{q}}{(2\pi)^3 2\omega_q} G(\mathbf{p}', s, \mathbf{q}) \mathcal{M}_2(q, s)}_{\text{“integration kernel”}} \, d(\mathbf{q}, s, \mathbf{p})$$

Need to resort to numerical solutions.

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Need to resort to numerical solutions.

There are three correlated challenges we will focus on and address:

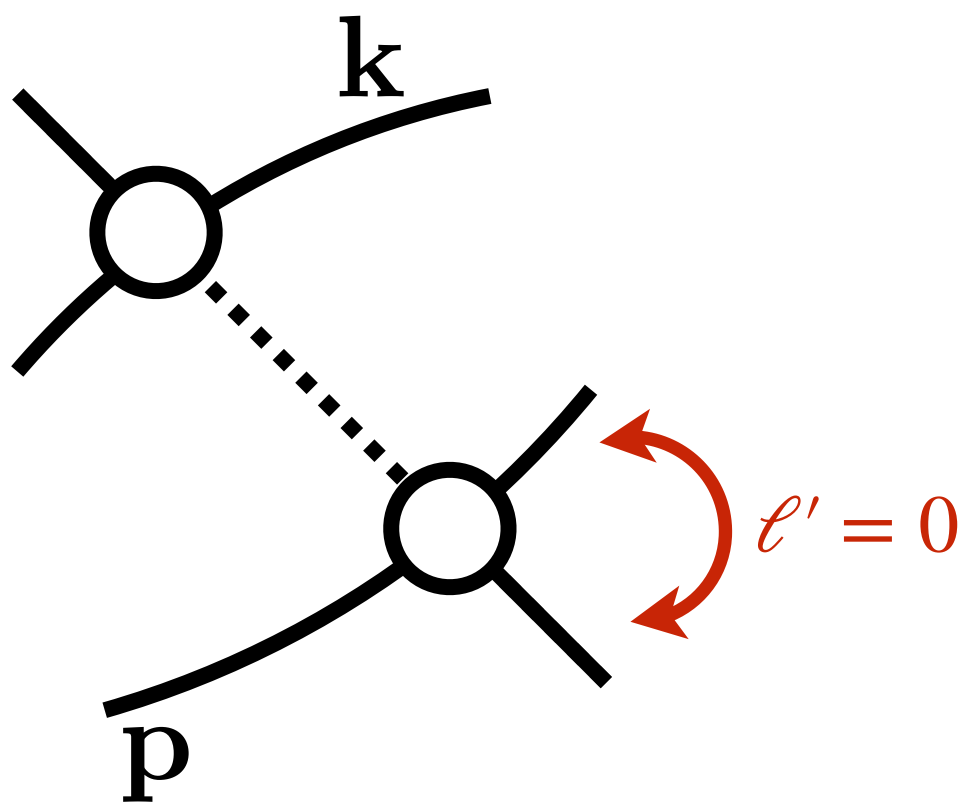
- ☐ 3D integral equation,
- ☐ need to project to **angular momentum and parity**,
- ☐ integration kernel is generally singular.

Starting assumption: the scattering amplitude satisfies the "whole" complex plane, including unphysical sheets.

Partial wave projections

The one-particle exchange is one of the main sources of singularities.

Let us consider the case where $\ell = 0$:



$$\sim G(\boldsymbol{p}, \boldsymbol{k}) = \frac{1}{(E - \omega_k - \omega_p) - (\boldsymbol{p} + \boldsymbol{k})^2 - m^2 + i\epsilon}$$

$$= \frac{1}{(E - \omega_k - \omega_p) - k^2 - p^2 - m^2 - 2pk\cos\theta + i\epsilon}$$

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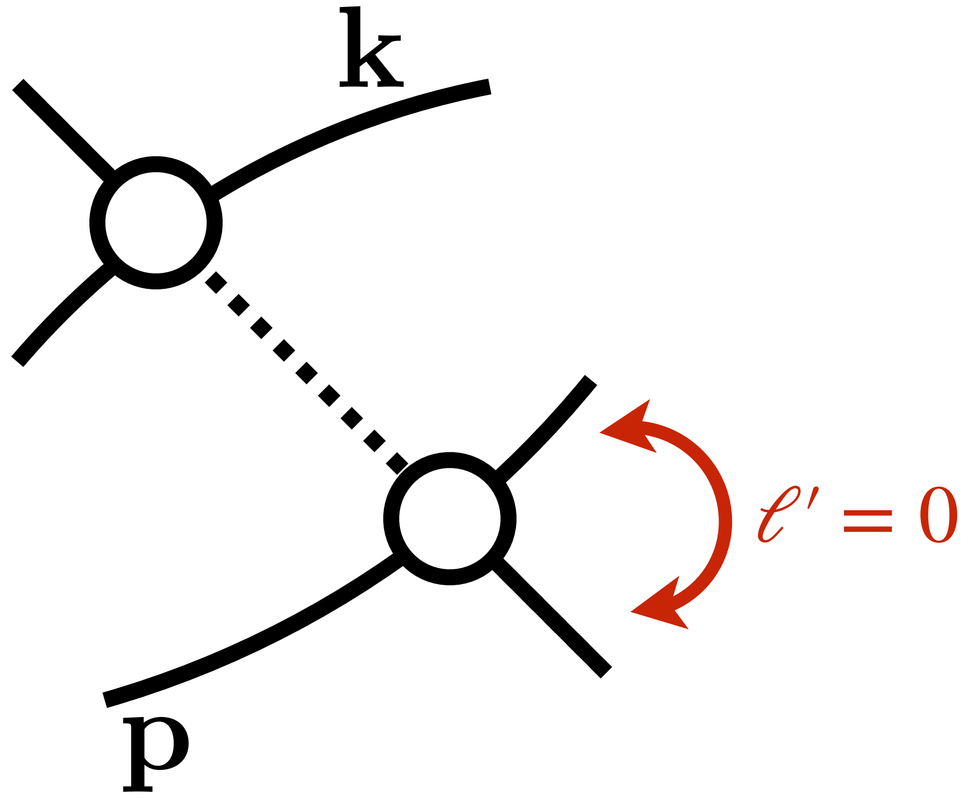

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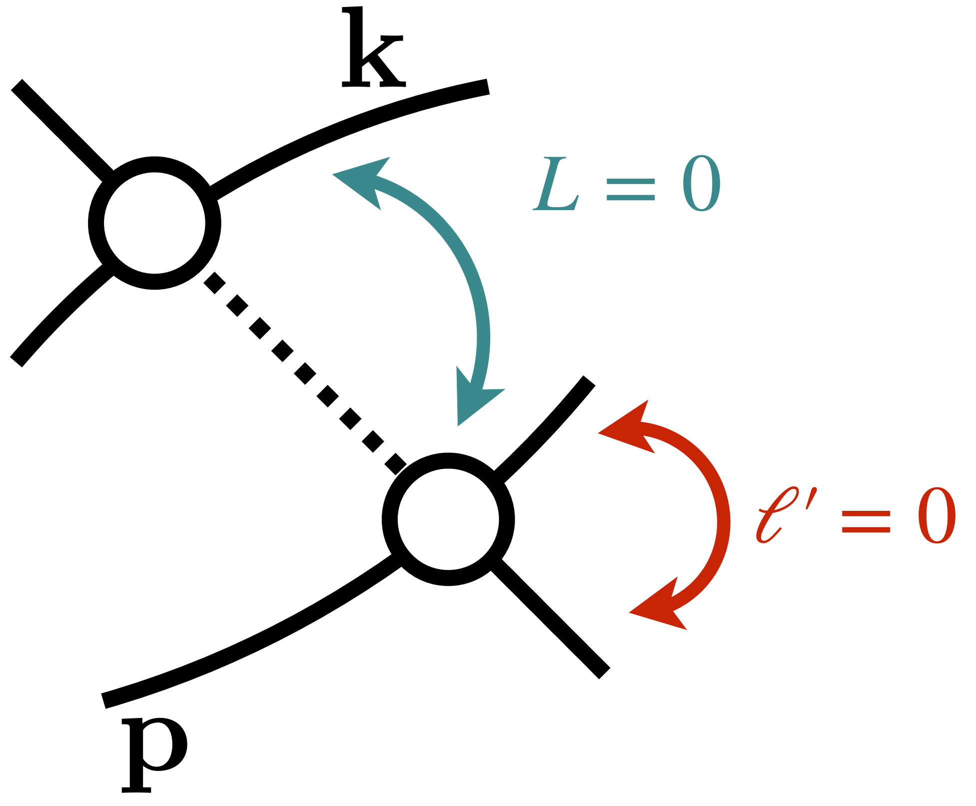
Projecting to total $J = 0$ amounts to integrating over all angles:



$$\sim G(p, k) = \frac{1}{2}\int_{-1}^{1} d\cos\theta \, G(\boldsymbol{p}, \boldsymbol{k}) = -\frac{1}{4pk}\log\frac{z_{pk} - 1}{z_{pk} + 1}$$

$$z(p, k) = \frac{(E - \omega_k - \omega_p)^2 - k^2 - p^2 - m^2}{2pk}$$

Partial wave projections

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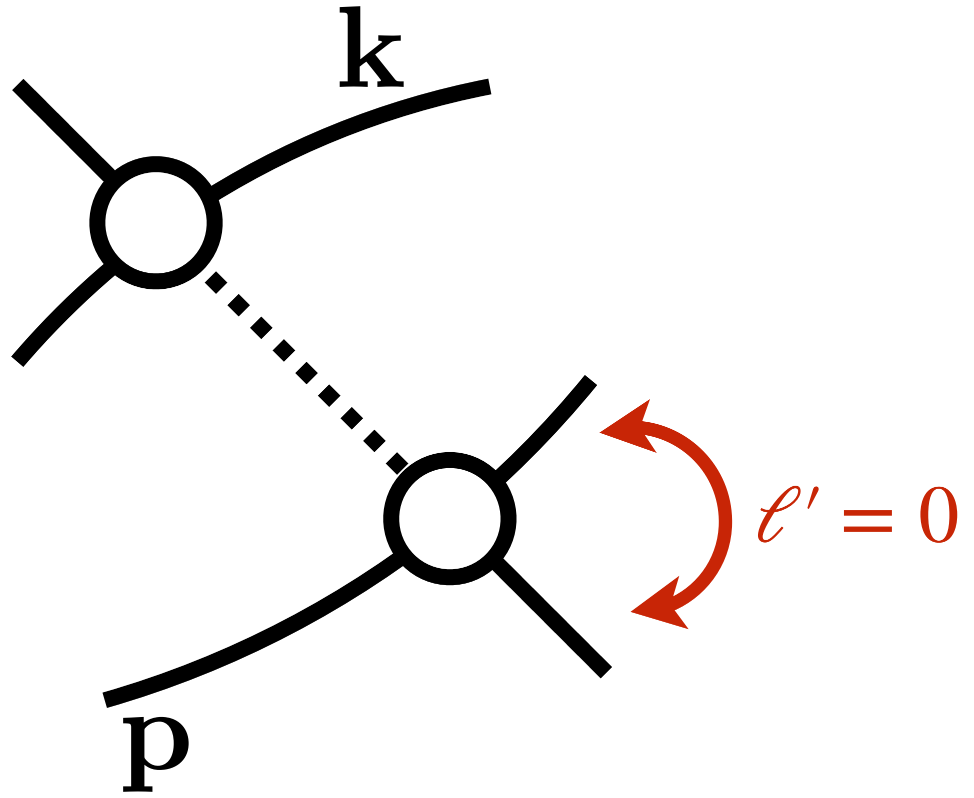

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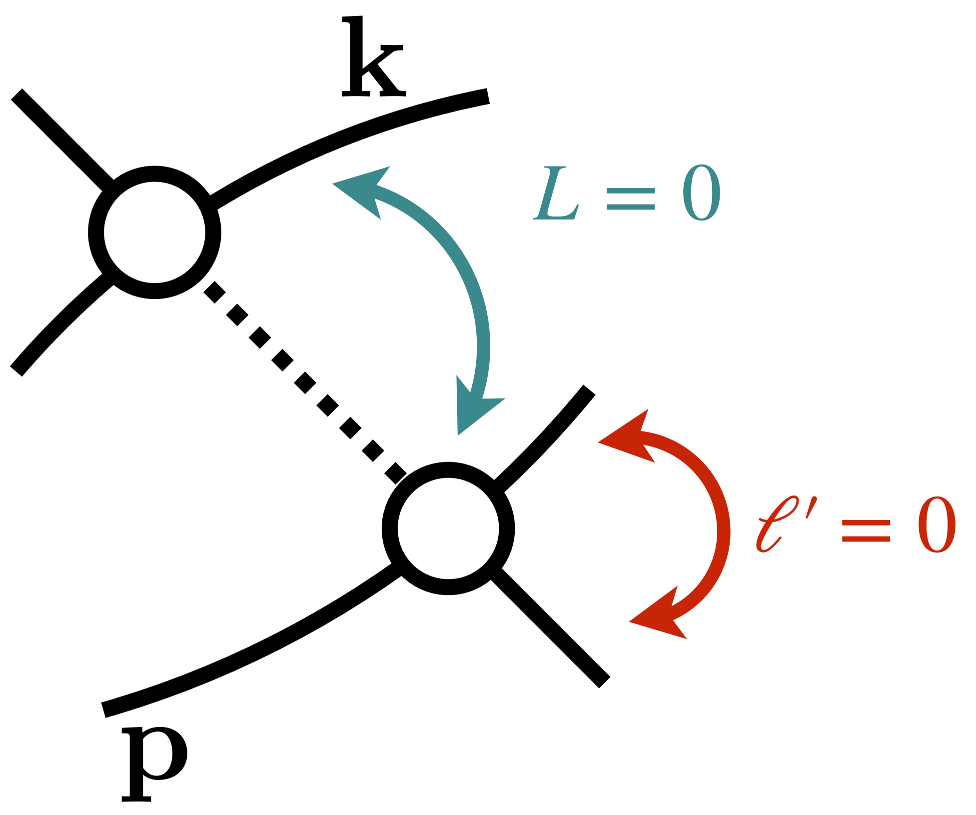

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$$z(p, k) = \frac{(E - \omega_k - \omega_p)^2 - k^2 - p^2 - m^2}{2pk}$$

Cut parametrization: $(E - \omega_k - \omega_p) - k^2 - p^2 - m^2 - 2pkx = 0$

$x = [-1, 1]$

Integral equations

We started with:

$$d(\mathbf{p}', s, \mathbf{p}) = -G(\mathbf{p}', s, \mathbf{p}) - \int_0^{q_{\max}} \frac{d^3\mathbf{q}}{(2\pi)^3 2\omega_q} G(\mathbf{p}', s, \mathbf{q}) \mathcal{M}_2(q, s)\, d(\mathbf{q}, s, \mathbf{p})$$

After partial wave projection, we get:

$$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \underbrace{\frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)}_{\text{“integration kernel”}}\, d(q, s, p)$$

We've now addressed two of three parts:

- [x] 3D integral equation,
- [x] need to project to **angular momentum and parity**,
- [ ] integration kernel is generally singular.

[the singularities of the kernel and the 3body amplitude depend on the two-particle scattering amplitude]

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A very interesting example

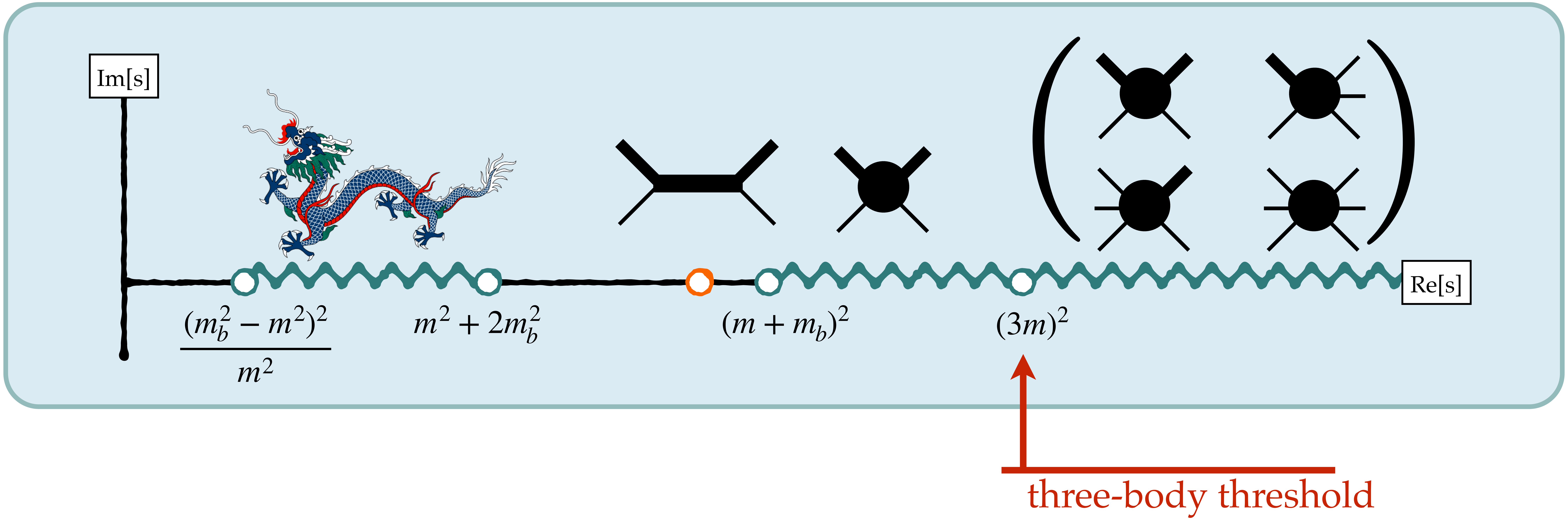
Consider a theory with a two-body bound state:

- Arguably the most singular example,
- testing formalism,
- exploring Efimov physics,
- towards nuclear physics,
- …

Can get a bound state using the effective range expansion at leading order:

- $\mathcal{M}_2(s) \sim \dfrac{1}{q\cot\delta(s) - iq} = \dfrac{1}{-\frac{1}{a} - iq}$
- if $a > 0$, can have a pole at $q = i\kappa = i/a$,
- bound-state mass $m_b = 2\sqrt{m^2 - 1/a^2}$,

