TEXAS A&M UNIVERSITY Engineering

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Model-Free Power System Control and Optimization via Zeroth-Order Methods

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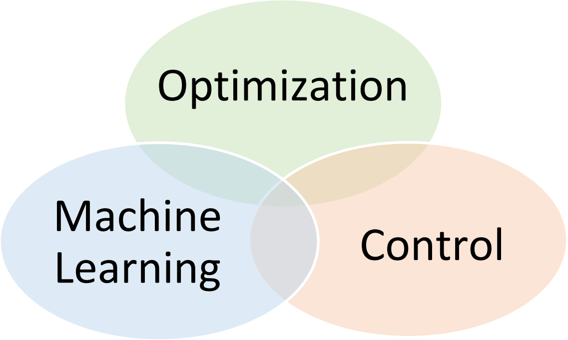
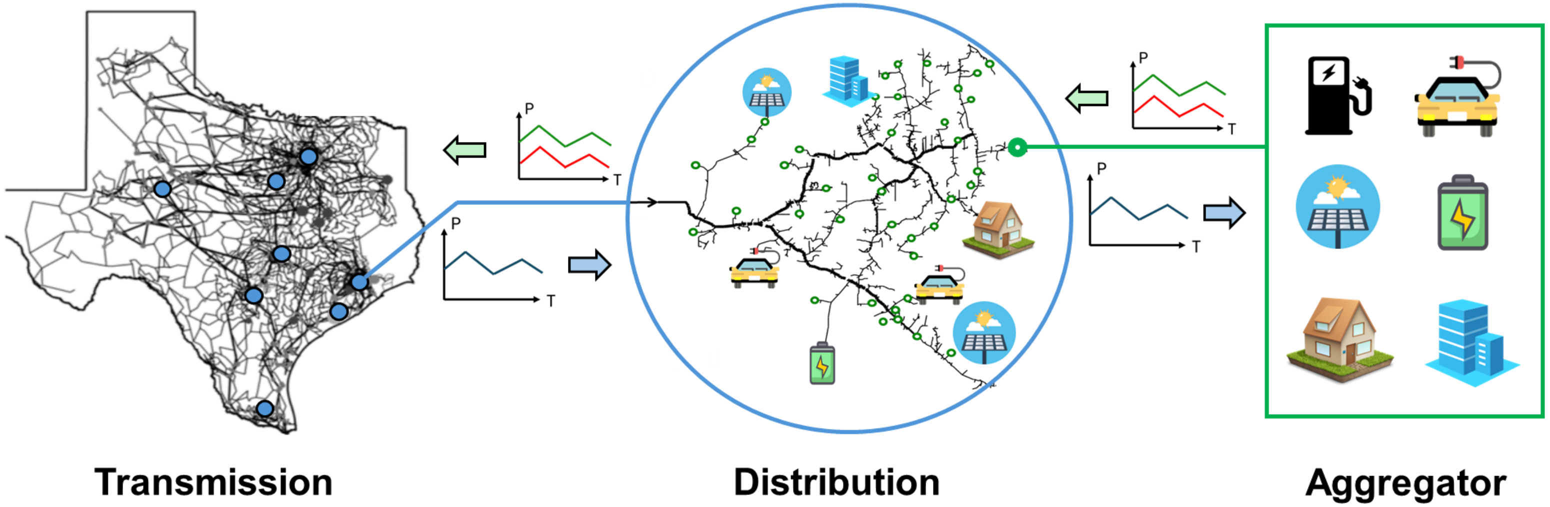
Smart Power, Energy and Decision-making (SPEED) Group



Smart Power & Energy Systems

Autonomous Coordination of Distributed Energy Resources (DERs)



Scalability

Uncertainty

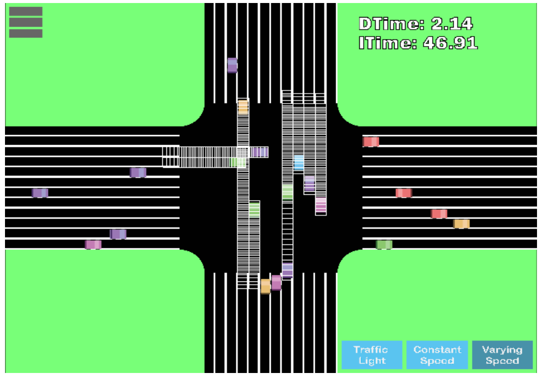
Unknown Model

Scalable Data-Driven Control and Optimization

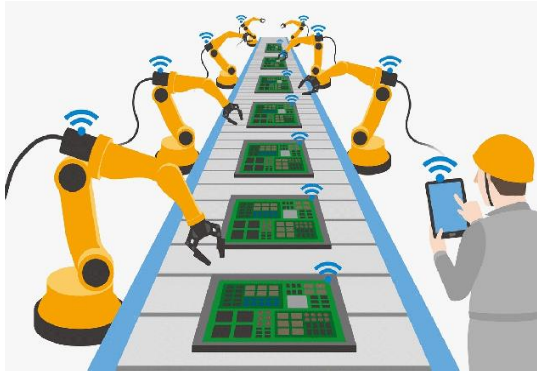
Motivation: Complex Human-Cyber-Physical Systems

Electricity Grid

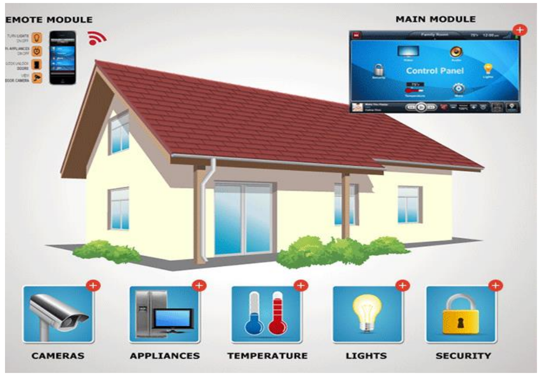
Transportation



Manufacturing



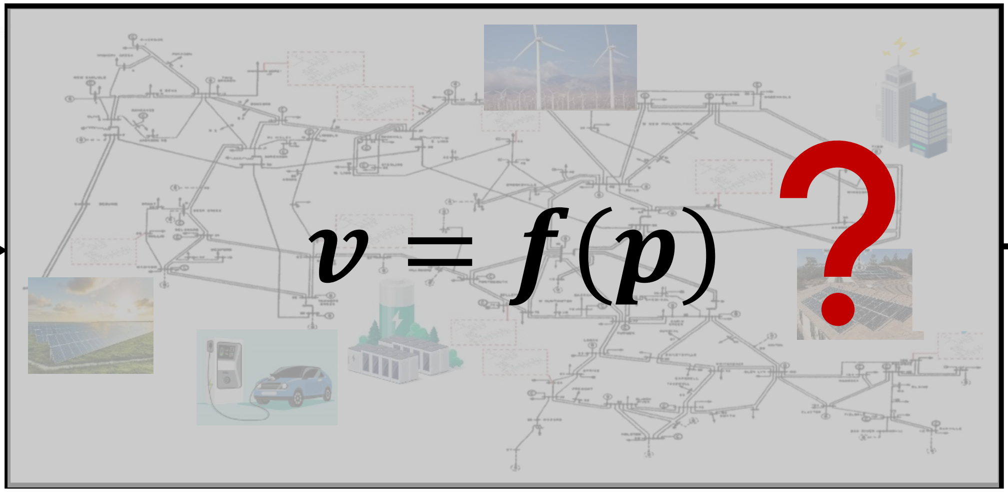
Buildings



Unknown physical system models + Hard-to-model human behaviors

Motivation: Complex Human-Cyber-Physical Systems

DER power injections p →



→ **Nodal voltages $v \in [\underline{v}, \overline{v}]$**

DER (Distributed Energy Resource)