The singularity landscape



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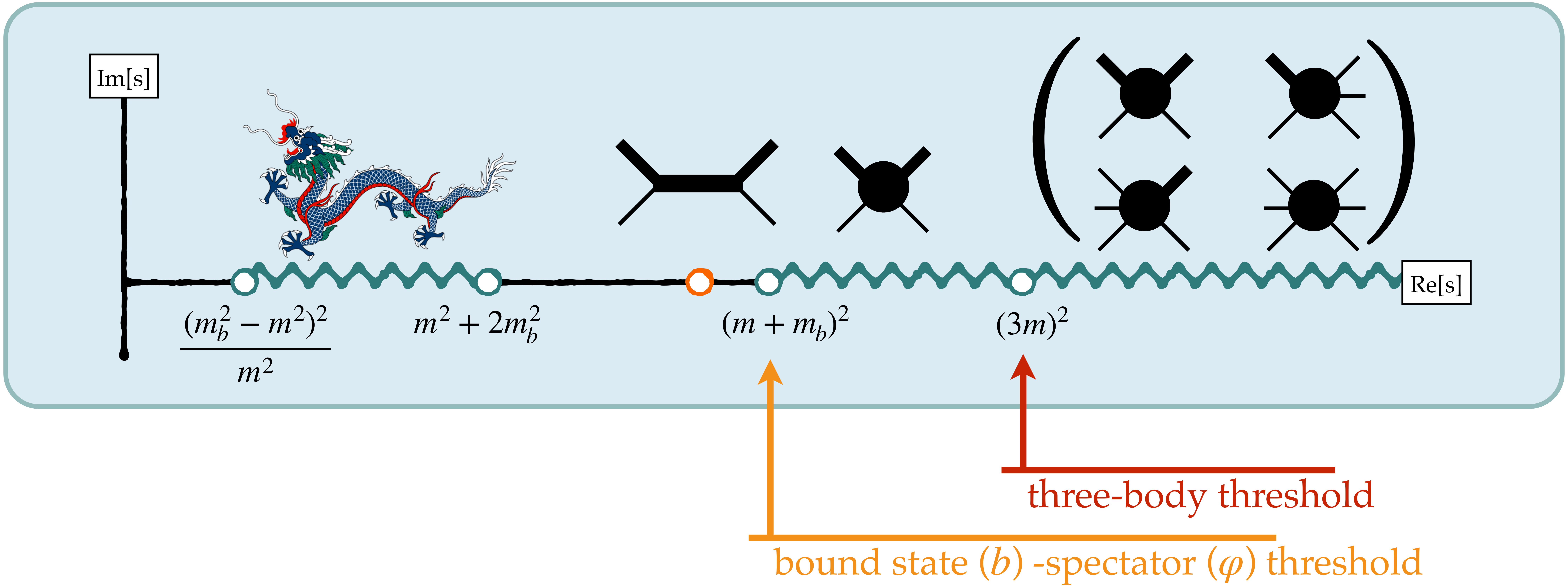

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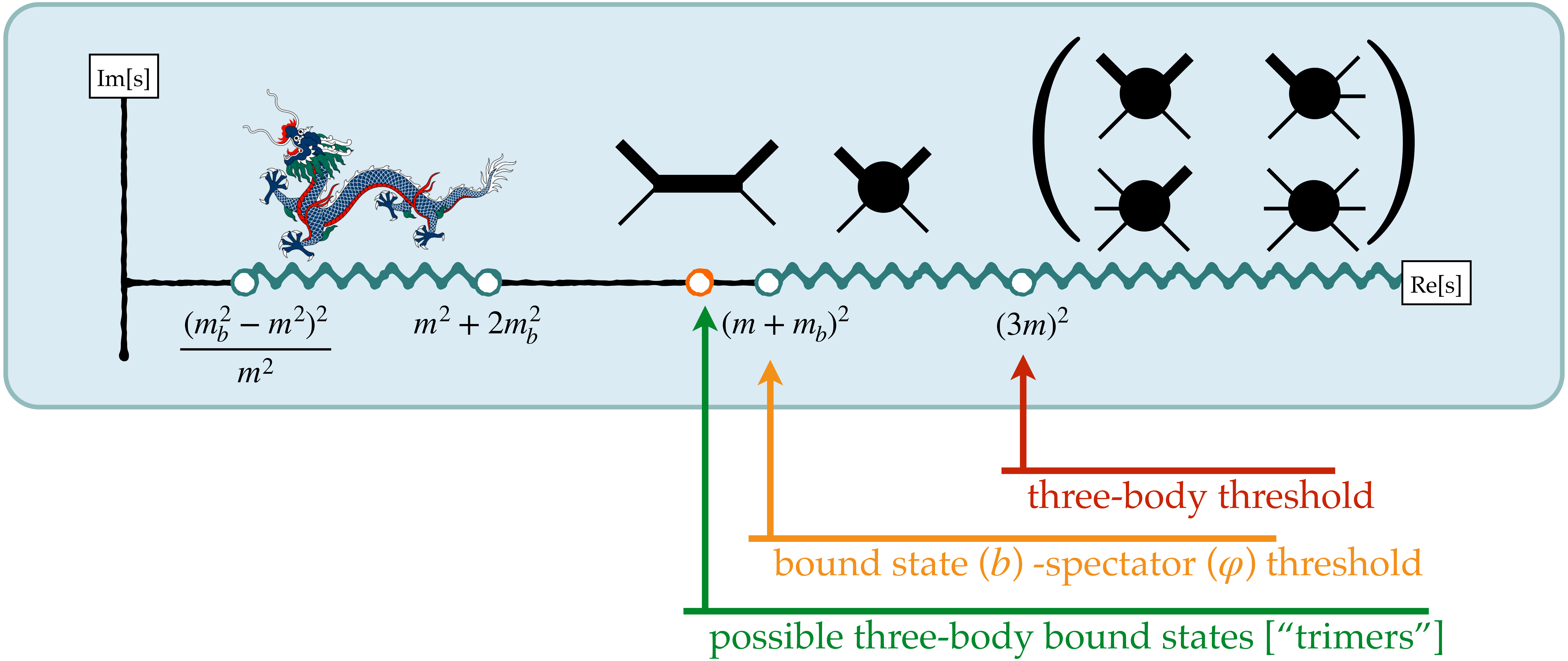

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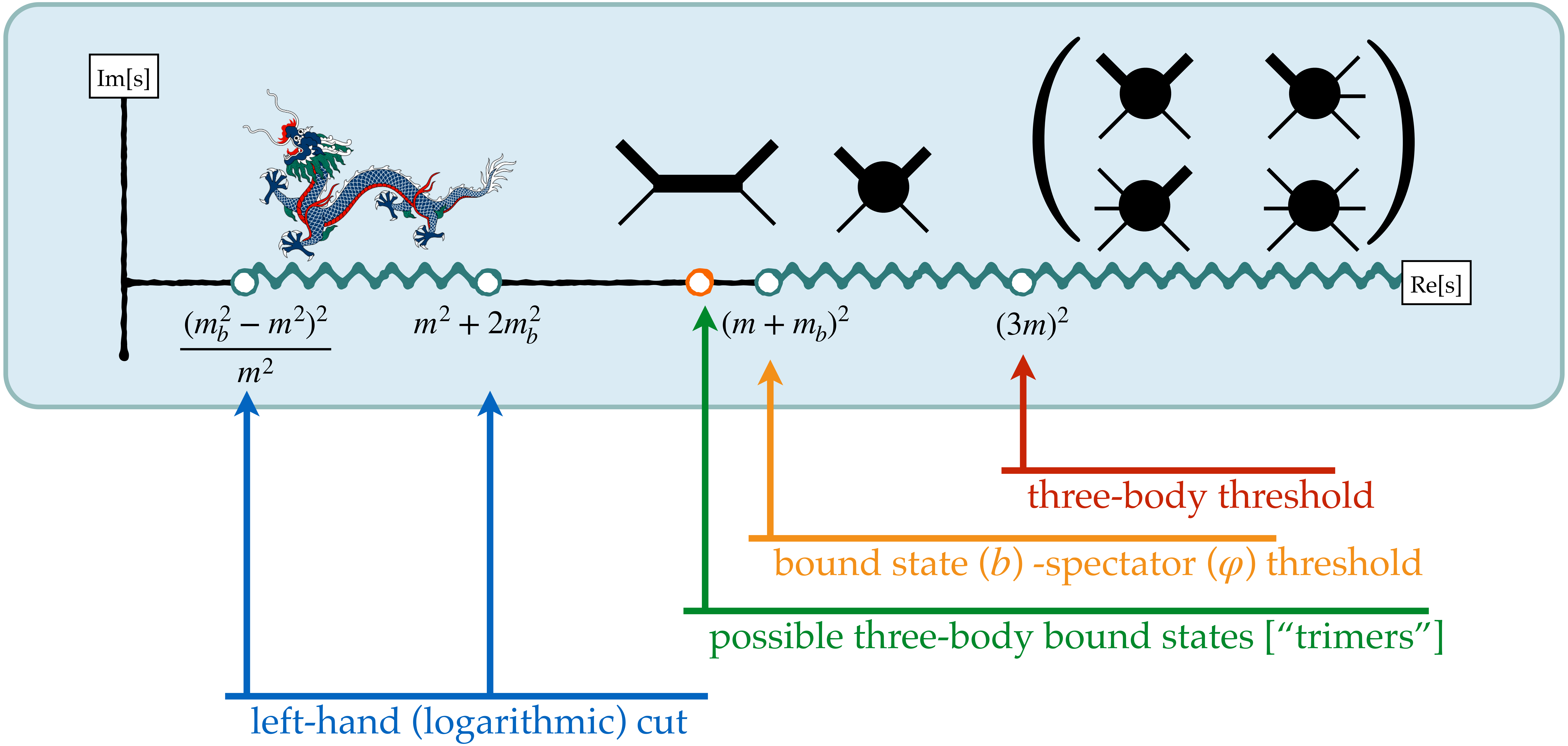

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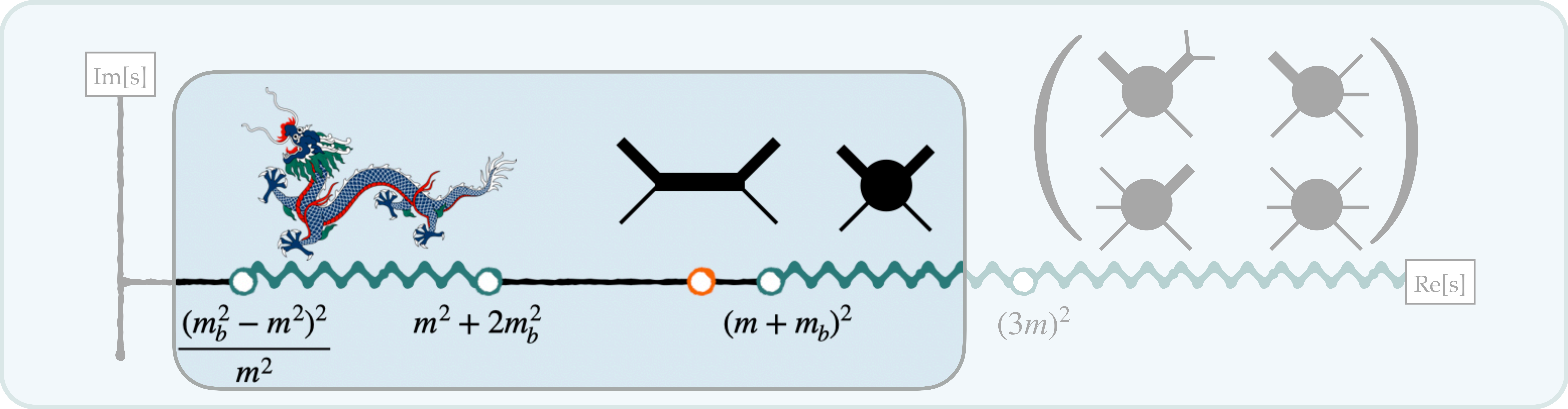


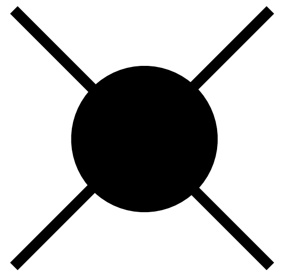
three-body / two-body duality

We must be able to describe this system as both a three-body and as a two-body system below the three-body threshold.

Obtaining the $b + \varphi$ amplitude

We can obtaine the $\mathcal{M}_{\varphi b}$ amplitudes using LSZ:

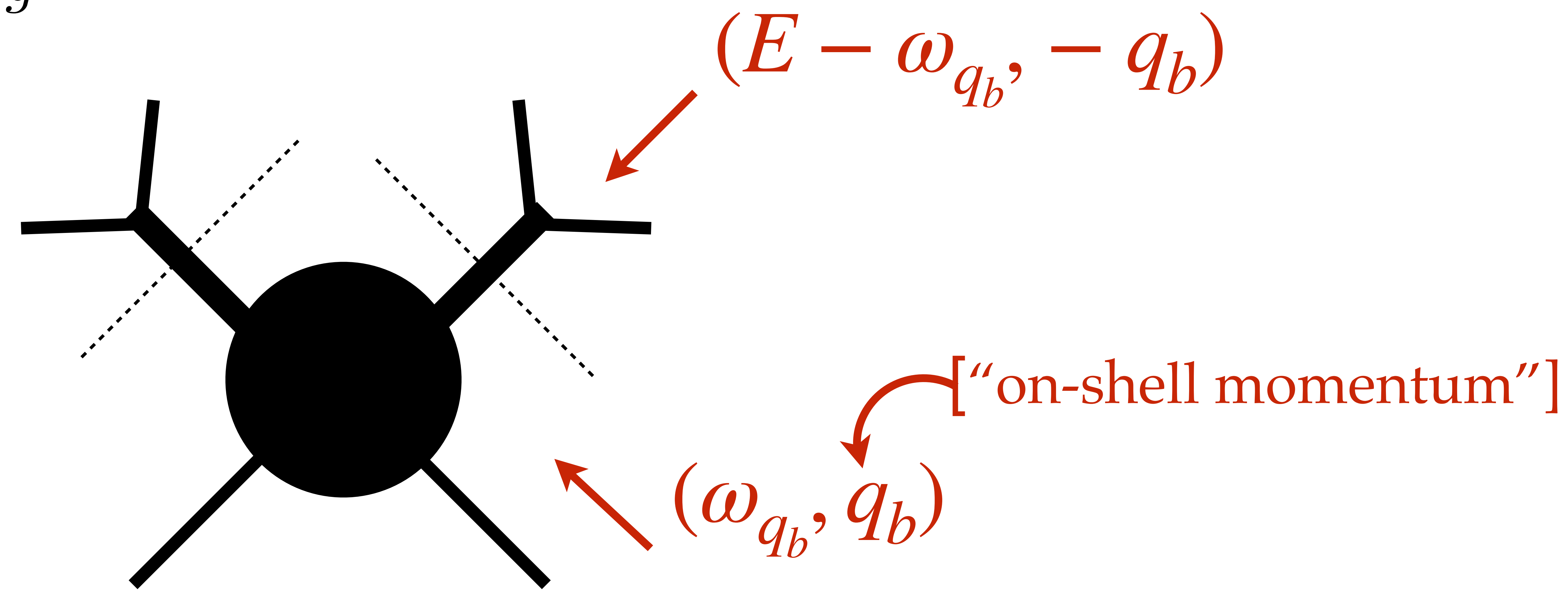
Two-body bound state:

$\mathcal{M}_2(s) =$  $\sim$  $\sim \dfrac{-g^2}{s - m_b^2}$

Bound state / spectator scattering amplitude

$$\mathcal{M}_{\varphi b}(s) = \frac{1}{\mathcal{K}_{\varphi b}^{-1}(s) - i\rho_{\varphi b}} = \lim_{\sigma_k, \sigma_p \to m_b^2} \mathcal{D}(s,k,p) \frac{(\sigma_k - m_b^2)(\sigma_p - m_b^2)}{g^2}$$

$$= \lim_{\sigma_k, \sigma_p \to m_b^2} d(s,k,p) g^2$$



Solving integral equations

- Deform contour to miss singularities and discretize momenta
  - sometimes useful / / sometimes critical
- Discretize momenta: $$d(p',s,p) = -G(p',s,p) - \int_0^{q_{\rm max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p',s,q)\mathcal{M}_2(q,s)\, d(q,s,p)$$
$$\approx -G(p',s,p) - \sum_{q=0}^{q_{\rm max}} K(p',s,q)\, d(q,s,p)$$

[**contains pole, logarithmic** and **square root cuts**]

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$$[1 + \boldsymbol{K}] \cdot \vec{d}_{\boldsymbol{s}ol}(s, p) = -\vec{G}(s, p)$$

- Use integral equation to interpolate or extrapolate:

$$d(p', s, p) \approx -G(p', s, p) - \vec{K}(p', s) \cdot \vec{d}_{\rm sol}(s, p)$$

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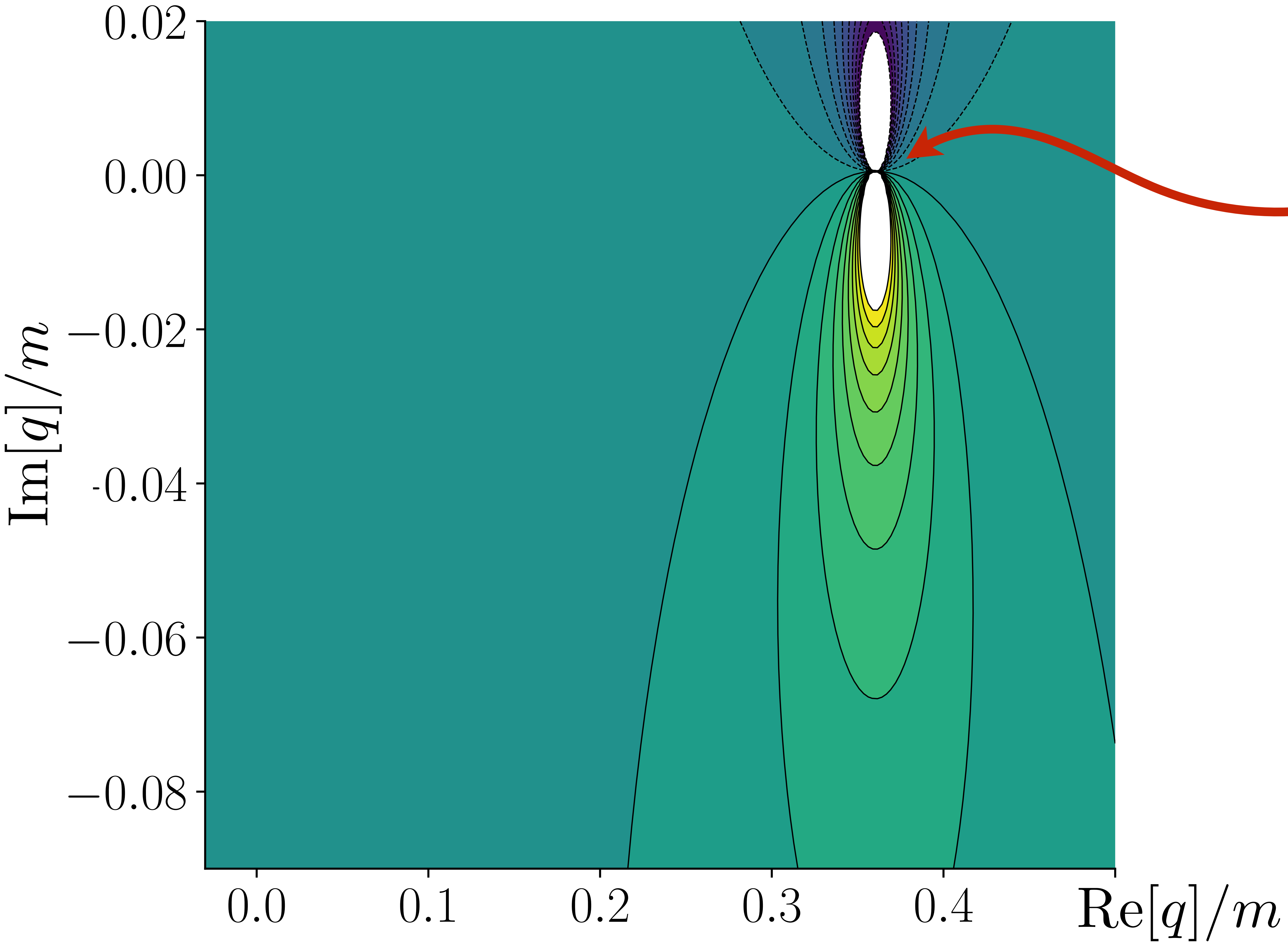
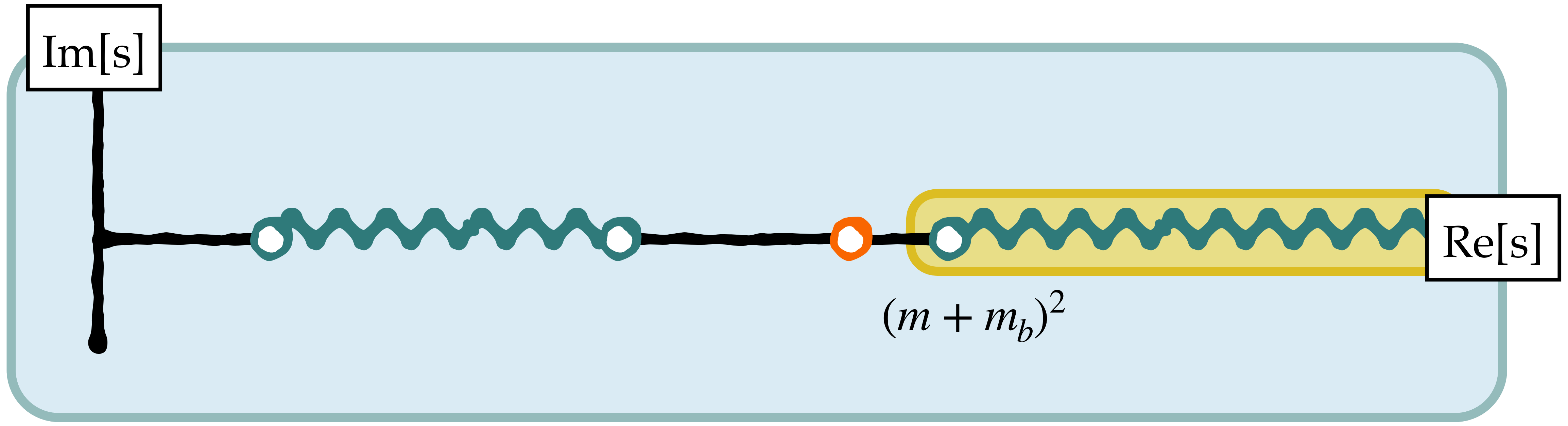
convergence tests

Contour plot of $\mathrm{Im}\mathcal{M}_2((P-q)^2)$



[pole in the two-body amplitude as a function of the spectator]



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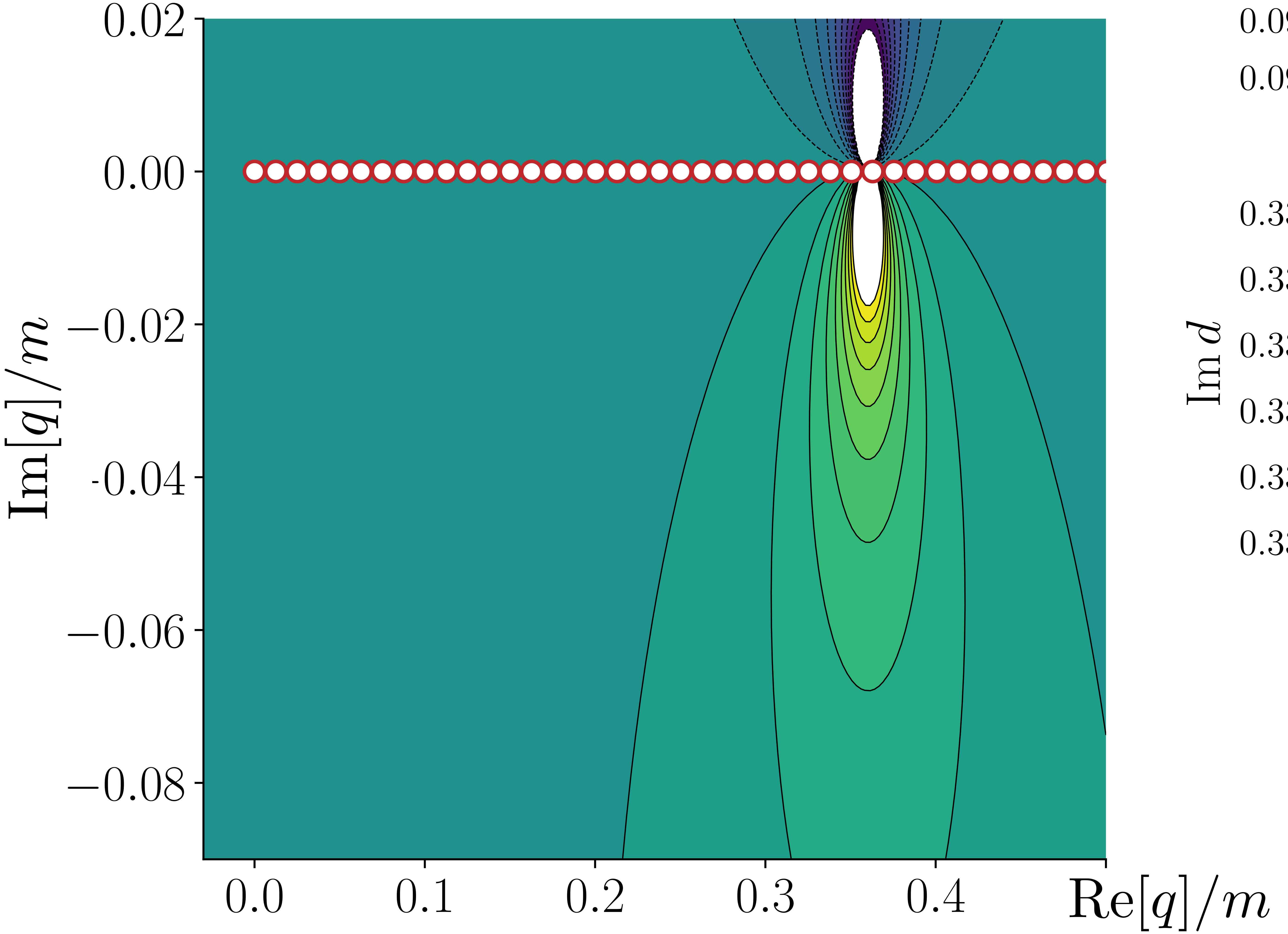