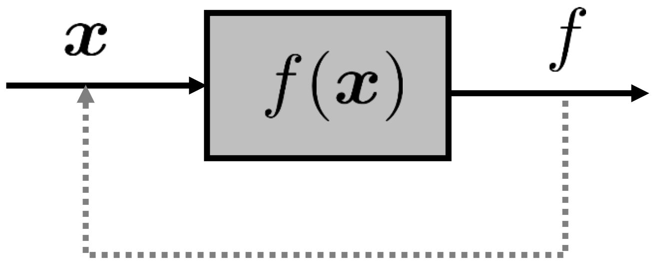
Voltage Control

Unknown physical system models **+** Hard-to-model human behaviors

Explosion of Data: real-time measurements / observations

Data-Driven / Model-Free Control and Optimization

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} f(\boldsymbol{x})$$



- Bandit learning
- Reinforcement learning
- Simulation-based control
- Black-box optimization
- Adversarial training
- Parameter tunning
- . . .

❖ (First-Order) Gradient Descent

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \eta \cdot \cancel{\nabla f(\boldsymbol{x})}$$

- Gradient information is unknown
- Only function evaluation is available

?

❖ Zeroth-Order Methods

solve using only function evaluations

- **Zeroth-order Optimization (ZO)**
- **Extremum Seeking Control (ESC)**

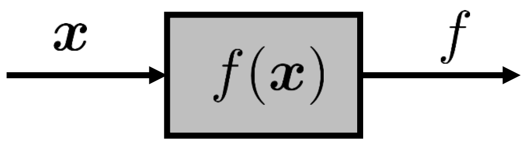
Preliminaries on Zeroth-Order Methods

[Flaxman et al. 2005] [Nesterov & Spokoiny 2017]

Zeroth-Order Optimization (ZO)

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \eta \cdot \mathsf{G}_f(\boldsymbol{x}_k; r, \boldsymbol{u}_k)$$

$$\min_{\boldsymbol{x}\in\mathbb{R}^d} f(\boldsymbol{x})$$



Extremum Seeking Control (ESC)

$$\dot{\boldsymbol{x}} = -\frac{2}{a} f(\boldsymbol{x} + a\sin(\boldsymbol{\omega}t))\sin(\boldsymbol{\omega}t)$$

Single-Point Gradient Estimator: $\mathsf{G}_f^{(1)}(\boldsymbol{x}; r, \boldsymbol{u}) := \frac{d}{r} f(\boldsymbol{x} + r\boldsymbol{u})\boldsymbol{u}$

Two-Point Gradient Estimator: $\mathsf{G}_f^{(2)}(\boldsymbol{x}; r, \boldsymbol{u}) := \frac{d}{r}(f(\boldsymbol{x}+r\boldsymbol{u}) - f(\boldsymbol{x}))\boldsymbol{u}$

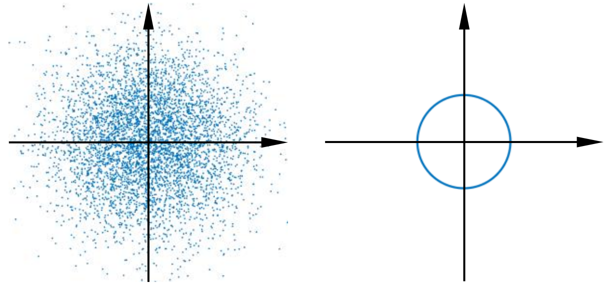
smoothing radius (r) random perturbation ($\boldsymbol{u}$)

Lemma 1.

$$\mathbb{E}_{\boldsymbol{u}}\left[\mathsf{G}_f(\boldsymbol{x}; r, \boldsymbol{u})\right] = \nabla f(\boldsymbol{x}) + \mathcal{O}(r)$$

Stochastic gradient estimation with nonzero (but controllable) bias.

Random perturbation Sampling



$\mathcal{N}(0, I/d)$ $\mathrm{Unif}(\mathbb{S}_{d-1})$

$$\mathbb{E}_{\boldsymbol{u}\sim\mathrm{Unif}(\mathbb{S}_{d-1})}\left[\mathsf{G}_f^{(1)}(\boldsymbol{x}; r, \boldsymbol{u})\right] = \nabla f_r(\boldsymbol{x}),$$

where $f_r(\boldsymbol{x}) := \mathbb{E}_{\bar{\boldsymbol{u}}\sim\mathrm{Unif}(\mathbb{B}_d)}\left[f(\boldsymbol{x} + r\bar{\boldsymbol{u}})\right].$

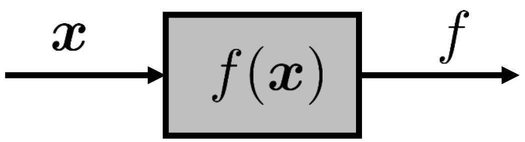
Preliminaries on Zeroth-Order Methods

[Flaxman et al. 2005] [Nesterov & Spokoiny 2017]

Zeroth-Order Optimization (ZO)

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \eta \cdot \mathsf{G}_f(\boldsymbol{x}_k; r, \boldsymbol{u}_k)$$

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} f(\boldsymbol{x})$$



Extremum Seeking Control (ESC)

$$\dot{\boldsymbol{x}} = -\frac{2}{a} f(\boldsymbol{x} + a\sin(\boldsymbol{\omega} t))\sin(\boldsymbol{\omega} t)$$

Single-Point Gradient Estimator: $\mathsf{G}_f^{(1)}(\boldsymbol{x}; r, \boldsymbol{u}) := \frac{d}{r} f(\boldsymbol{x} + r\boldsymbol{u})\boldsymbol{u}$

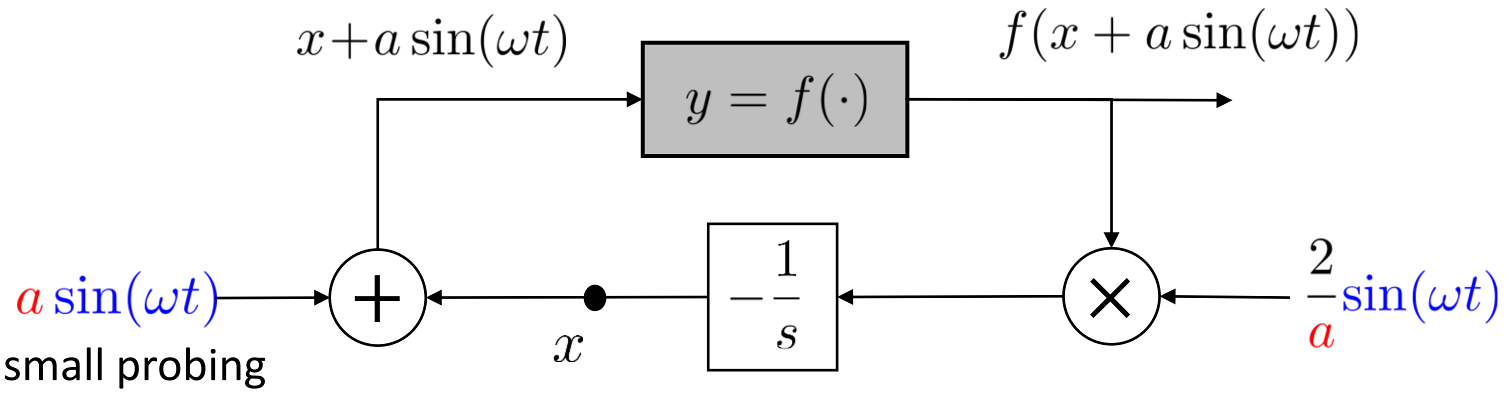
Two-Point Gradient Estimator: $\mathsf{G}_f^{(2)}(\boldsymbol{x}; r, \boldsymbol{u}) := \frac{d}{r}(f(\boldsymbol{x} + r\boldsymbol{u}) - f(\boldsymbol{x}))\boldsymbol{u}$

smoothing radius

random perturbation

Using only output feedback, ES system drives the state ***x*** to an optimum.