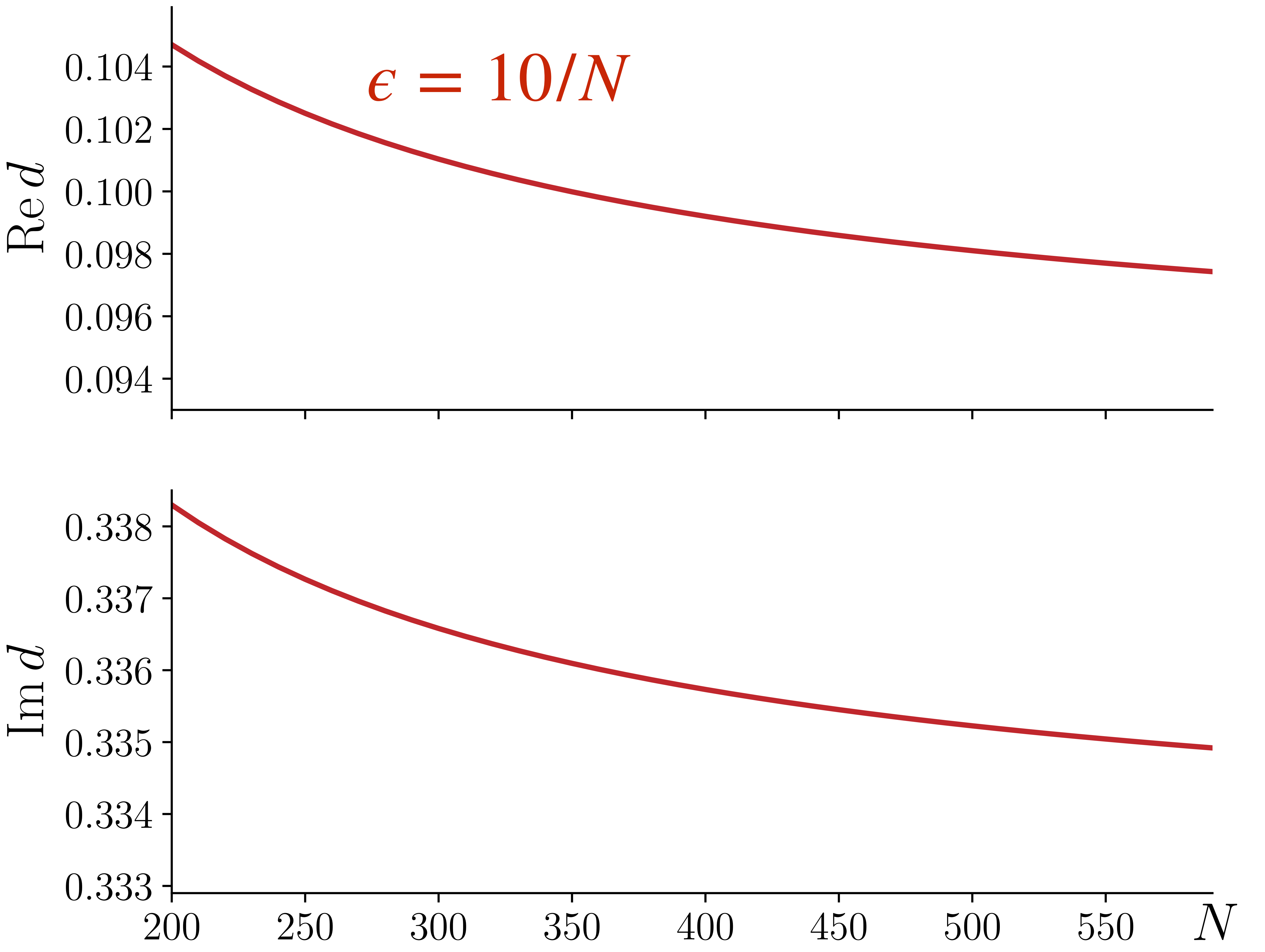

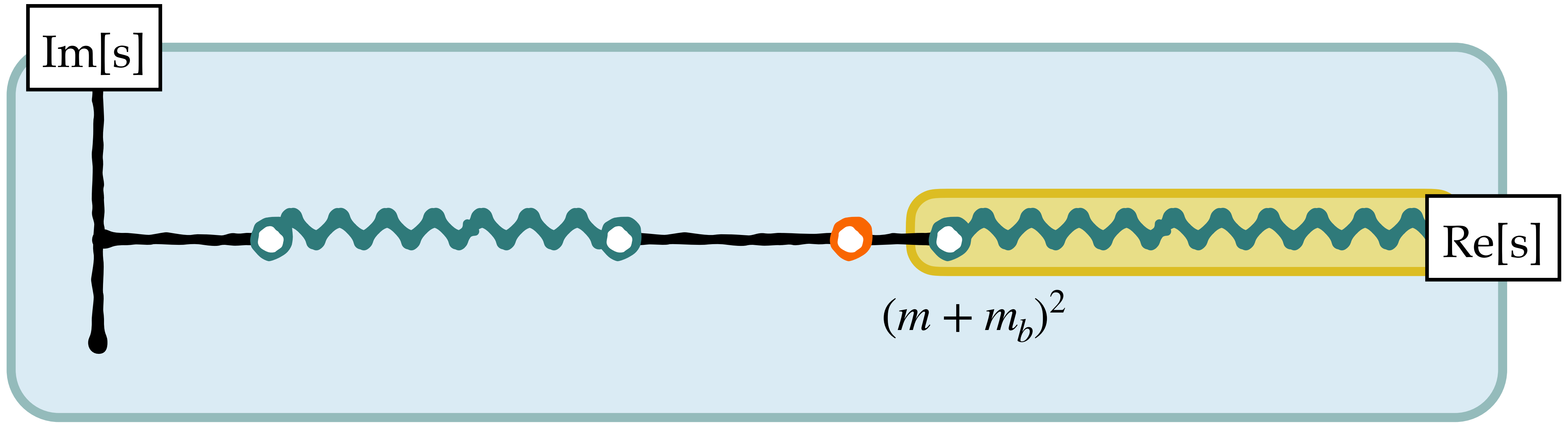

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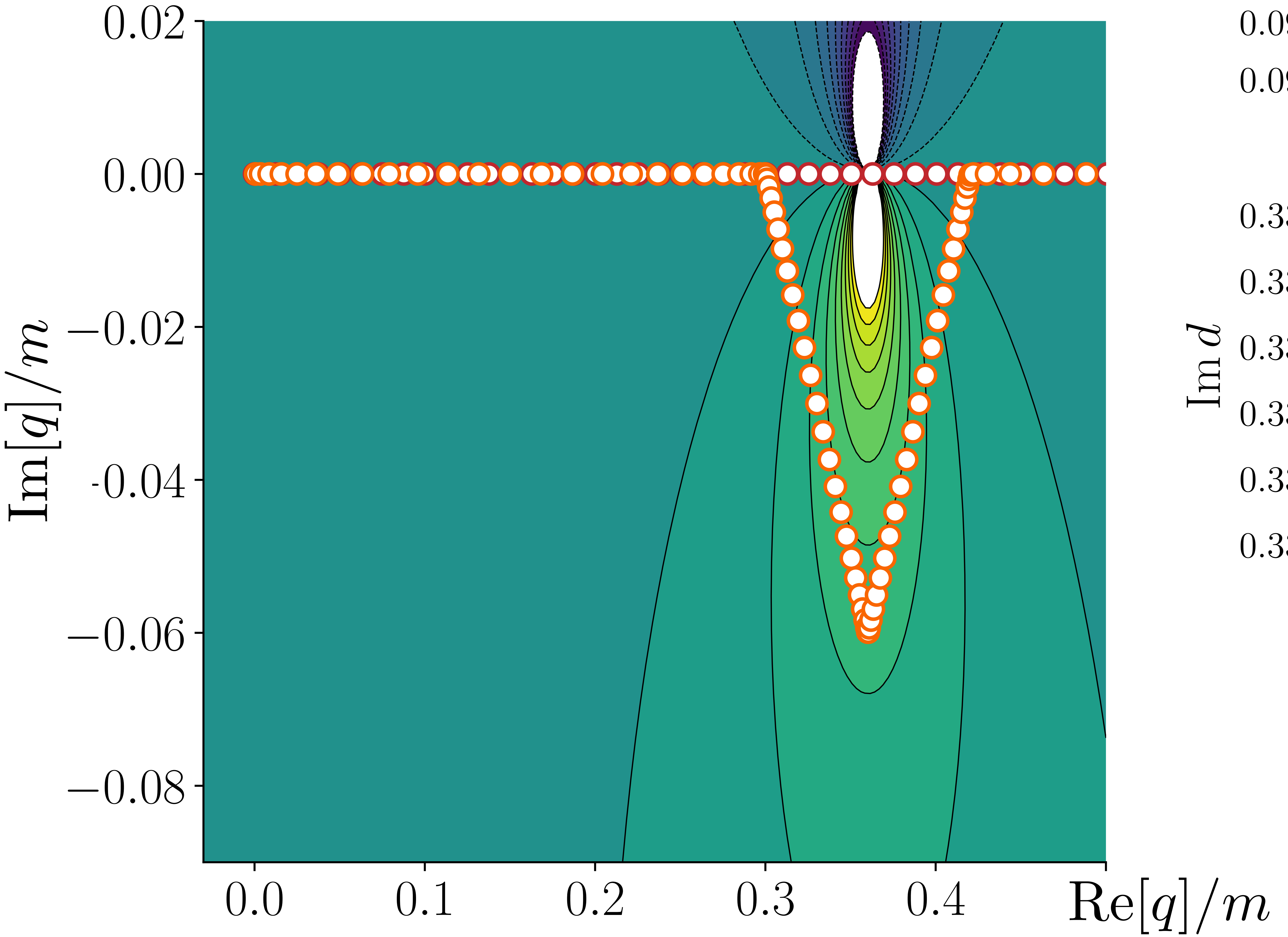
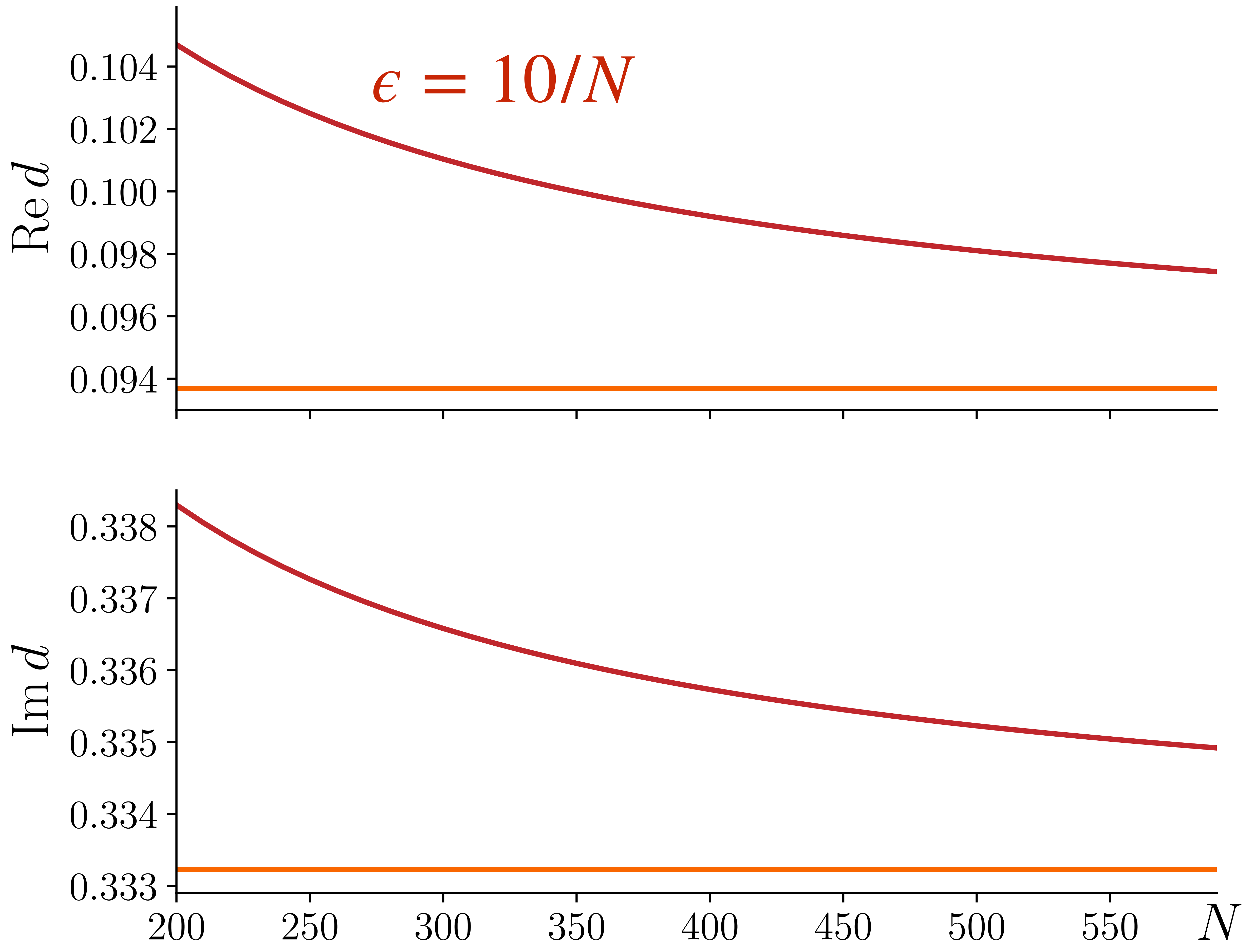
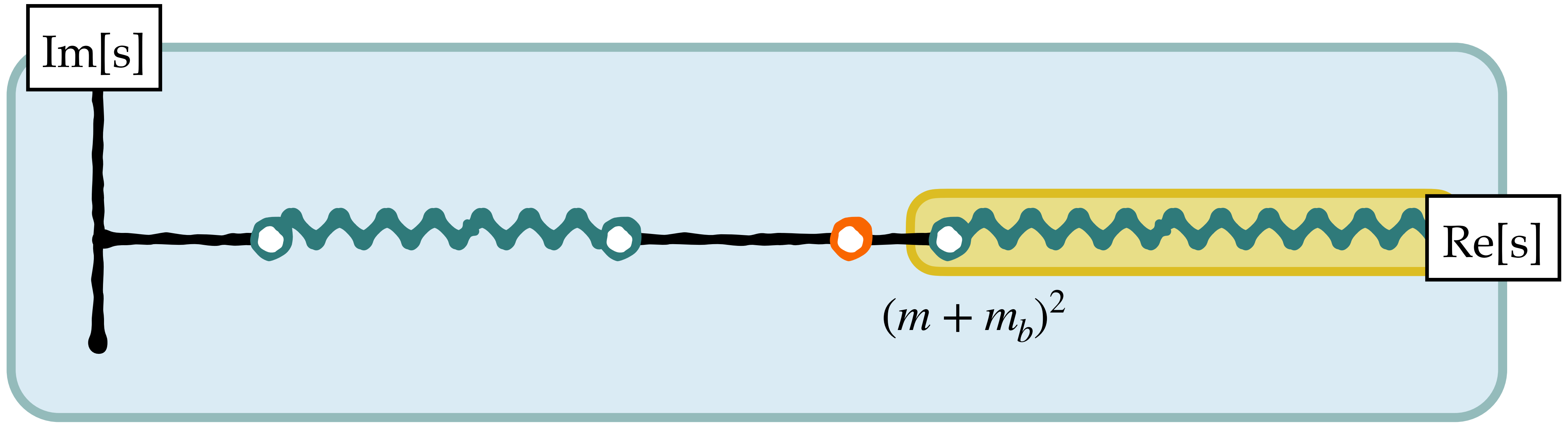
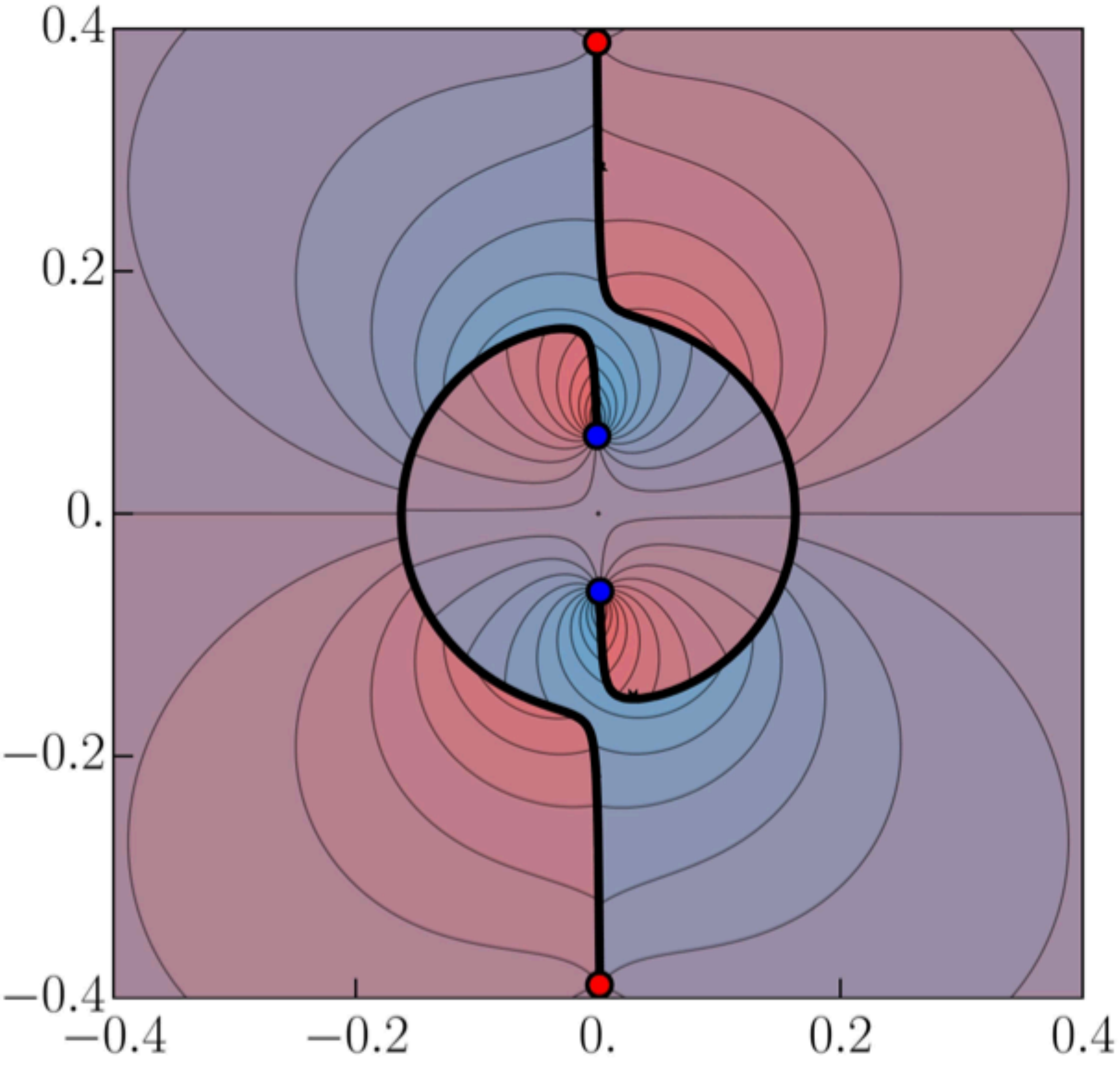
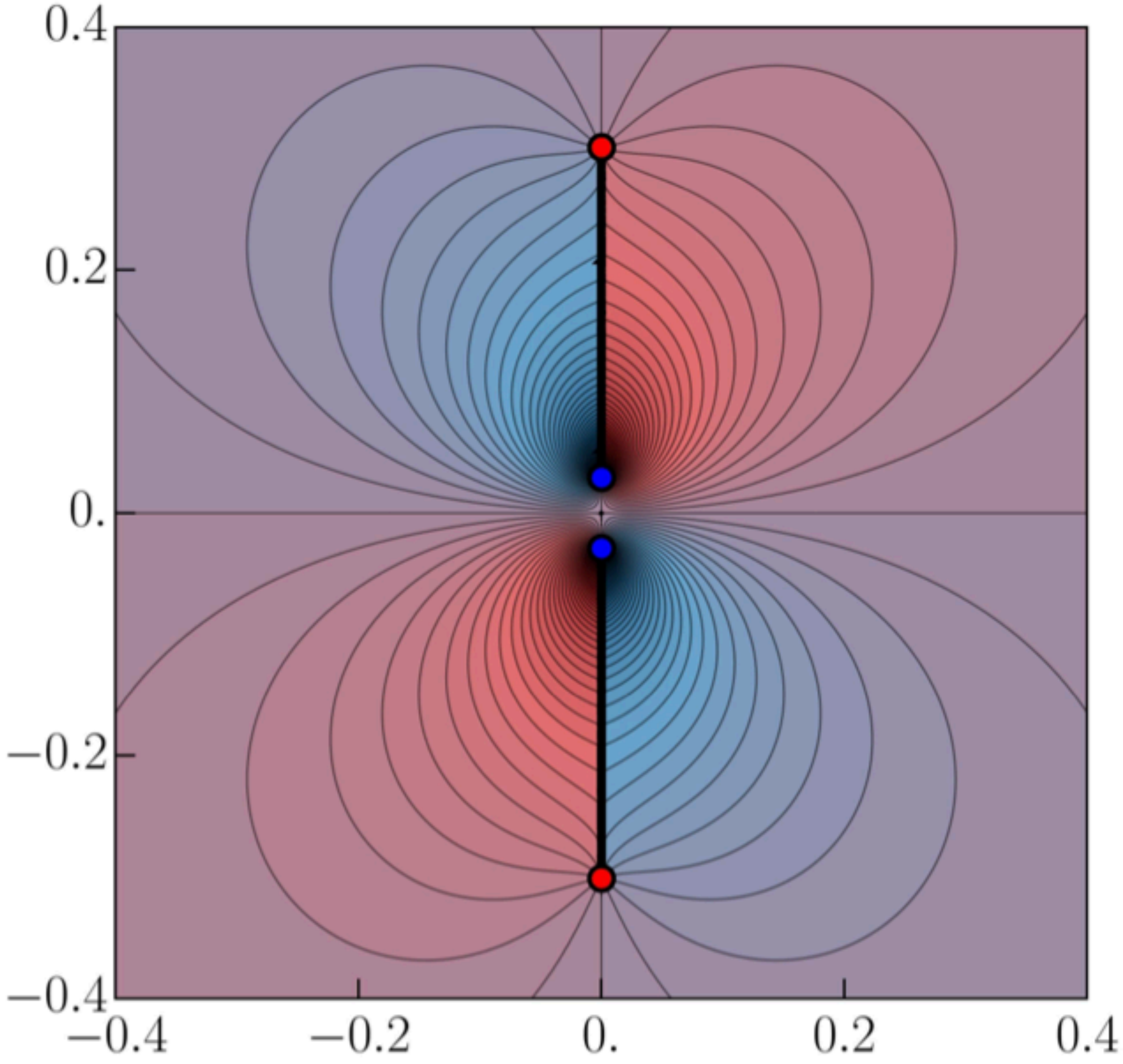