Lemma 2. $\dot{\boldsymbol{x}} = -\frac{2}{a} f(\boldsymbol{x} + a\sin(\boldsymbol{\omega} t))\sin(\boldsymbol{\omega} t) \approx \dot{\boldsymbol{x}} = -\nabla f(\boldsymbol{x}) + \mathcal{O}(a)$



ESC System

Preliminaries on Zeroth-Order Methods

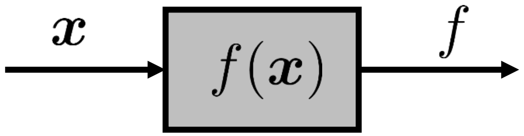
[Flaxman et al. 2005] [Nesterov & Spokoiny 2017]

Zeroth-Order Optimization (ZO)

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \eta \cdot \mathsf{G}_f(\boldsymbol{x}_k; r, \boldsymbol{u}_k)$$

(Discrete-Time & Random Perturbation)

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} f(\boldsymbol{x})$$



Estimate unknown gradients using perturbed function evaluations (or system measurements)

Extremum Seeking Control (ESC)

$$\dot{\boldsymbol{x}} = -\frac{2}{a} f(\boldsymbol{x} + a \sin(\boldsymbol{\omega} t)) \sin(\boldsymbol{\omega} t)$$

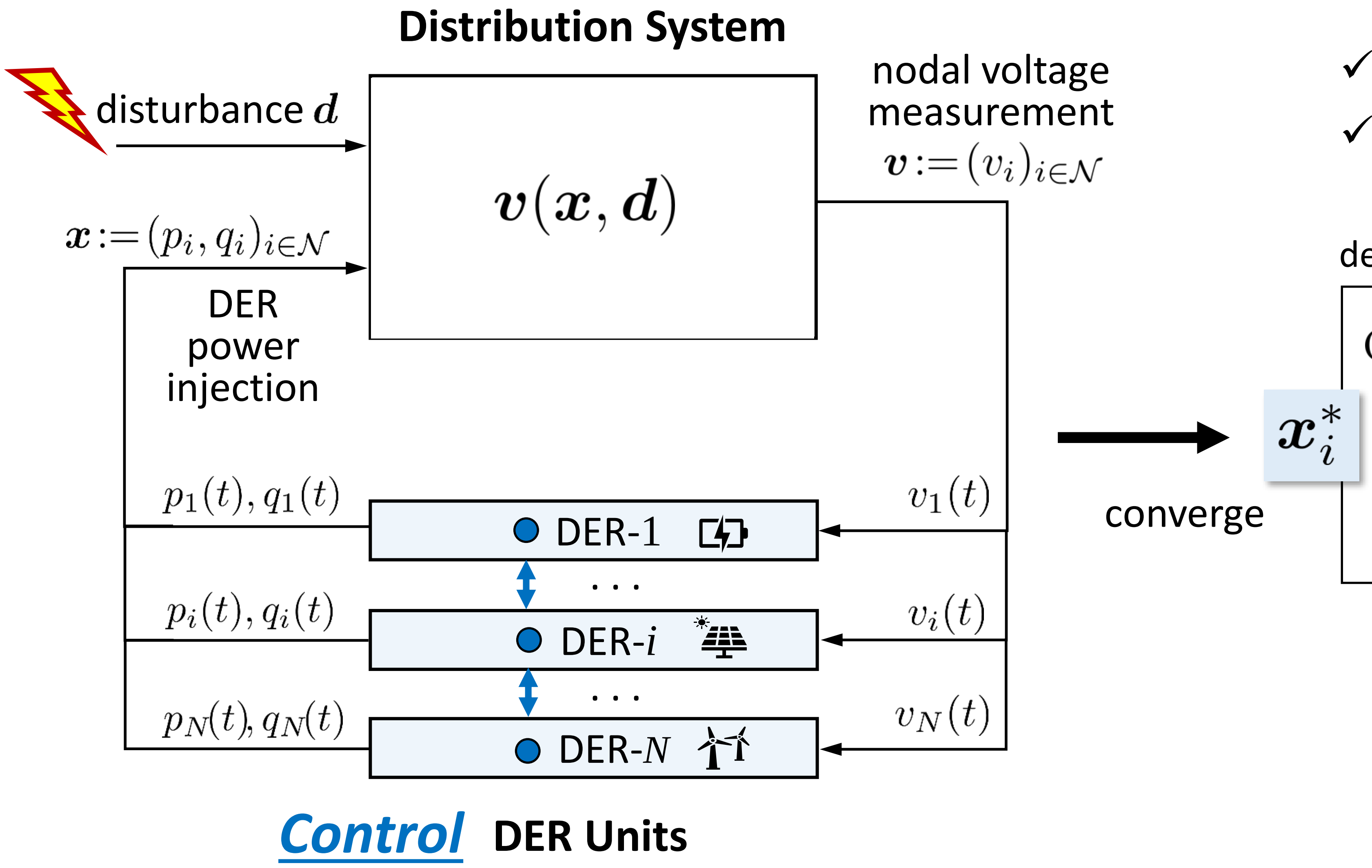
(Continuous-Time & Deterministic Perturbation)

This Talk

I. Model-free Optimal Voltage Control in Power Distribution Systems

II. Improve Single-point Zeroth-order Optimization Using Filters

Real-Time Optimal Voltage Control (OVC)



✓ Control Optimality

✓ Voltage Safety

decision $\boldsymbol{x}_i := (p_i, q_i)$

OVC

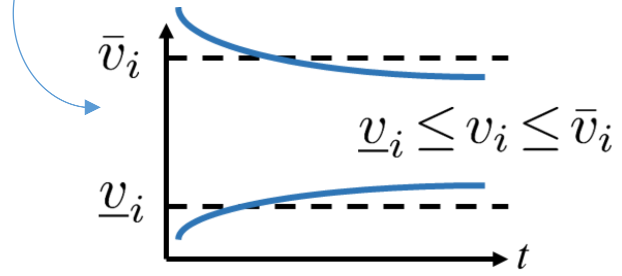
$$\text{Obj.} \quad \min_{\boldsymbol{x}} \sum_{i\in\mathcal{N}} c_i(\boldsymbol{x}_i)$$

$$\text{s.t.} \quad \boldsymbol{x}_i \in \mathcal{X}_i, \quad i \in \mathcal{N}$$

$$\underline{v}_i \leq v_i(\boldsymbol{x}, \boldsymbol{d}) \leq \bar{v}_i, \quad i \in \mathcal{N}$$