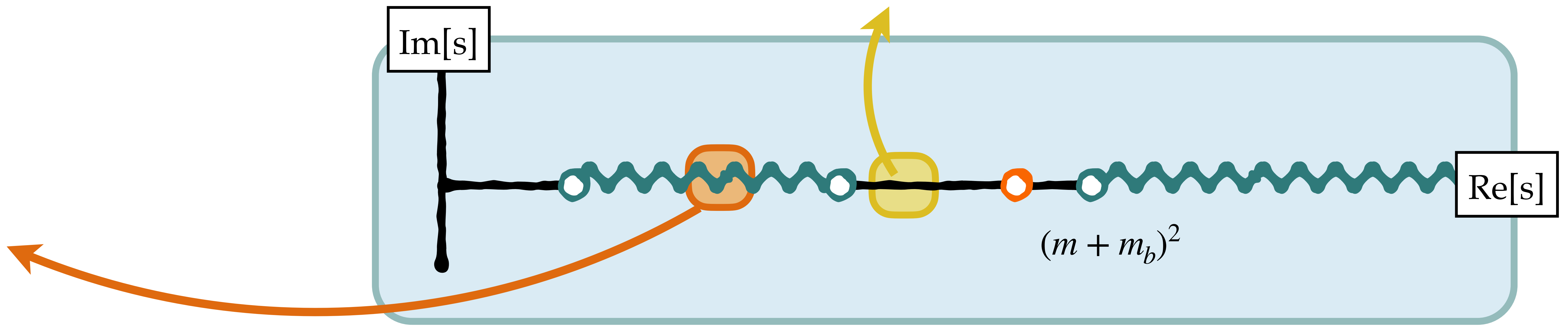




Logarithmic cut







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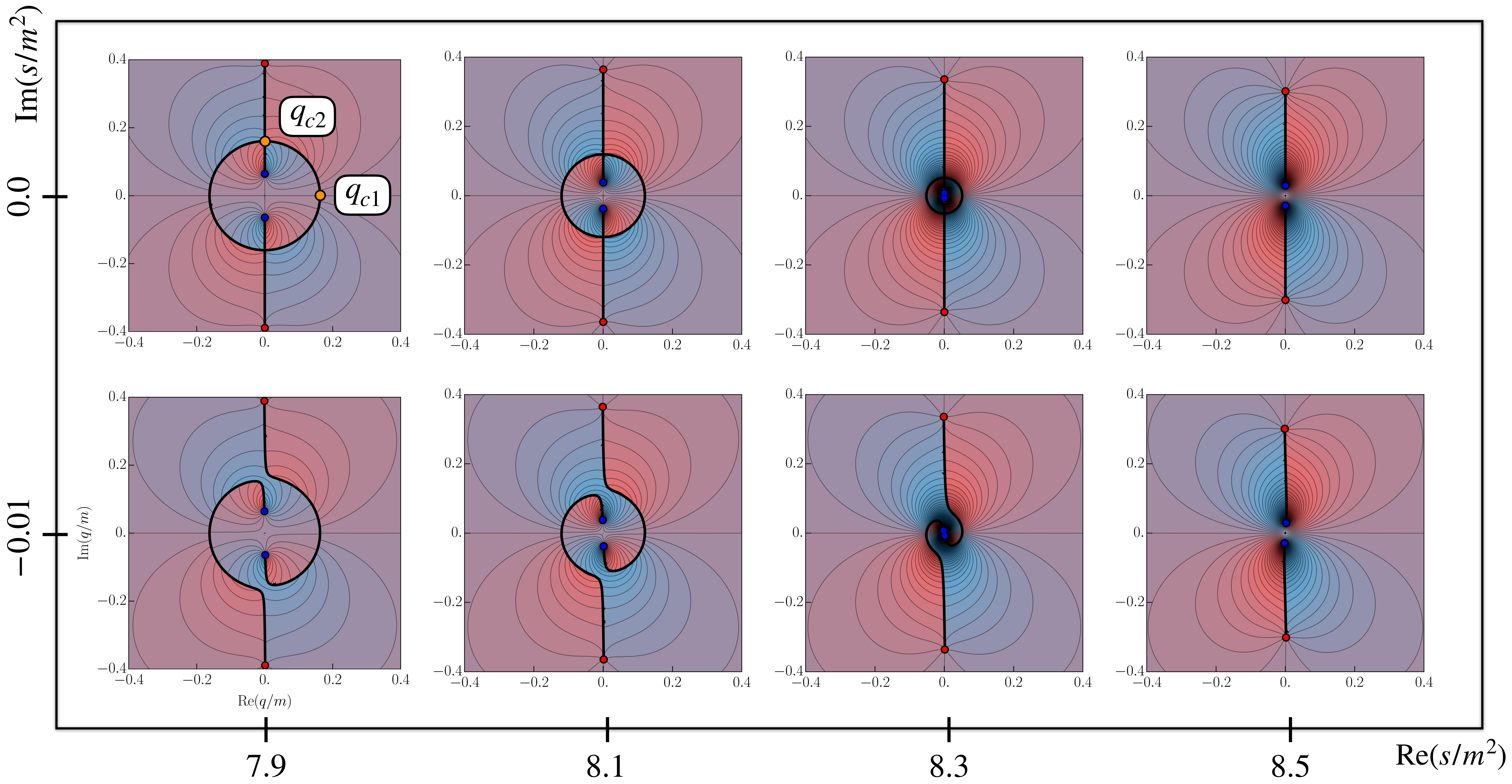
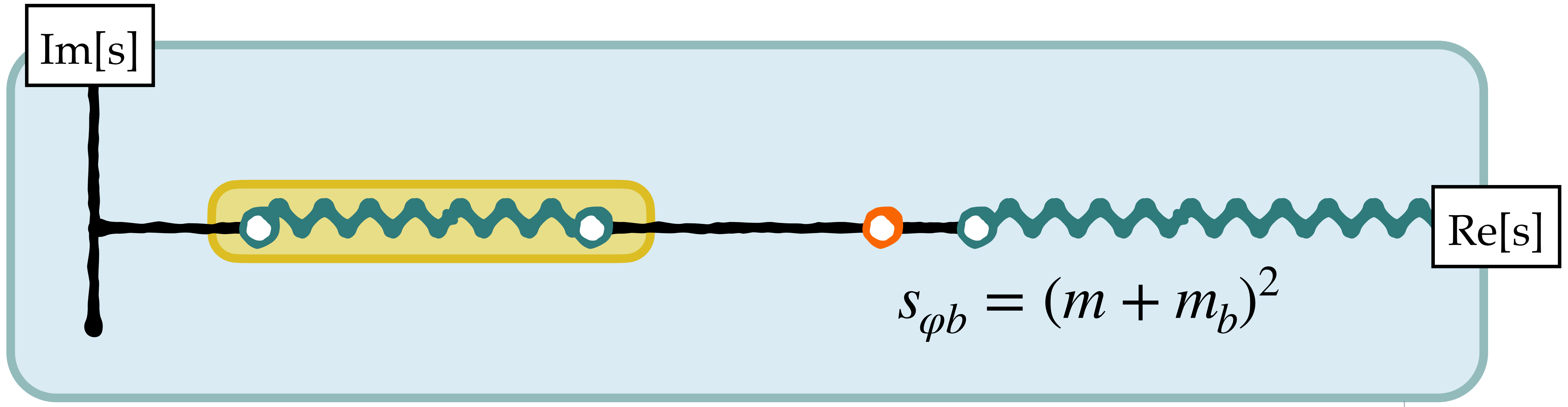
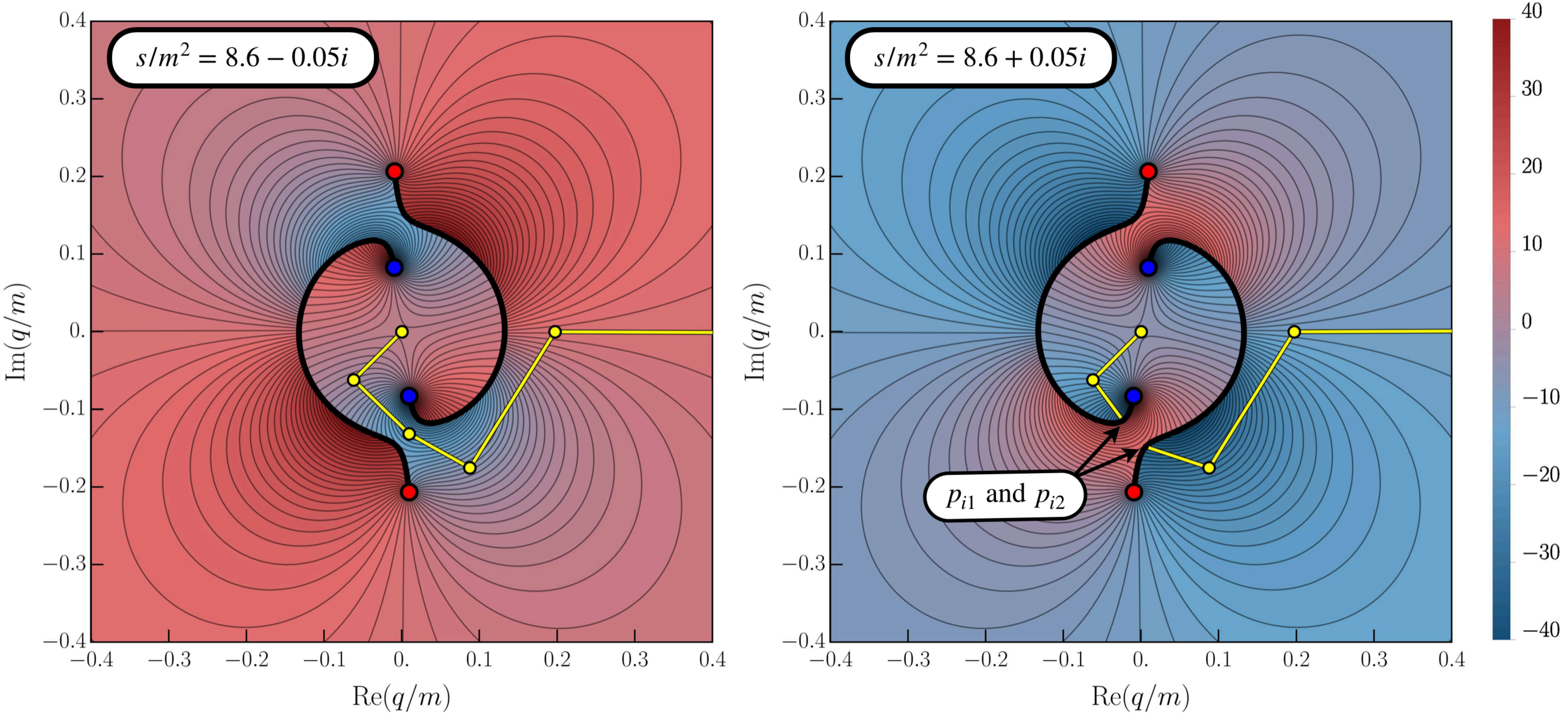
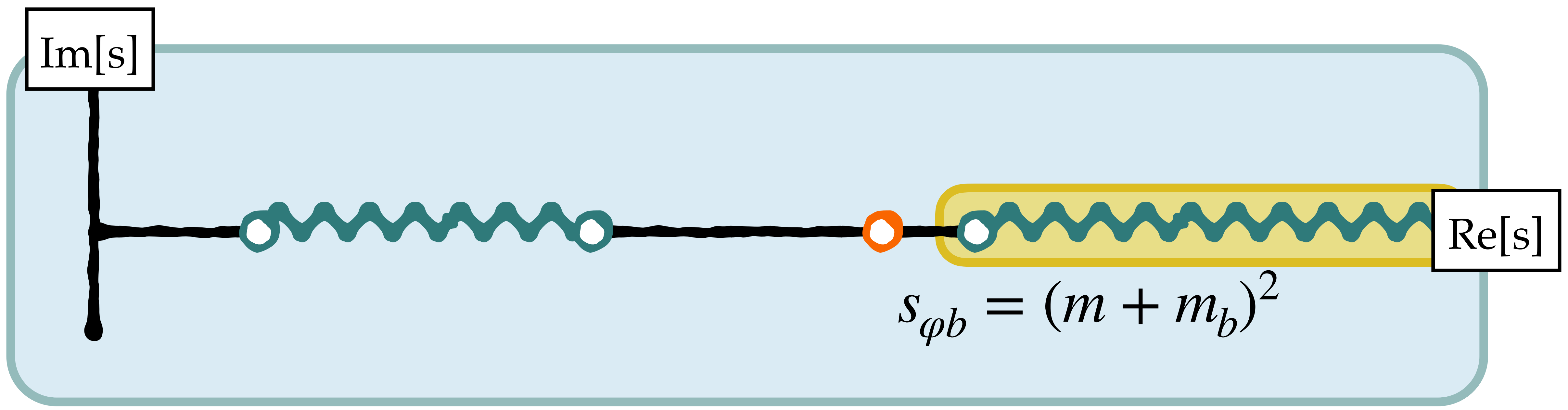


Logarithmic cut

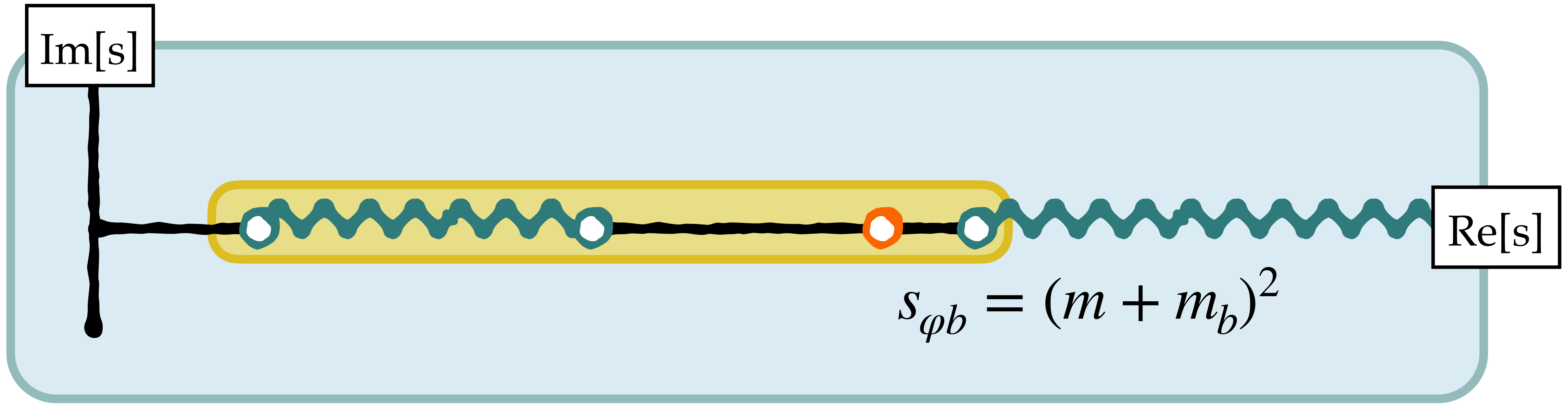


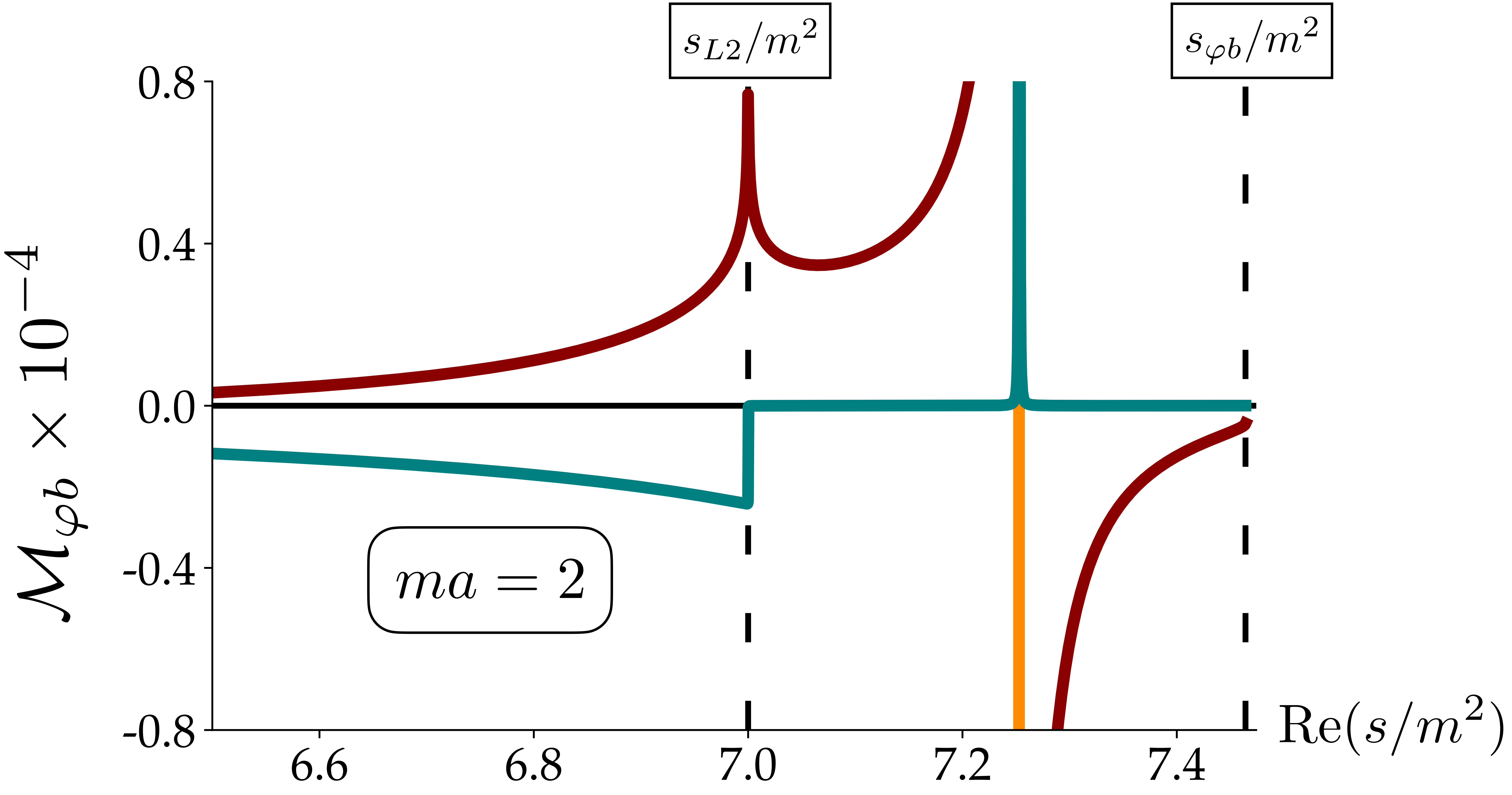


Some amplitude results



Some amplitude results





outline

- [x] an example of the full procedure being done
- [x] integral equations
- [x] angular momentum projection
- [x] toy model calculations
- [ ] consistency checks and the breakdown of Lüscher
- [ ] Efimov physics