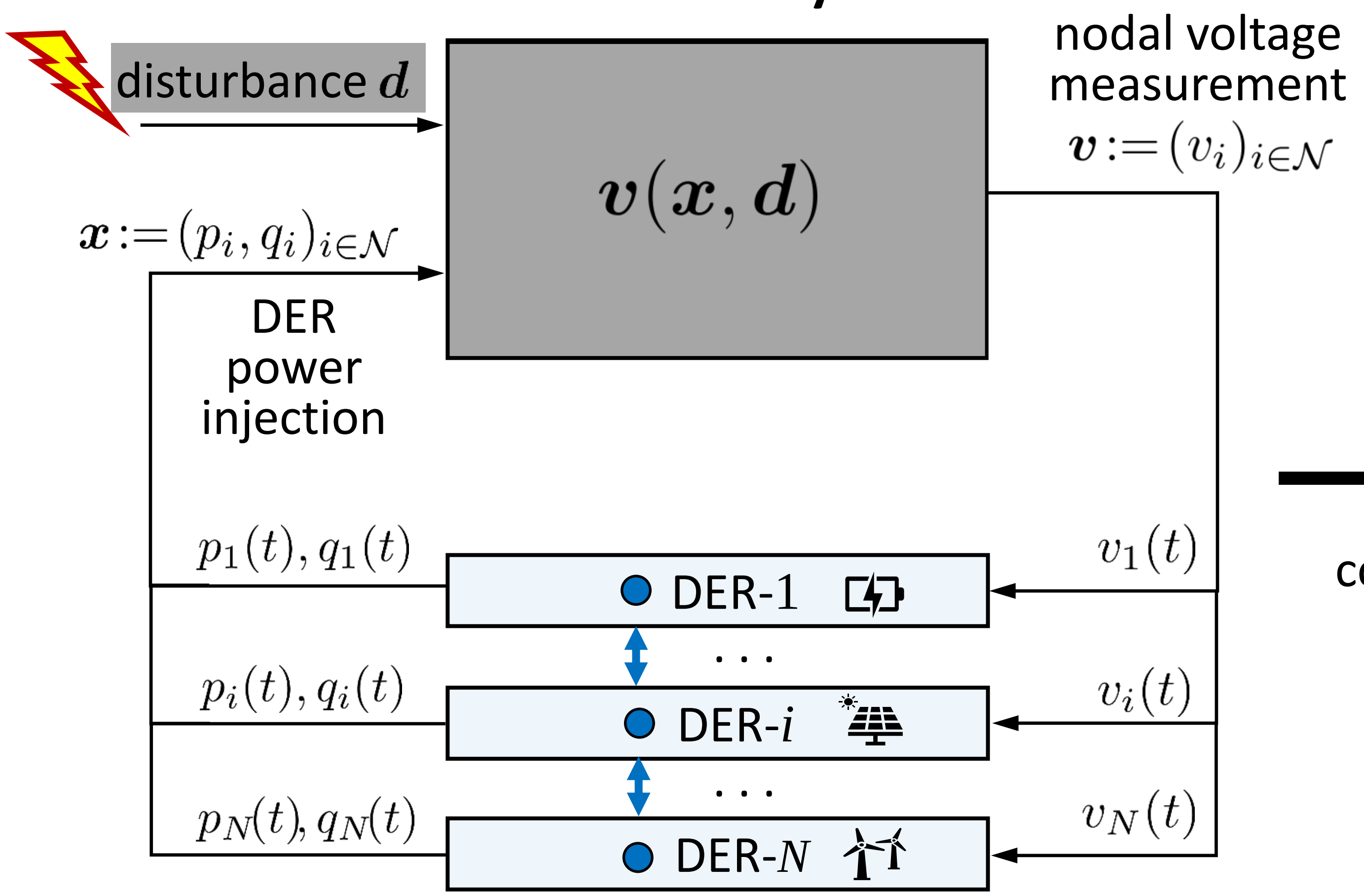
control cost

DER power capacity set



Real-Time Optimal Voltage Control (OVC)

Unknown Model **Distribution System**



Control **DER Units**

converge $\rightarrow$ $\boldsymbol{x}_i^*$

- ✓ Control Optimality
- ✓ Voltage Safety

decision $\boldsymbol{x}_i := (p_i, q_i)$

OVC

$$\text{Obj.} \quad \min_{\boldsymbol{x}} \sum_{i \in \mathcal{N}} c_i(\boldsymbol{x}_i)$$

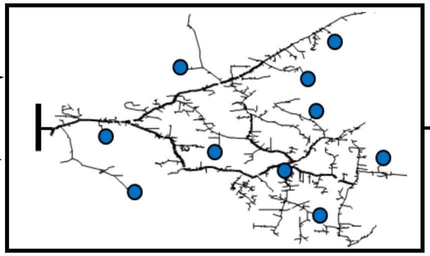
$$\text{s.t.} \quad \boldsymbol{x}_i \in \mathcal{X}_i, \qquad i \in \mathcal{N}$$

$$\underline{v}_i \leq v_i(\boldsymbol{x}, \boldsymbol{d}) \leq \bar{v}_i, \qquad i \in \mathcal{N}$$

? **Scalability**

Optimal Voltage Control Design

Distribution System



disturbance $\boldsymbol{d}$

voltage measurement

power injection

$\boldsymbol{x}_i := (p_i, q_i)$

$v_i^{\mathrm{m}}(t)$

P-PDGD

● **DER *i***

converge

$\boldsymbol{x}_i^*$

OVC

$$\text{Obj.} \quad \min_{\boldsymbol{x}} \sum_{i \in \mathcal{N}} c_i(\boldsymbol{x}_i)$$

$$\text{s.t.} \quad \boldsymbol{x}_i \in \mathcal{X}_i, \qquad i \in \mathcal{N}$$

$$\underline{v}_i \leq v_i(\boldsymbol{x}, \boldsymbol{d}) \leq \bar{v}_i, \qquad i \in \mathcal{N}$$

dual variable

$$\boldsymbol{\lambda} := (\lambda_i^+, \lambda_i^-)_{i \in \mathcal{N}}$$

Projected Primal-Dual Gradient Dynamics (P-PDGD)

$$\dot{\boldsymbol{x}}_i = k_x \left[\mathrm{Proj}_{\mathcal{X}_i} \left(\boldsymbol{x}_i - \alpha_x (\nabla c_i(\boldsymbol{x}_i) + \boldsymbol{\ell}_i) \right) - \boldsymbol{x}_i \right]$$

$$\dot{\lambda}_i^+ = k_\lambda \left[\mathrm{Proj}_{\mathbb{R}_+} (\lambda_i^+ + \alpha_\lambda (v_i(\boldsymbol{x}, \boldsymbol{d}) - \bar{v}_i)) - \lambda_i^+ \right]$$

$$\dot{\lambda}_i^- = k_\lambda \left[\mathrm{Proj}_{\mathbb{R}_+} (\lambda_i^- + \alpha_\lambda (\underline{v}_i - v_i(\boldsymbol{x}, \boldsymbol{d}))) - \lambda_i^- \right]$$

$$\boldsymbol{\ell}_i = \sum_{j \in \mathcal{N}} (\lambda_j^+ - \lambda_j^-) \frac{\partial v_j(\boldsymbol{x}, \boldsymbol{d})}{\partial \boldsymbol{x}_i}$$