Romero-López, Sharpe, Blanton, RB, & Hansen (2019)
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Jackura, RB (to appear)
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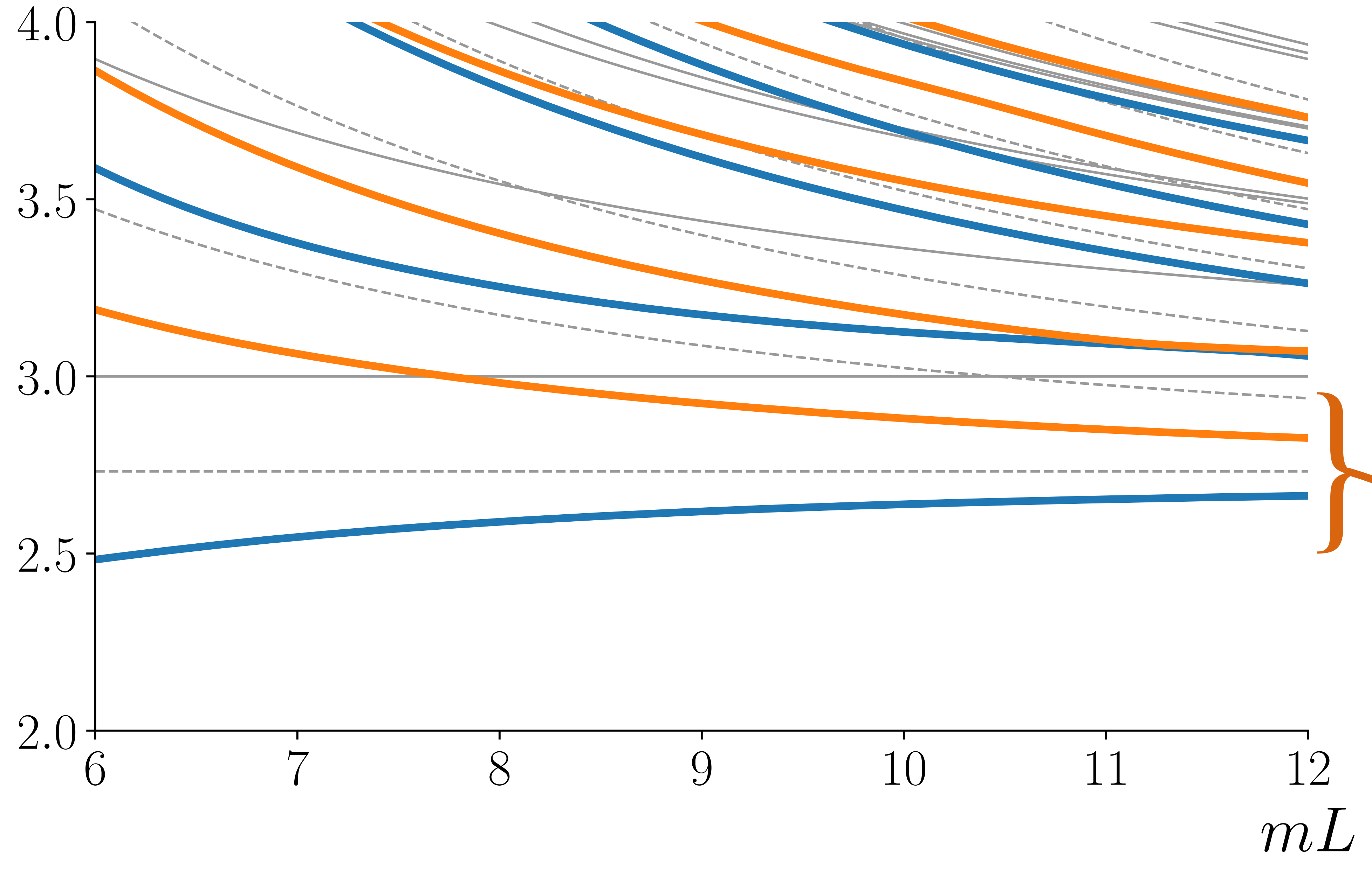
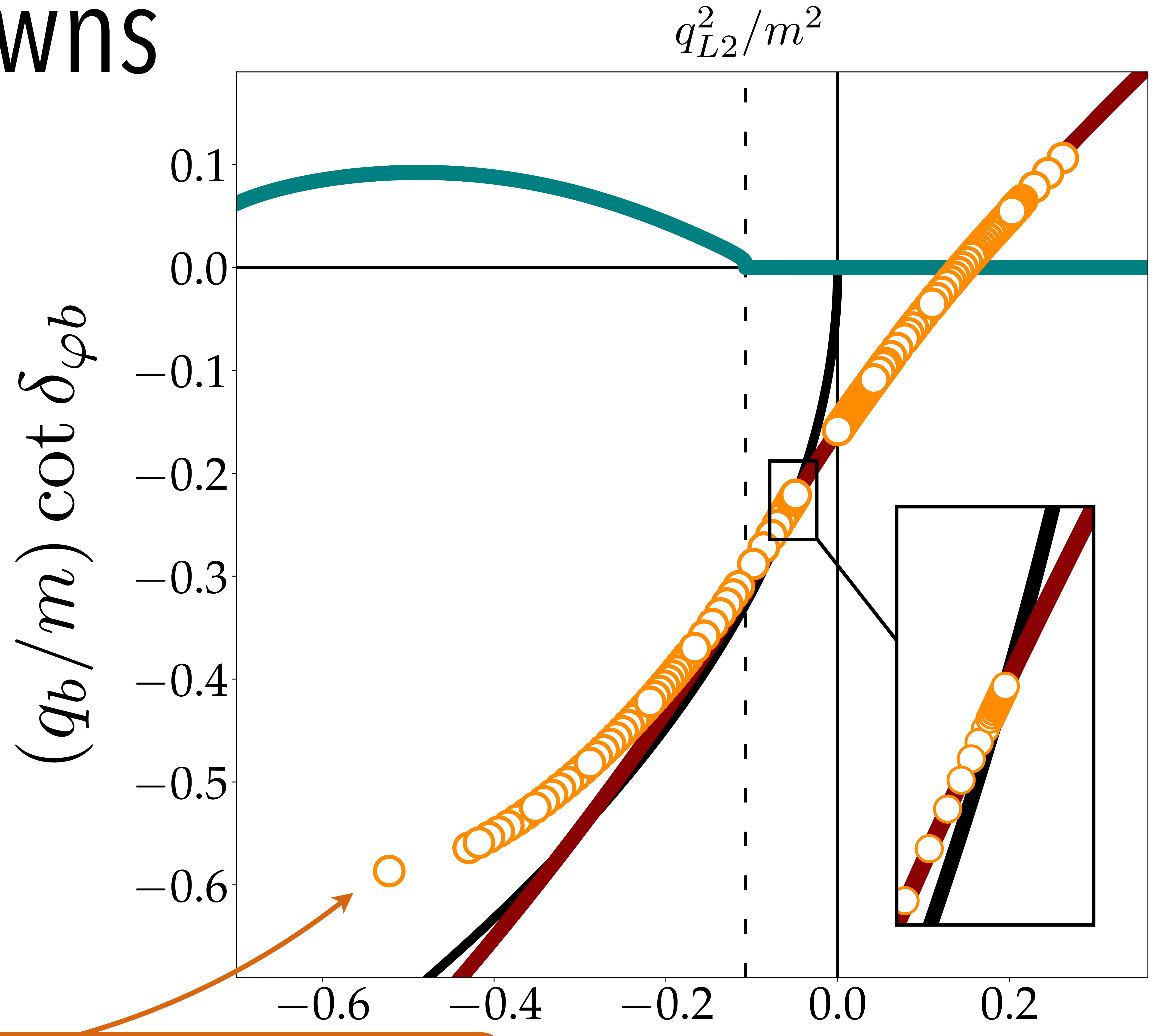
Consistencies and breakdowns

Hansen & Sharpe (2014)

$F_3^{-1}(E_L, L) = 0$



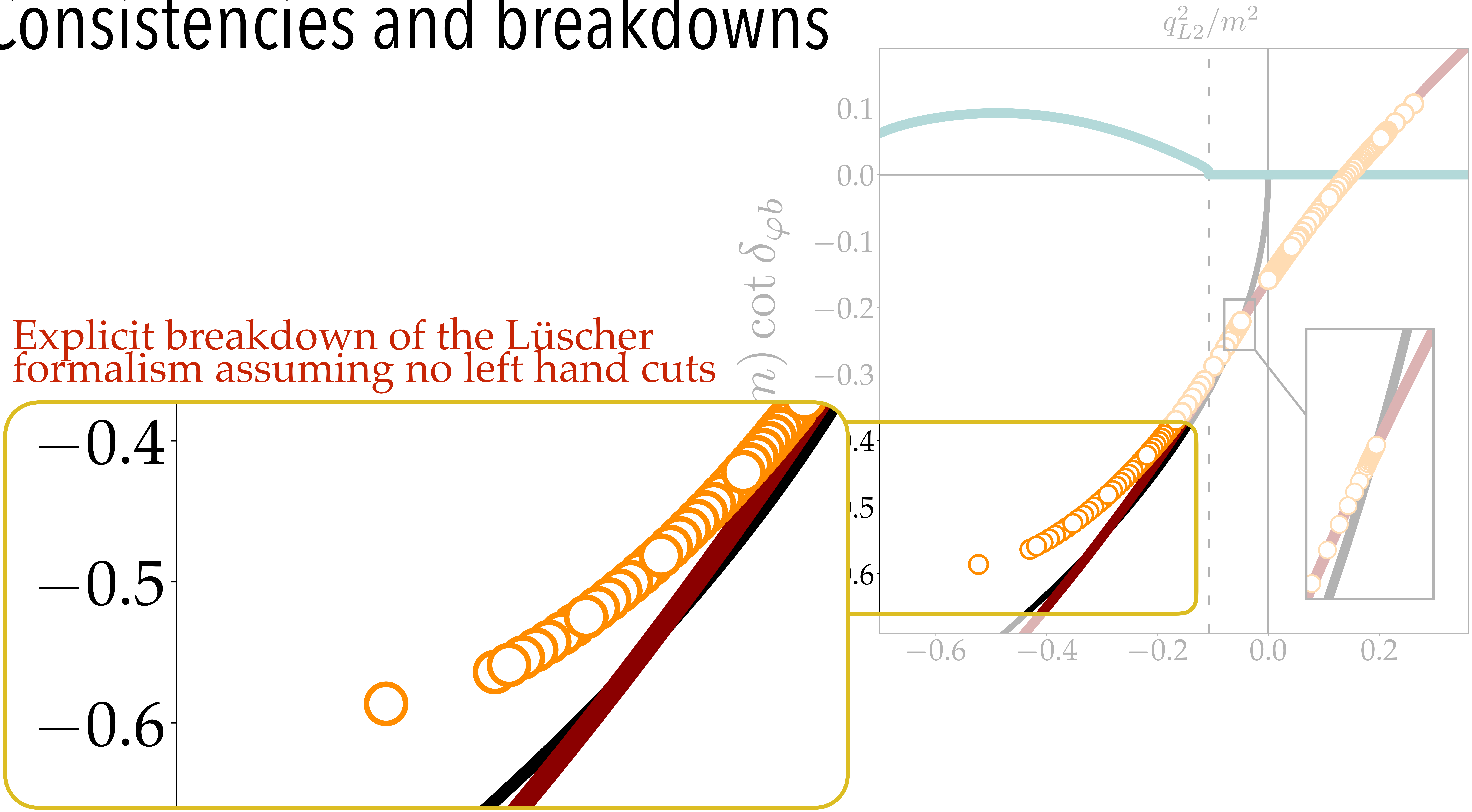


$F(E_L, L) + \mathcal{M}^{-1}(E_L) = 0$

Dawid, Islam, & RB (2023)
Jackura, RB, Dawid, Islam, & McCarty (2020)
Romero-López, Sharpe, Blanton, RB, & Hansen (2019)

Consistencies and breakdowns

Explicit breakdown of the Lüscher formalism assuming no left hand cuts



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Efimov physics

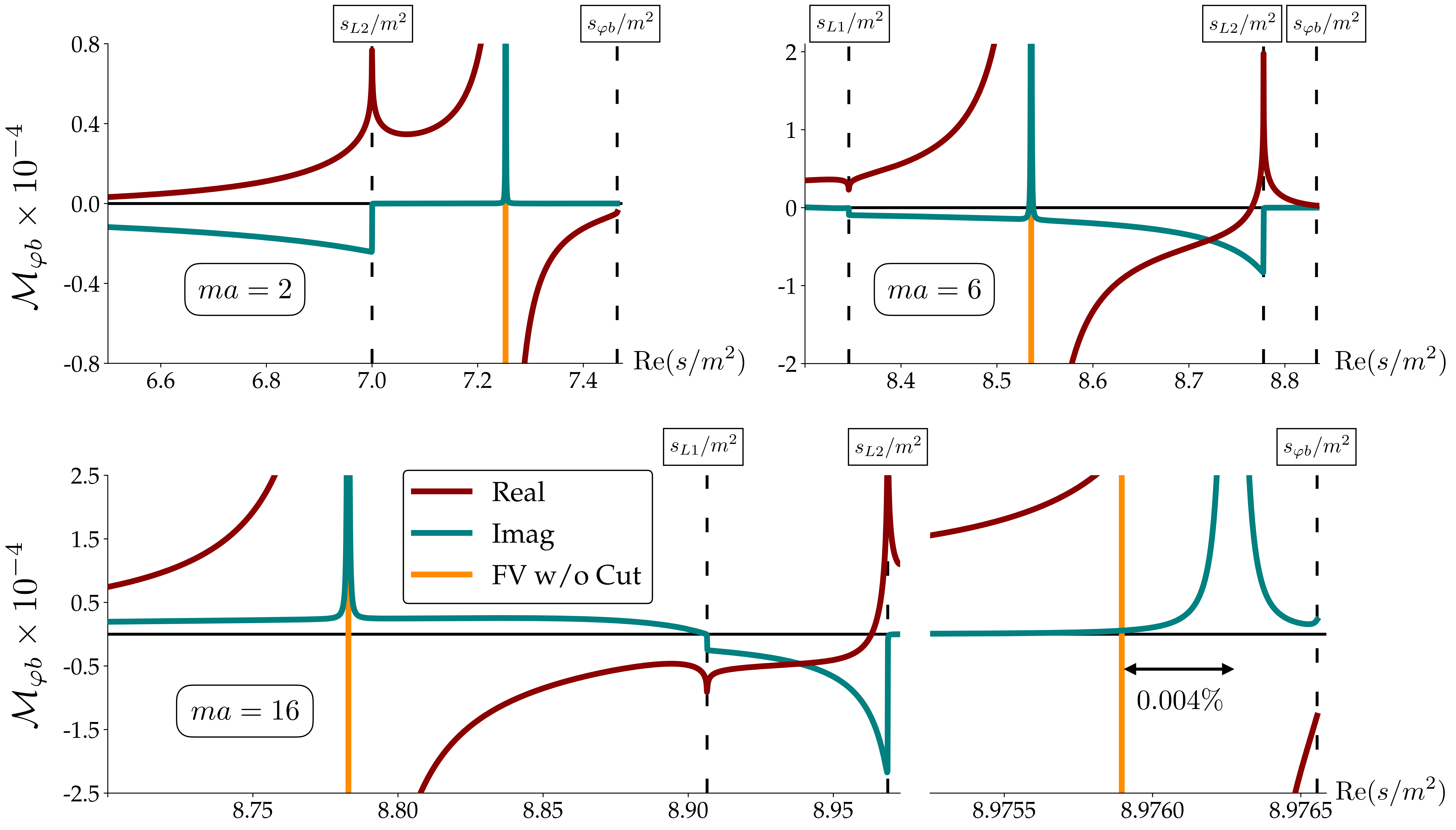
Unitary limit: $a \to \infty$

Pole in the two-body scattering amplitude at threshold: $\mathcal{M} \sim \dfrac{1}{p\cot\delta - ip} = \dfrac{1}{ip}$

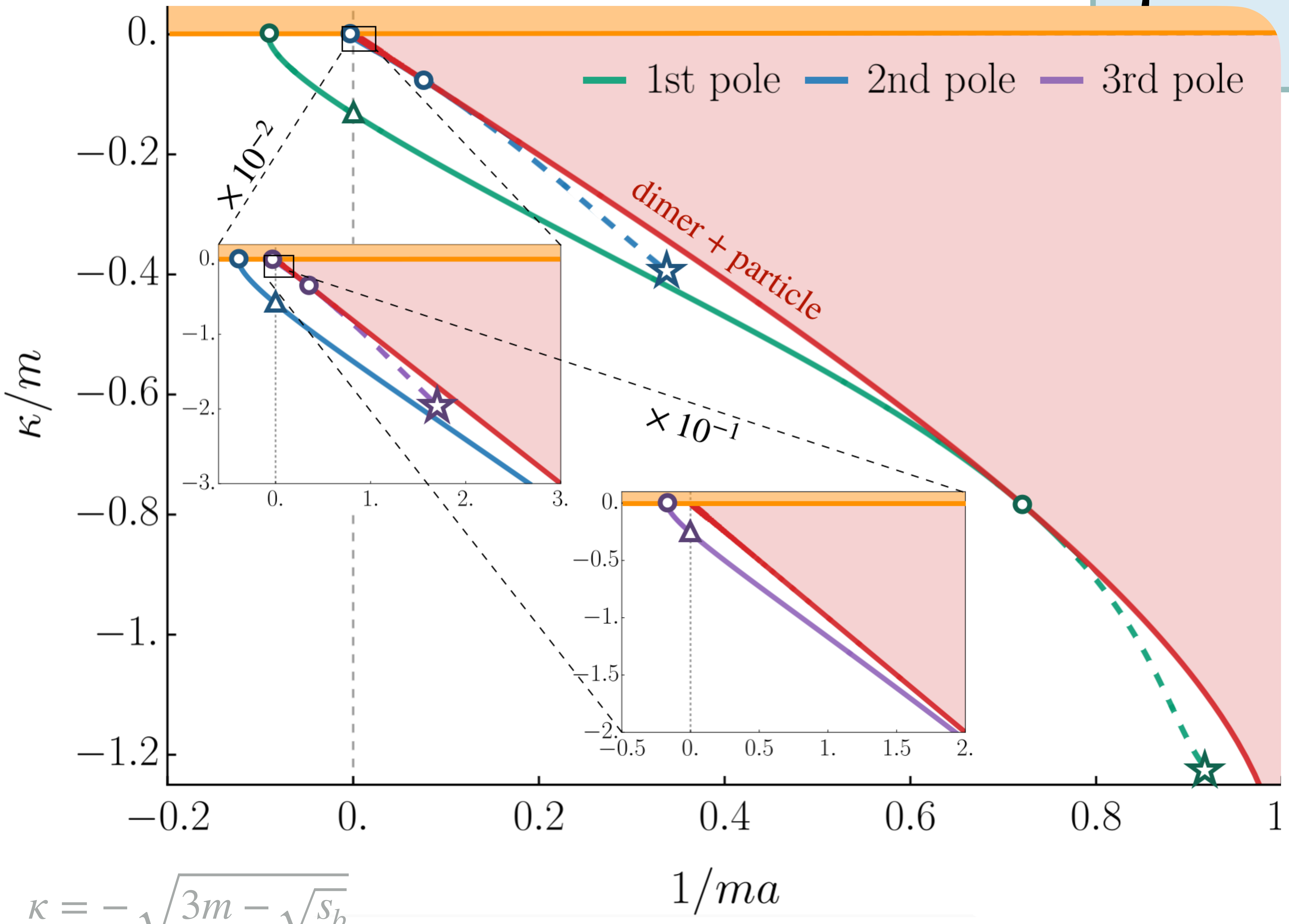
Infinite tower of geometrically-separated three-body bound states: $E_N / E_{N+1} = \lambda^2$ where $\lambda = 22.69438...$

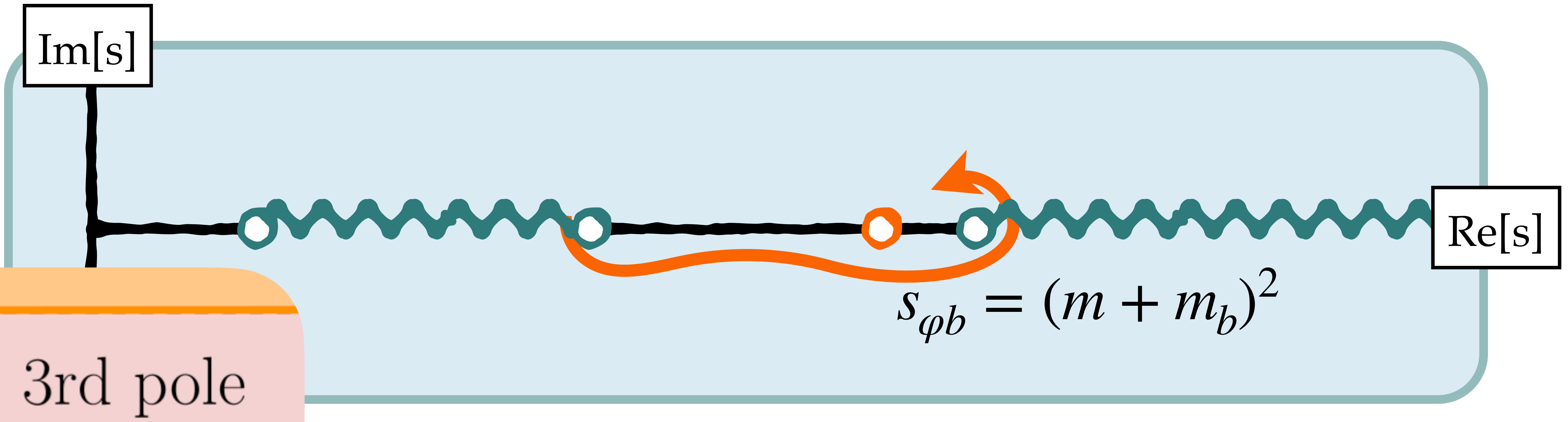
Vitaly Efimov

increasing a



the *Efimov evolution*

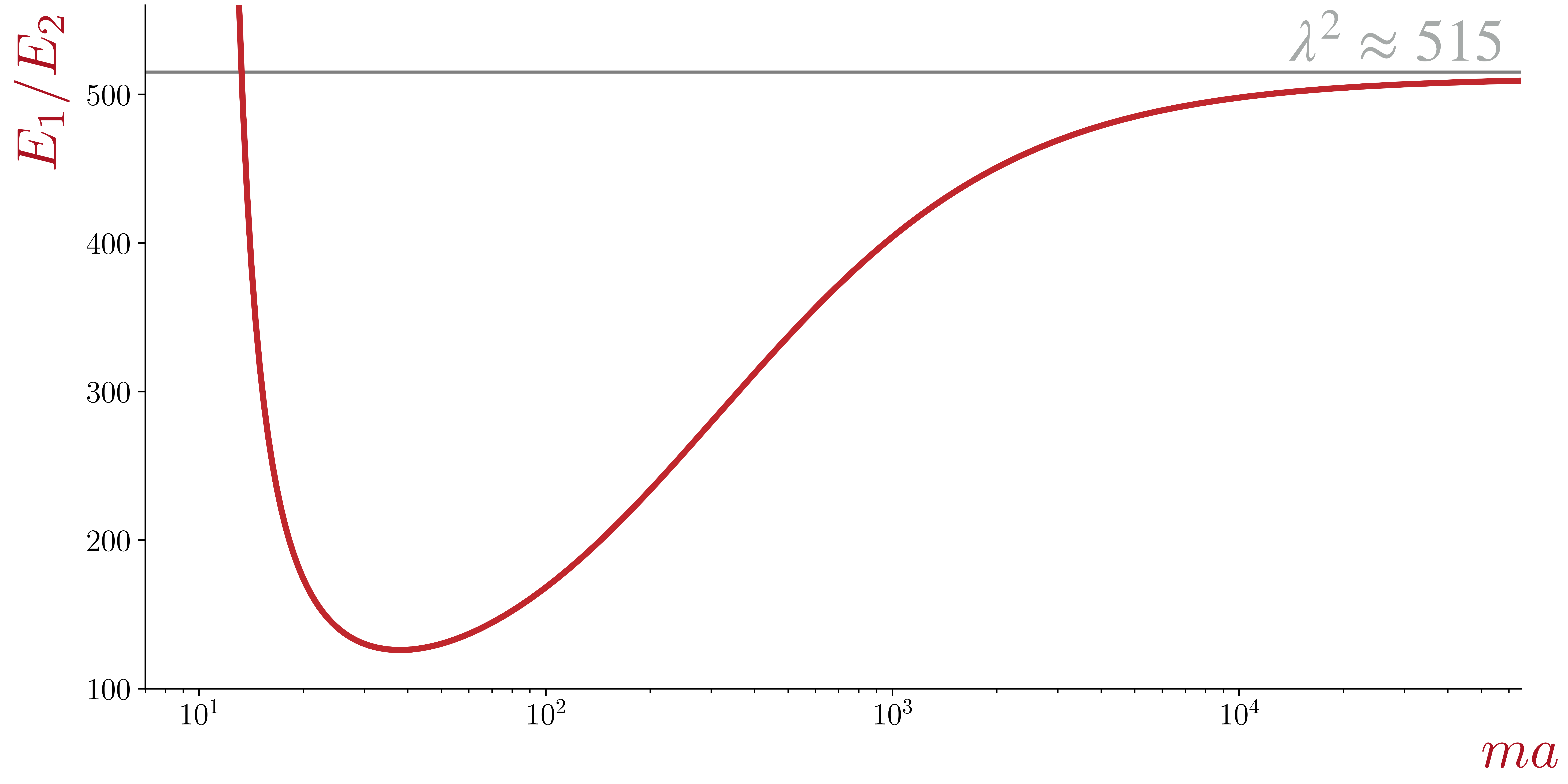




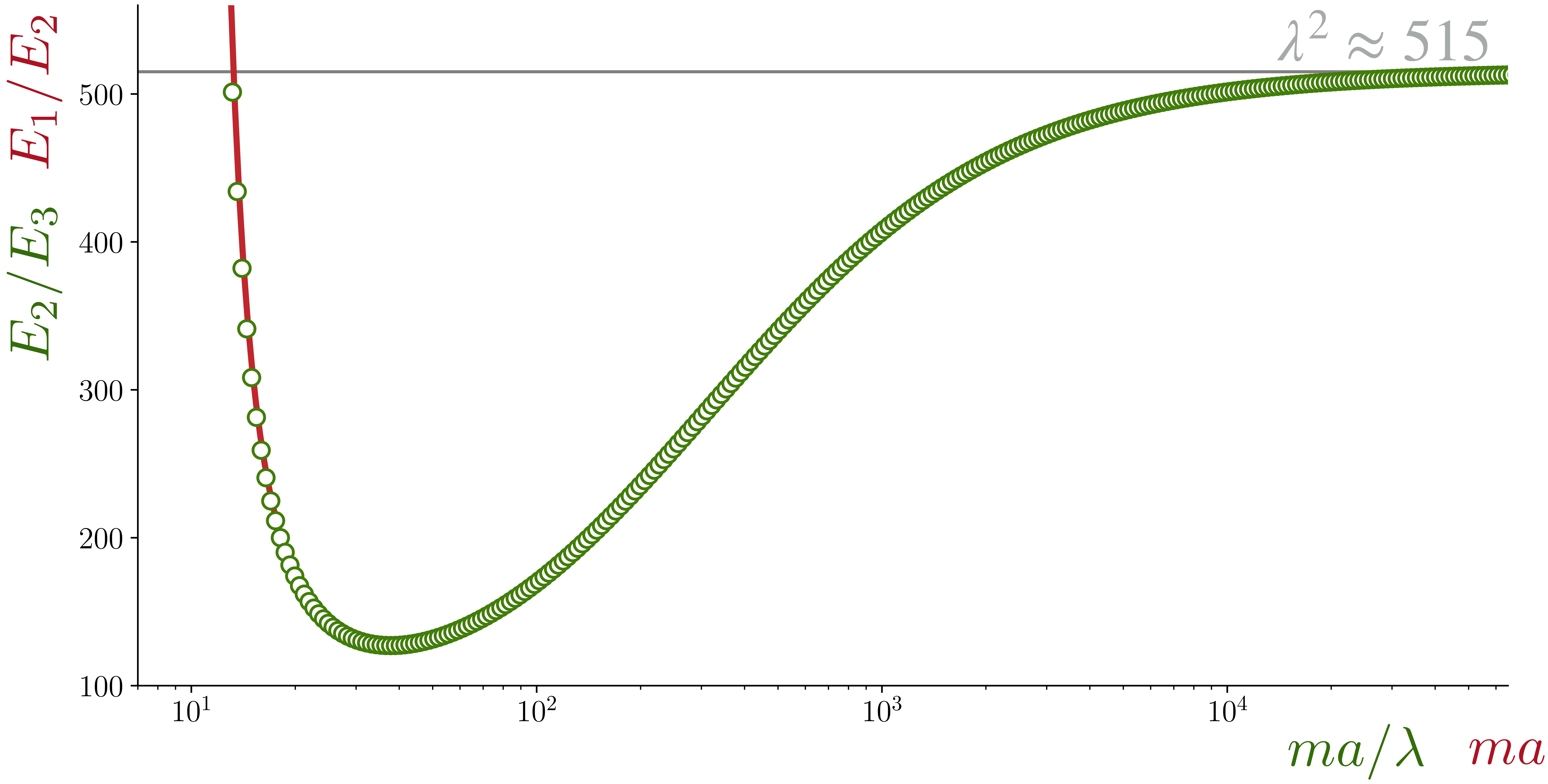
$\kappa = -\sqrt{3m - \sqrt{s_b}}$

Dawid, RB, Islam, Jackura, (to appear)

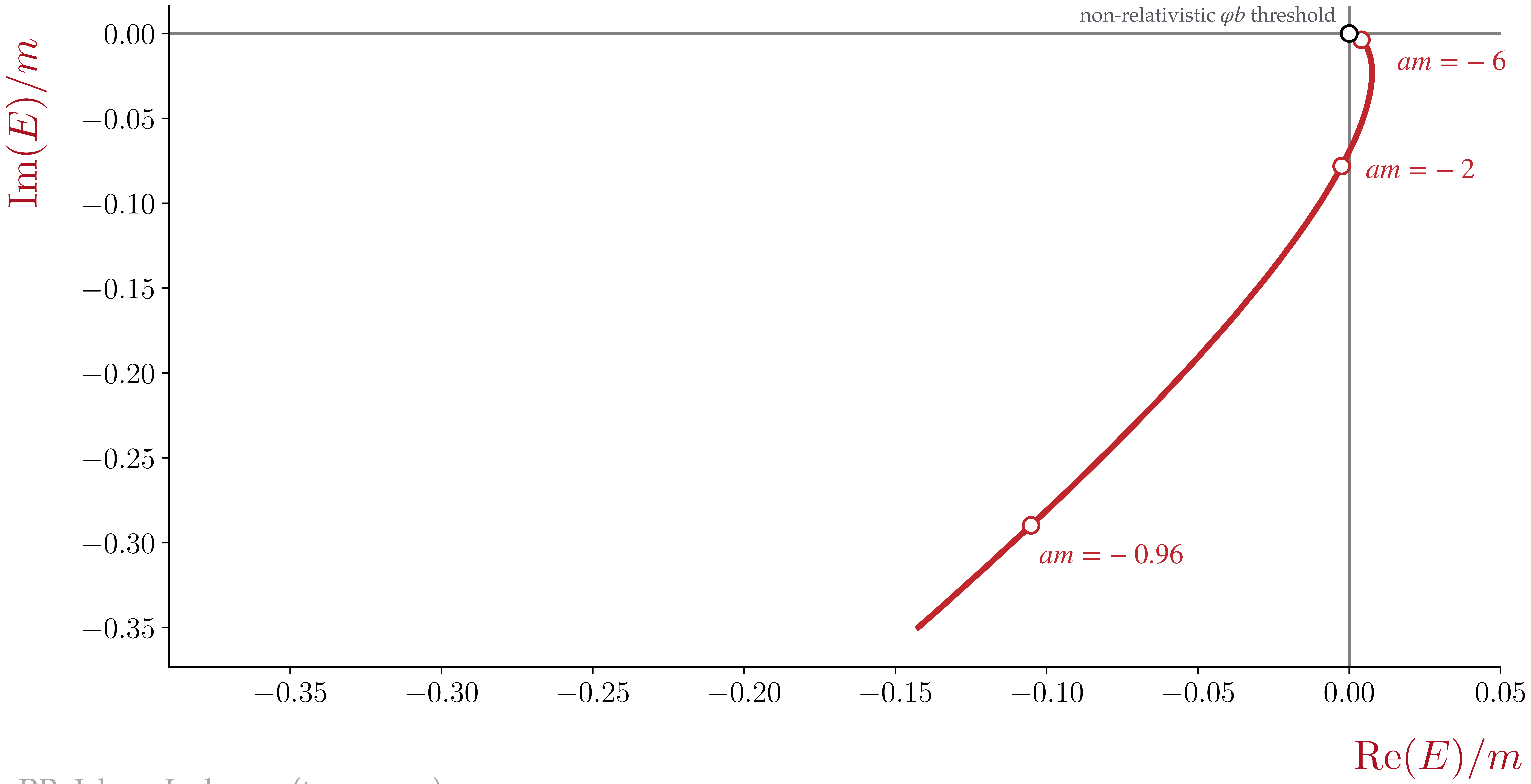
Efimov bound states



Efimov bound states



Efimov resonances $a < 0$



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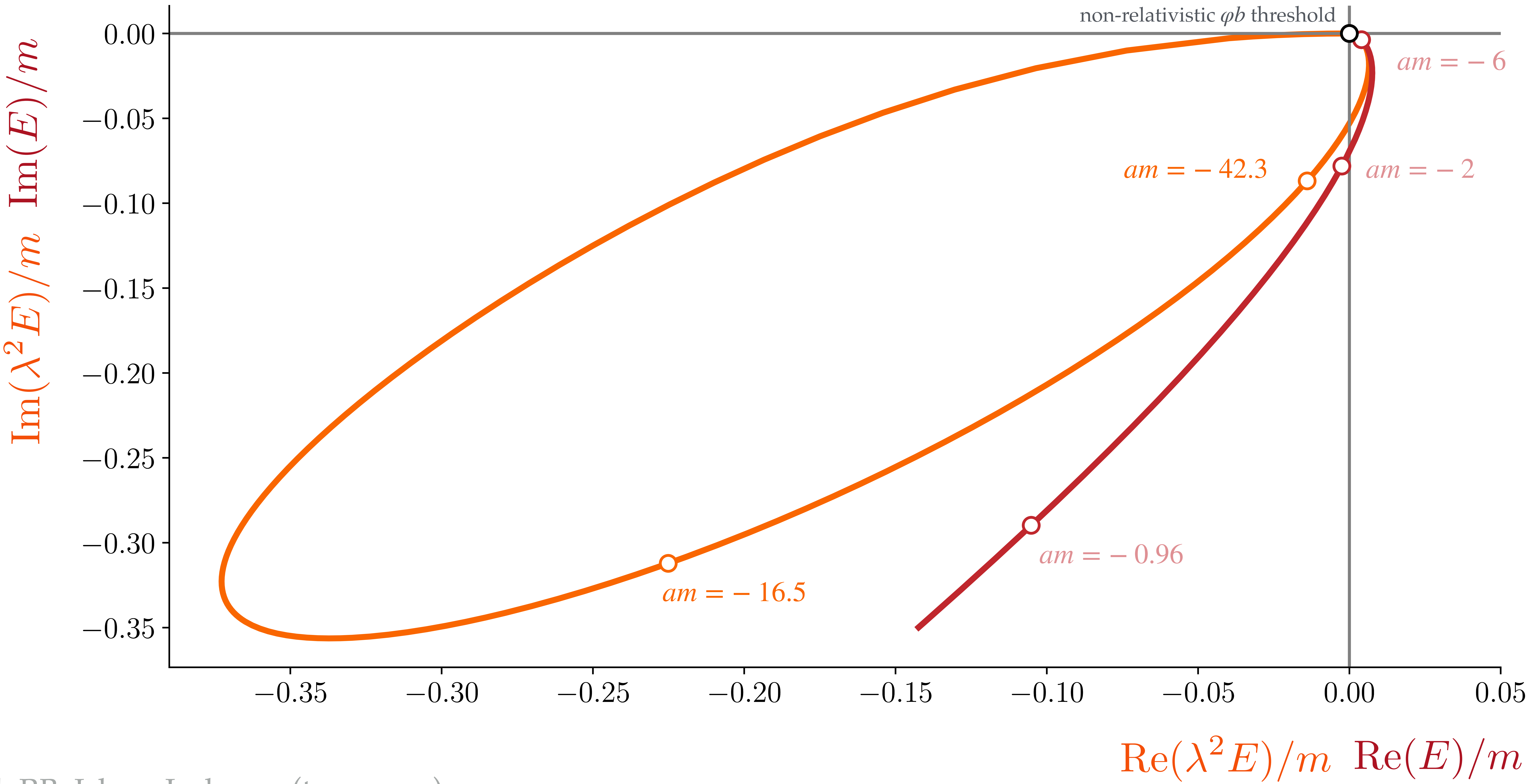

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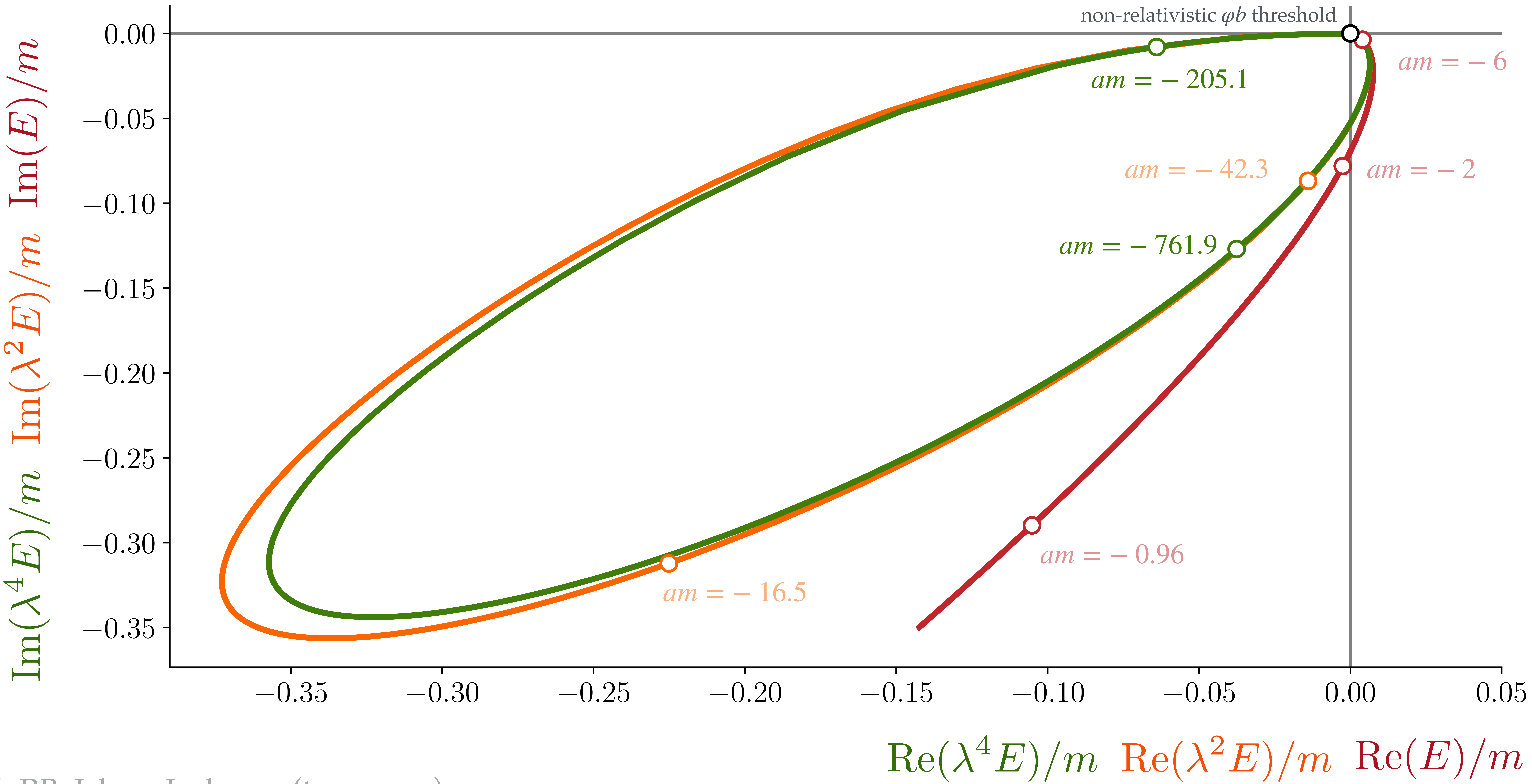