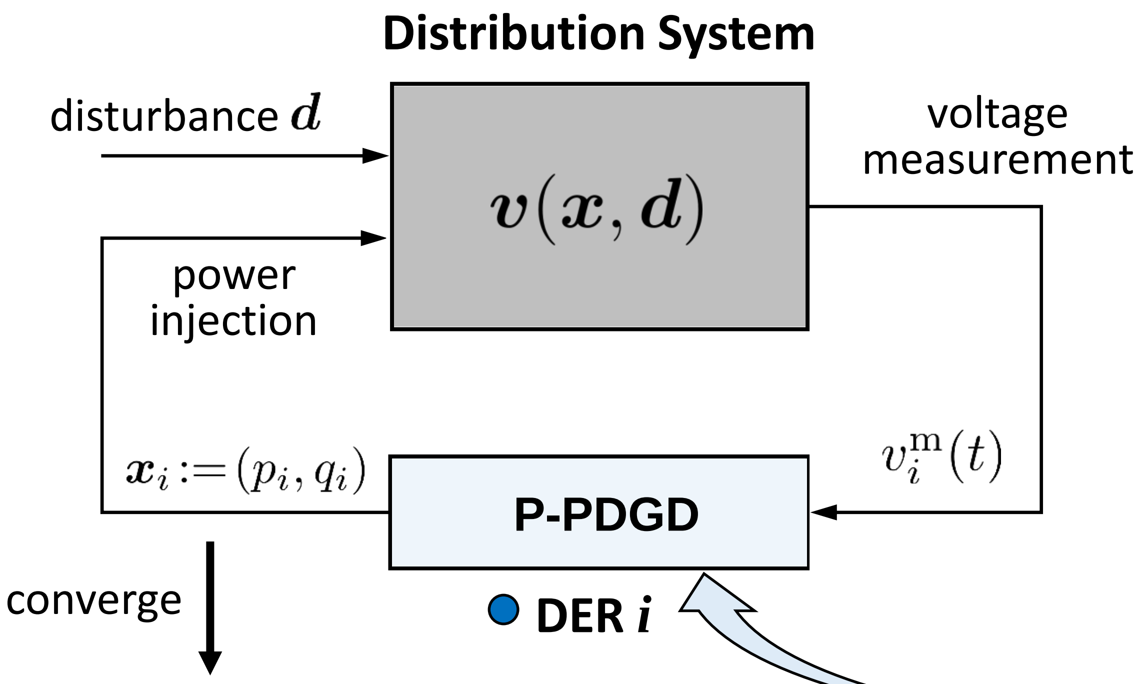
● $i \in \mathcal{N}$

solved by

Saddle Point Problem

$$\max_{\boldsymbol{\lambda} \geq 0} \min_{\boldsymbol{x} \in \mathcal{X}} L(\boldsymbol{x}, \boldsymbol{\lambda})$$

Optimal Voltage Control Design



$\boldsymbol{x}_i^*$

OVC

$$
\begin{aligned}
\text{Obj.} \quad & \min_{\boldsymbol{x}} \sum_{i \in \mathcal{N}} c_i(\boldsymbol{x}_i) \\
\text{s.t.} \quad & \boldsymbol{x}_i \in \mathcal{X}_i, && i \in \mathcal{N} \\
& \underline{v}_i \leq v_i(\boldsymbol{x}, \boldsymbol{d}) \leq \bar{v}_i, && i \in \mathcal{N}
\end{aligned}
$$

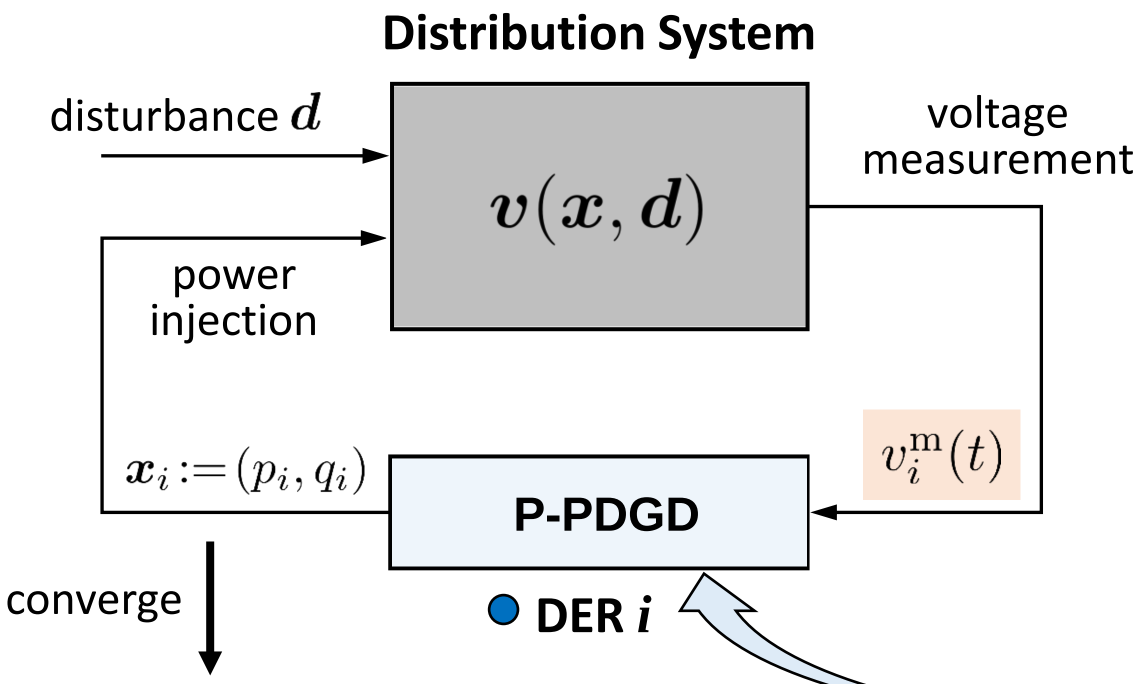
Projected Primal-Dual Gradient Dynamics (P-PDGD)

$$
\begin{aligned}
\dot{\boldsymbol{x}}_i &= k_x \left[\mathrm{Proj}_{\mathcal{X}_i} \left(\boldsymbol{x}_i - \alpha_x (\nabla c_i(\boldsymbol{x}_i) + \boldsymbol{\ell}_i) \right) - \boldsymbol{x}_i \right] \\
\dot{\lambda}_i^+ &= k_\lambda \left[\mathrm{Proj}_{\mathbb{R}_+} (\lambda_i^+ + \alpha_\lambda (v_i(\boldsymbol{x}, \boldsymbol{d}) - \bar{v}_i)) - \lambda_i^+ \right] \\
\dot{\lambda}_i^- &= k_\lambda \left[\mathrm{Proj}_{\mathbb{R}_+} (\lambda_i^- + \alpha_\lambda (\underline{v}_i - v_i(\boldsymbol{x}, \boldsymbol{d}))) - \lambda_i^- \right] \\
\boldsymbol{\ell}_i &= \sum_{j \in \mathcal{N}} (\lambda_j^+ - \lambda_j^-) \frac{\partial v_j(\boldsymbol{x}, \boldsymbol{d})}{\partial \boldsymbol{x}_i}
\end{aligned}
$$

$i \in \mathcal{N}$

not implementable
due to **unknown system model**

Optimal Voltage Control Design



Projected Primal-Dual Gradient Dynamics (P-PDGD)

$$\dot{\boldsymbol{x}}_i = k_x \left[\mathrm{Proj}_{\mathcal{X}_i}\big(\boldsymbol{x}_i - \alpha_x(\nabla c_i(\boldsymbol{x}_i) + \boldsymbol{\ell}_i)\big) - \boldsymbol{x}_i \right]$$

$$\dot{\lambda}_i^+ = k_\lambda \left[\mathrm{Proj}_{\mathbb{R}_+}(\lambda_i^+ + \alpha_\lambda(v_i^{\mathrm{m}}(t) - \bar{v}_i)) - \lambda_i^+ \right]$$

$$\dot{\lambda}_i^- = k_\lambda \left[\mathrm{Proj}_{\mathbb{R}_+}(\lambda_i^- + \alpha_\lambda(\underline{v}_i - v_i^{\mathrm{m}}(t))) - \lambda_i^- \right]$$

$$\boldsymbol{\ell}_i = \sum_{j\in\mathcal{N}} (\lambda_j^+ - \lambda_j^-) \frac{\partial v_j(\boldsymbol{x},\boldsymbol{d})}{\partial \boldsymbol{x}_i} \qquad i \in \mathcal{N}$$

not implementable
due to **unknown system model**

$\boldsymbol{x}_i^*$

OVC

$$\text{Obj.} \quad \min_{\boldsymbol{x}} \sum_{i\in\mathcal{N}} c_i(\boldsymbol{x}_i)$$

$$\text{s.t.} \quad \boldsymbol{x}_i \in \mathcal{X}_i, \qquad i \in \mathcal{N}$$

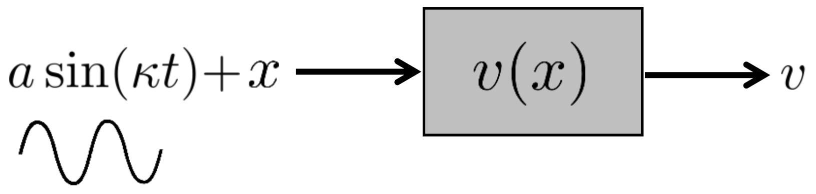
$$\underline{v}_i \le v_i(\boldsymbol{x},\boldsymbol{d}) \le \bar{v}_i, \qquad i \in \mathcal{N}$$

Our Solution:

- Replace $v_i(\boldsymbol{x},\boldsymbol{d})$ by real-time measurement $v_i^{\mathrm{m}}(t)$
- **Zeroth-order method** to estimate gradient $\frac{\partial v_j(\boldsymbol{x},\boldsymbol{d})}{\partial \boldsymbol{x}_i}$

Optimal Voltage Control Design

Zeroth-order Method:



Zeroth-order estimation **First-order gradient**

$$\frac{2}{a} v(x + a\sin(\kappa t)) \sin(\kappa t) \approx \frac{dv(x)}{dx}$$

real-time measurement $v^{\mathrm{m}}(t)$

Projected Primal-Dual Gradient Dynamics (P-PDGD)

$$\dot{\boldsymbol{x}}_i = k_x \left[\mathrm{Proj}_{\mathcal{X}_i} \left(\boldsymbol{x}_i - \alpha_x (\nabla c_i(\boldsymbol{x}_i) + \boldsymbol{\ell}_i) \right) - \boldsymbol{x}_i \right]$$

$$\dot{\lambda}_i^+ = k_\lambda \left[\mathrm{Proj}_{\mathbb{R}_+} (\lambda_i^+ + \alpha_\lambda (v_i^{\mathrm{m}}(t) - \bar{v}_i)) - \lambda_i^+ \right]$$

$$\dot{\lambda}_i^- = k_\lambda \left[\mathrm{Proj}_{\mathbb{R}_+} (\lambda_i^- + \alpha_\lambda (\underline{v}_i - v_i^{\mathrm{m}}(t))) - \lambda_i^- \right]$$

$$\boldsymbol{\ell}_i = \sum_{j \in \mathcal{N}} (\lambda_j^+ - \lambda_j^-) \frac{\partial v_j(\boldsymbol{x}, \boldsymbol{d})}{\partial \boldsymbol{x}_i}$$

$i \in \mathcal{N}$

not implementable
due to **unknown system model**

Our Solution:

- Replace $v_i(\boldsymbol{x}, \boldsymbol{d})$ by real-time measurement $v_i^{\mathrm{m}}(t)$
- **Zeroth-order method** to estimate gradient $\frac{\partial v_j(\boldsymbol{x}, \boldsymbol{d})}{\partial \boldsymbol{x}_i}$

Projected Primal-Dual Zeroth-Order Dynamics (P-PDZD)

$$\dot{\boldsymbol{x}}_i = k_x \left[\mathrm{Proj}_{\hat{\mathcal{X}}_i} \left(\boldsymbol{x}_i - \alpha_x (\nabla c_i(\boldsymbol{x}_i) + \boldsymbol{\ell}_i) \right) - \boldsymbol{x}_i \right]$$

$$\dot{\lambda}_i^+ = k_\lambda \left[\mathrm{Proj}_{\mathbb{R}_+} (\lambda_i^+ + \alpha_\lambda (\mu_i - \bar{v}_i)) - \lambda_i^+ \right]$$

$$\dot{\lambda}_i^- = k_\lambda \left[\mathrm{Proj}_{\mathbb{R}_+} (\lambda_i^- + \alpha_\lambda (\underline{v}_i - \mu_i)) - \lambda_i^- \right]$$

$$\dot{\boldsymbol{\ell}}_i = \frac{1}{\epsilon} \left[-\boldsymbol{\ell}_i + \frac{2}{a} \cdot N \cdot y_i(t) \sin(\boldsymbol{\kappa}_i t) \right]$$

$$\dot{\mu}_i = \frac{1}{\epsilon} \left[-\mu_i + v_i^{\mathrm{m}}(t) \right]$$

$$\dot{p}_i = \frac{1}{\epsilon_p} \cdot \sum_{ij \in \mathcal{E}} a_{ij} (y_i(t) - y_j(t))$$