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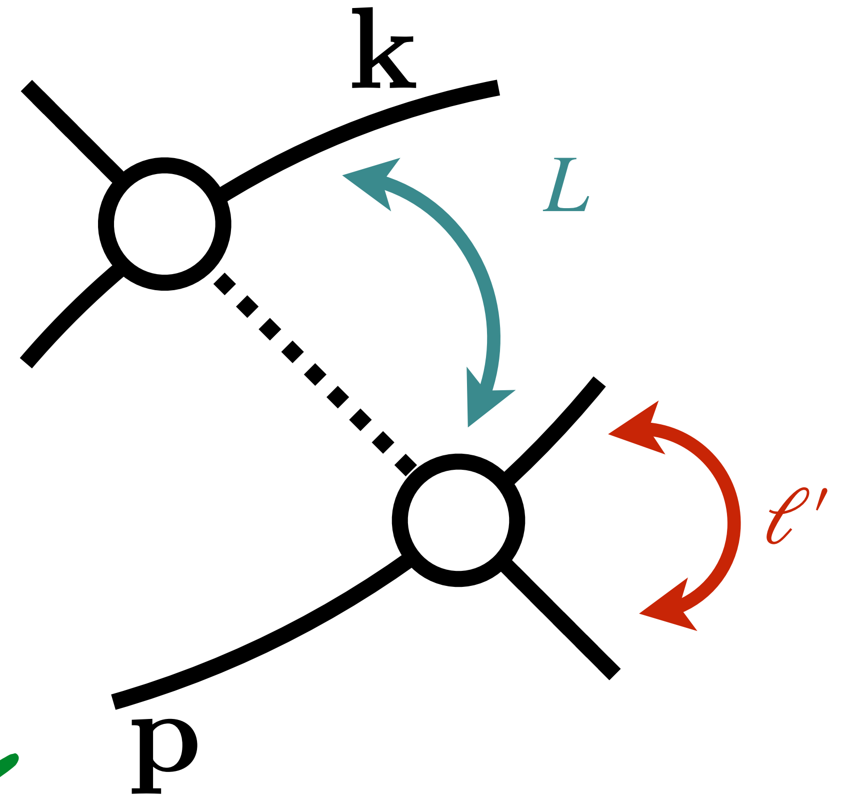
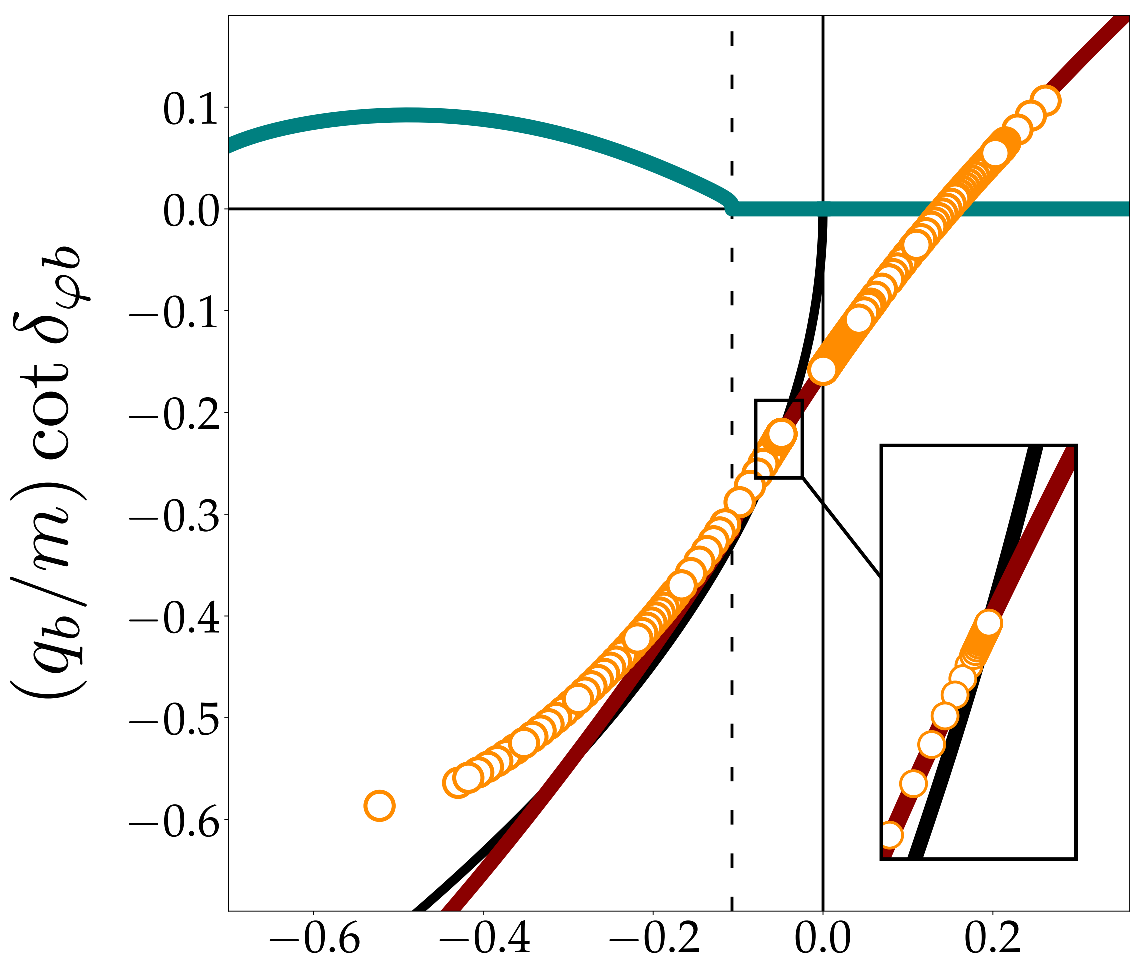
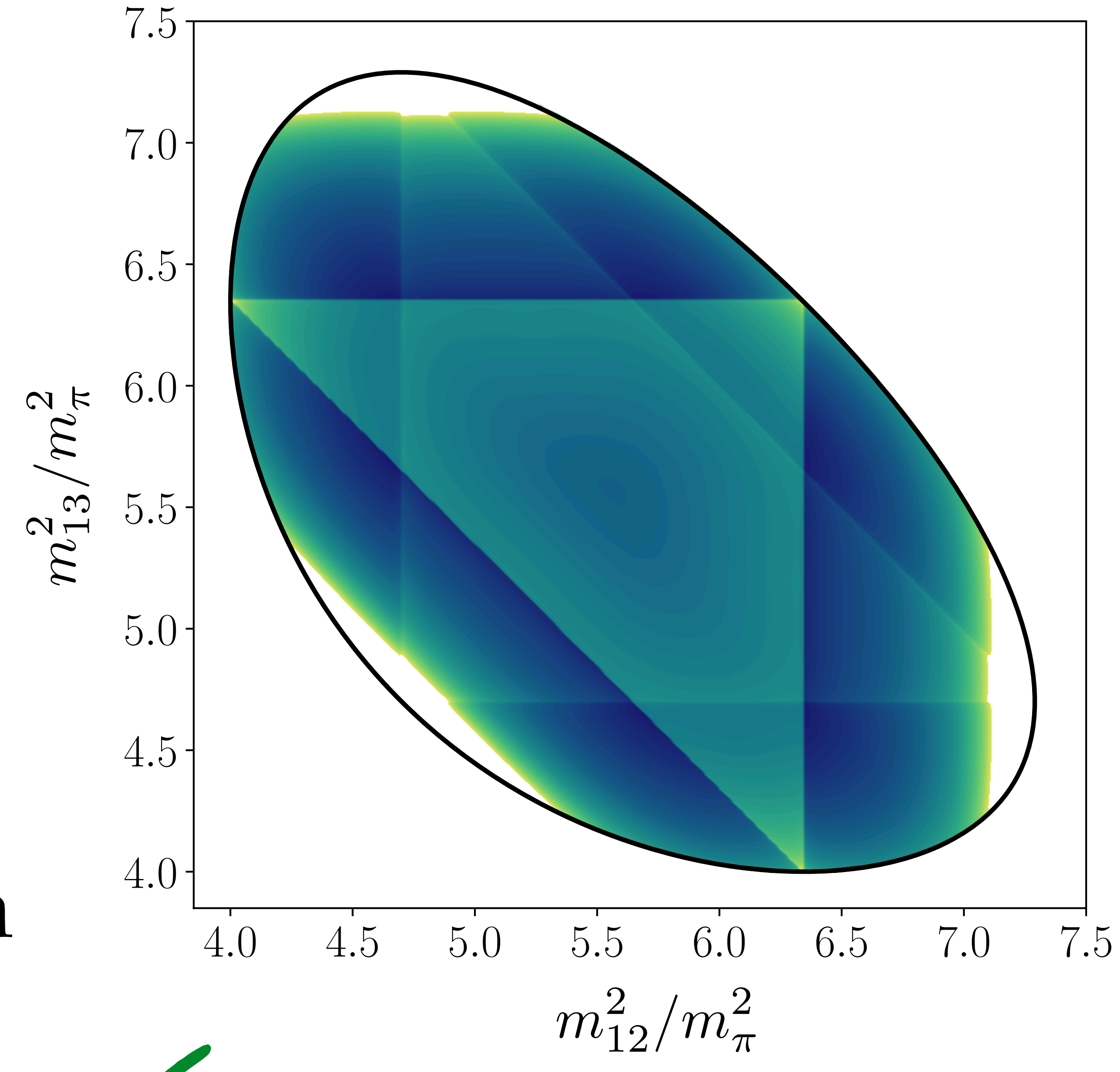
rapidly developing field!



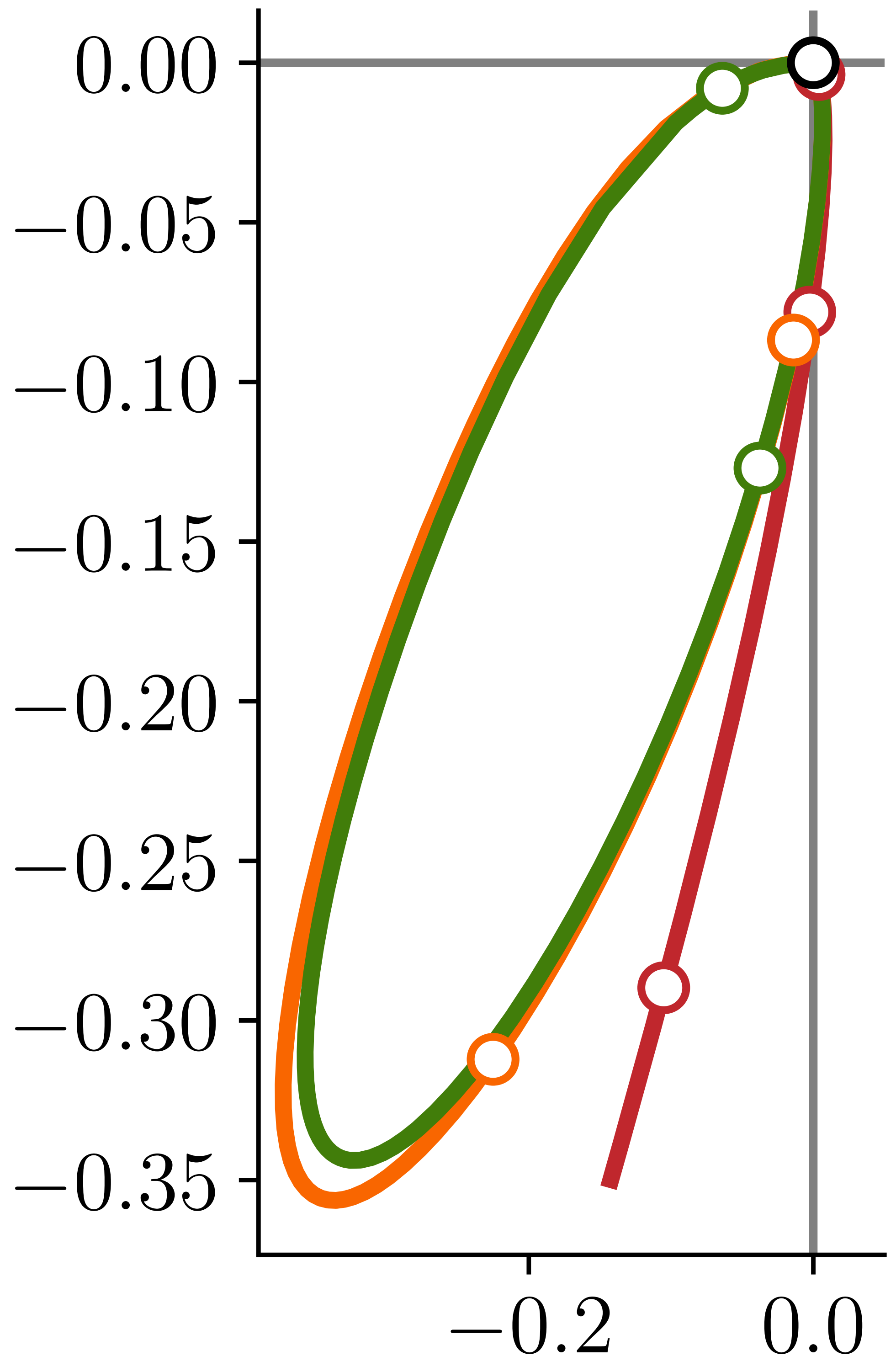
formal developments



checks of the formalism



actual lattice calculations



further explorations

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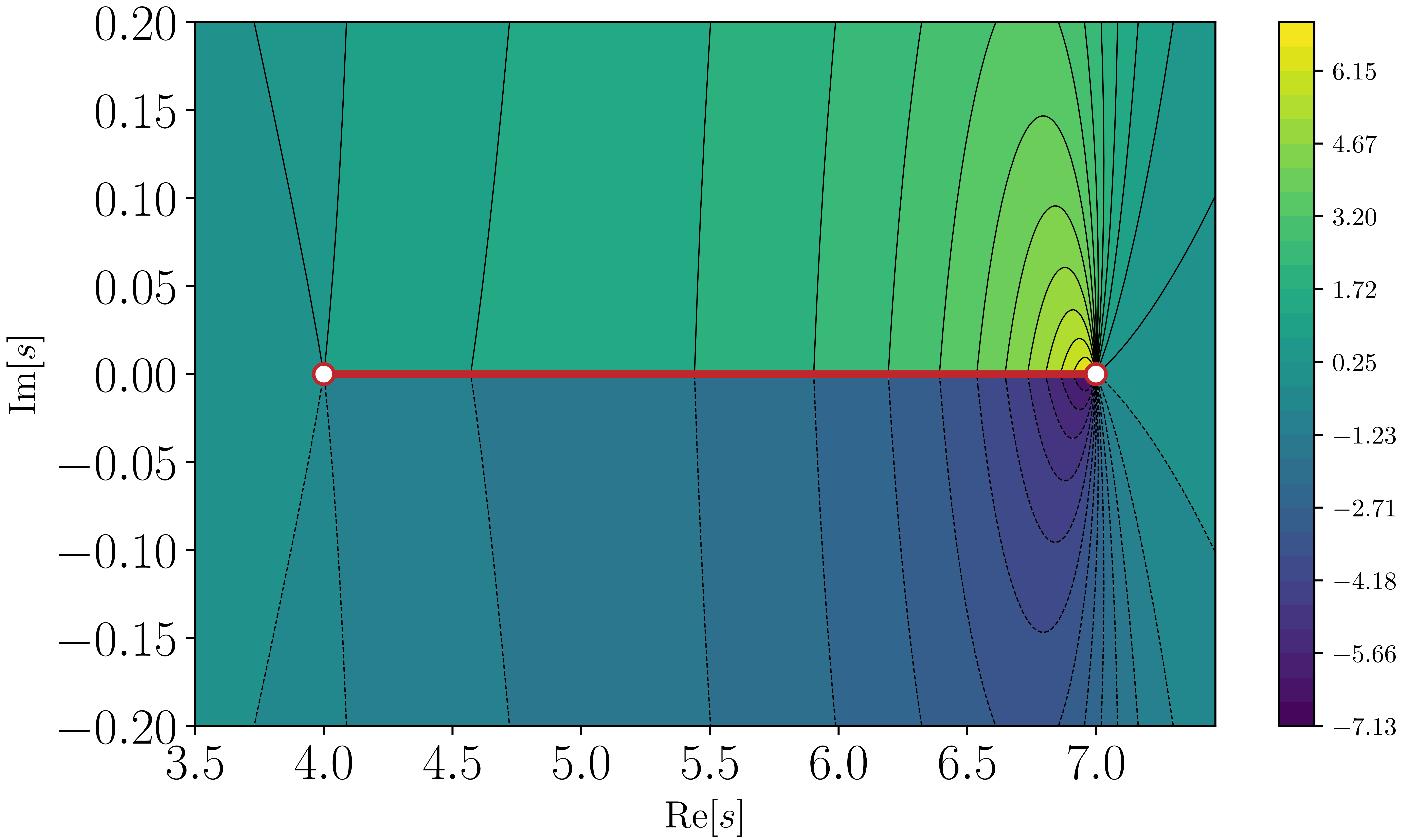
0.20
0.15
0.10
0.05
0.00
−0.05
−0.10
−0.15
−0.20
3.5
4.0
4.5
5.0
5.5
6.0
6.5
7.0
Re[s]
Im[s]
6.15
4.67
3.20
1.72
0.25
−1.23
−2.71
−4.18
−5.66
−7.13