$$y_i = (\lambda_i^+ - \lambda_i^-) \; v_i^{\mathrm{m}}(t) \; - p_i$$

- Each DER unit $i \in \mathcal{N}$

Projected Primal-Dual Dynamics

✓ Optimality & Safety

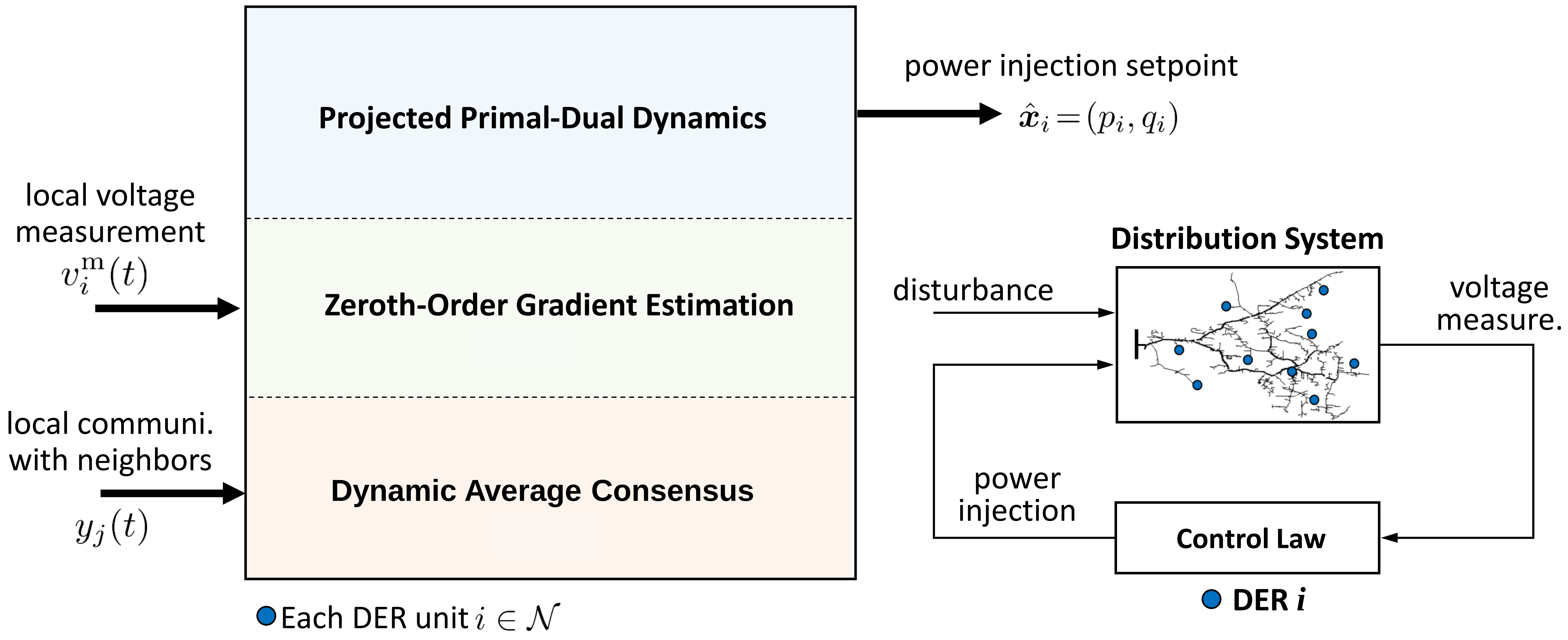
Zeroth-Order Gradient Estimation

✓ Model-Free

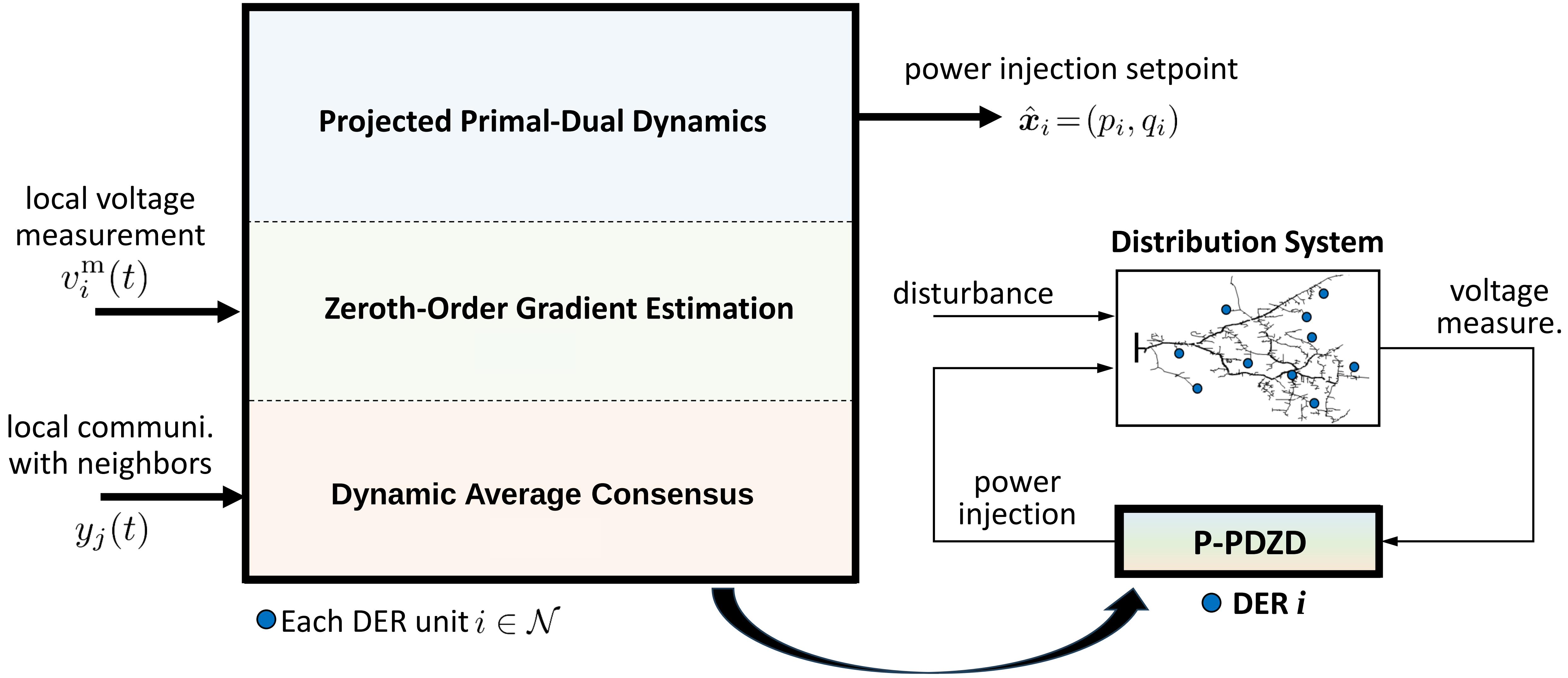
Dynamic Average Consensus

✓ Distributed Control

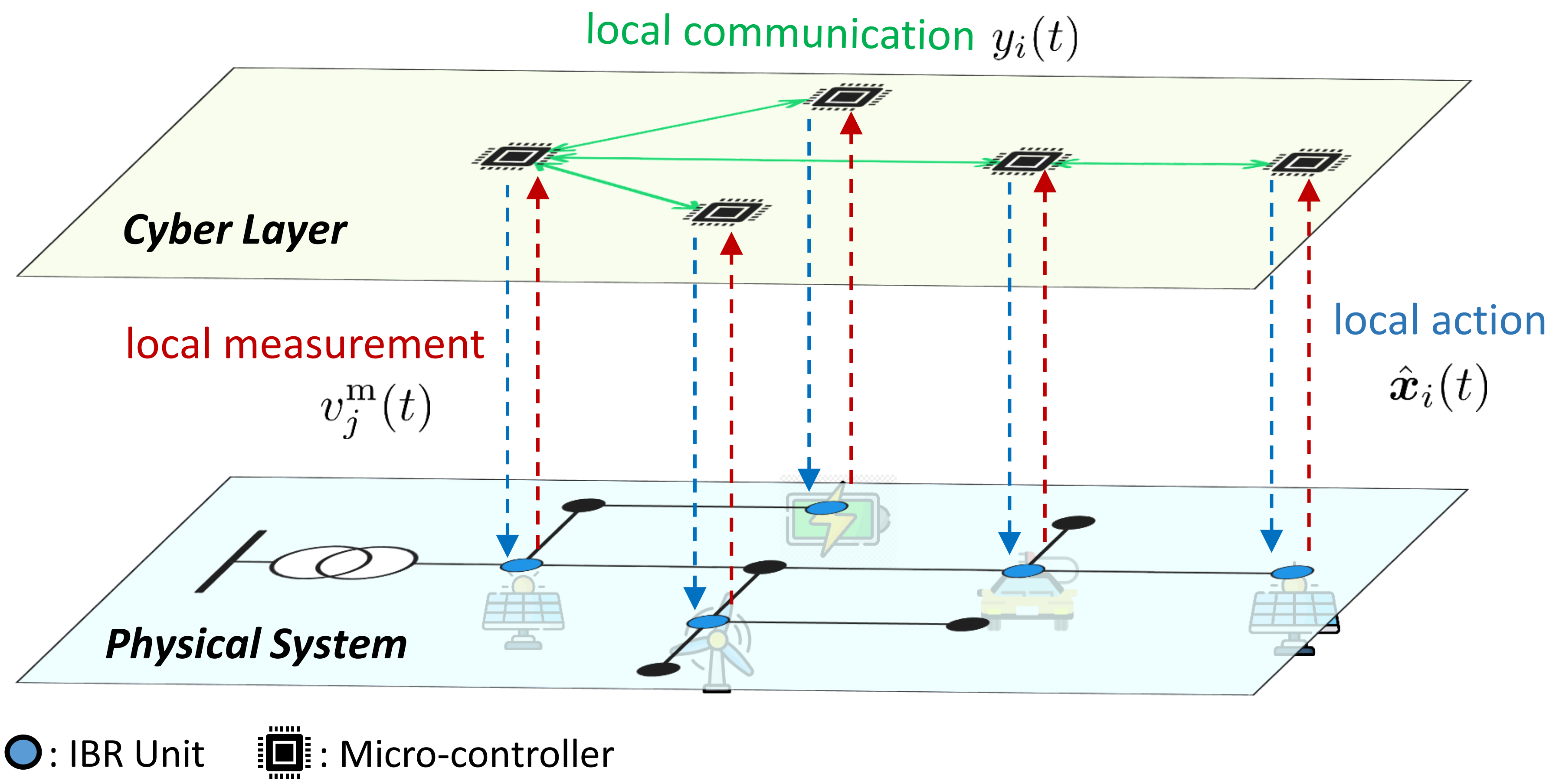
Projected Primal-Dual Zeroth-Order Dynamics (P-PDZD)



Projected Primal-Dual Zeroth-Order Dynamics (P-PDZD)



Control Process Illustration



- ✓ **Optimality**
- ✓ **Safety**
- ✓ **Model-Free**
- ✓ **Distributed**
- ✓ **Self-adaptive**

Real-time physical system feedback + Zeroth-order gradient learning

Theoretical Guarantees

[**Chen**, et al, arXiv 2023] [**Chen**, et al, CDC 2021]

- **(Semi-Global Practical Asymptotical Stability)**

Theorem 1. (informal) Under assumptions of convexity, with feasible initial condition in a compact set, there exists a class-*KL* function β such that for any precision $\nu > 0$, with sufficiently small $(\epsilon, a, \varepsilon_\omega, \epsilon_p)$, the trajectory $\boldsymbol{z}(t)$ of the **P-PDZD** satisfies

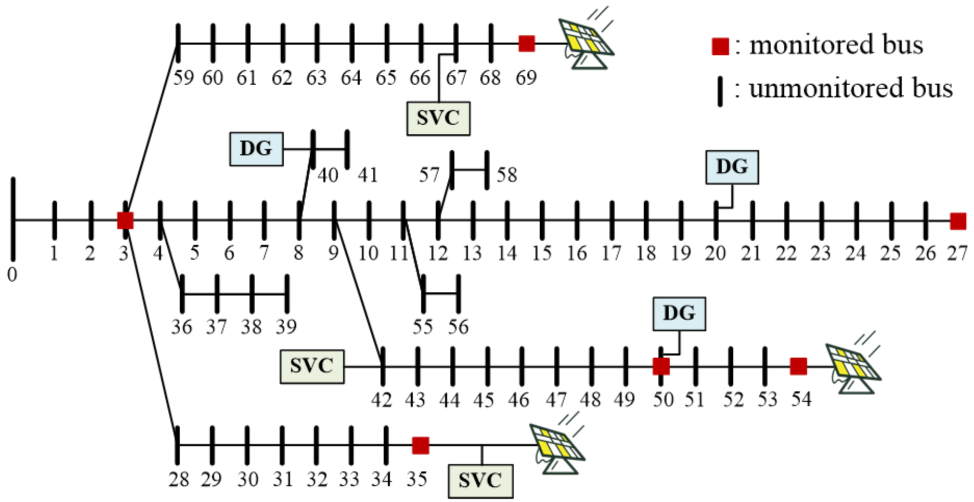
$$||\boldsymbol{z}(t)||_{\hat{\mathcal{A}}} \leq \beta(||\boldsymbol{z}(0)||_{\hat{\mathcal{A}}}, t) + \nu, \quad \forall t \geq 0.$$

- **(Structural Robustness to Small Measurement Noise)**

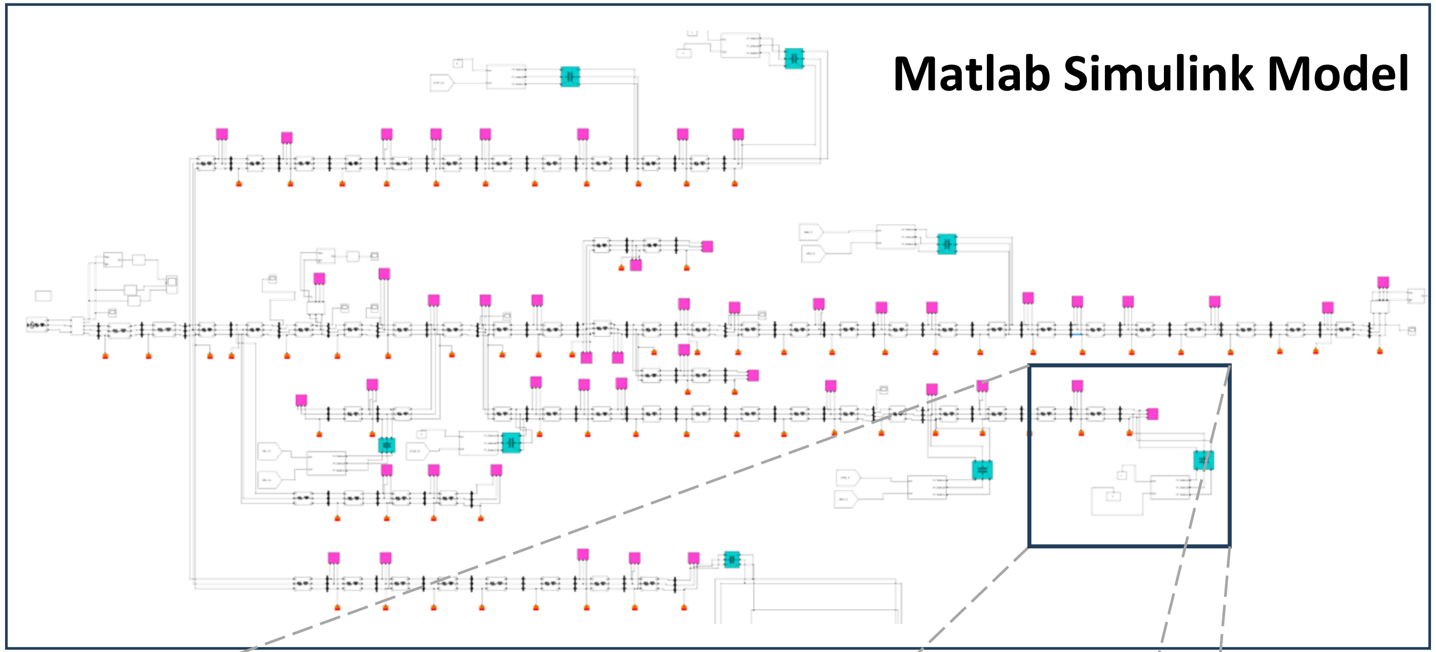
Theorem 2. (informal) Under the same conditions in Theorem 1, there exists $\rho^* > 0$ such that for any measurement noise $\boldsymbol{d}$ with $\sup_{t \geq 0} ||\boldsymbol{d}(t)|| \leq \rho^*$, the trajectory $\boldsymbol{z}(t)$ of the **P-PDZD** with additive measurement noise satisfies

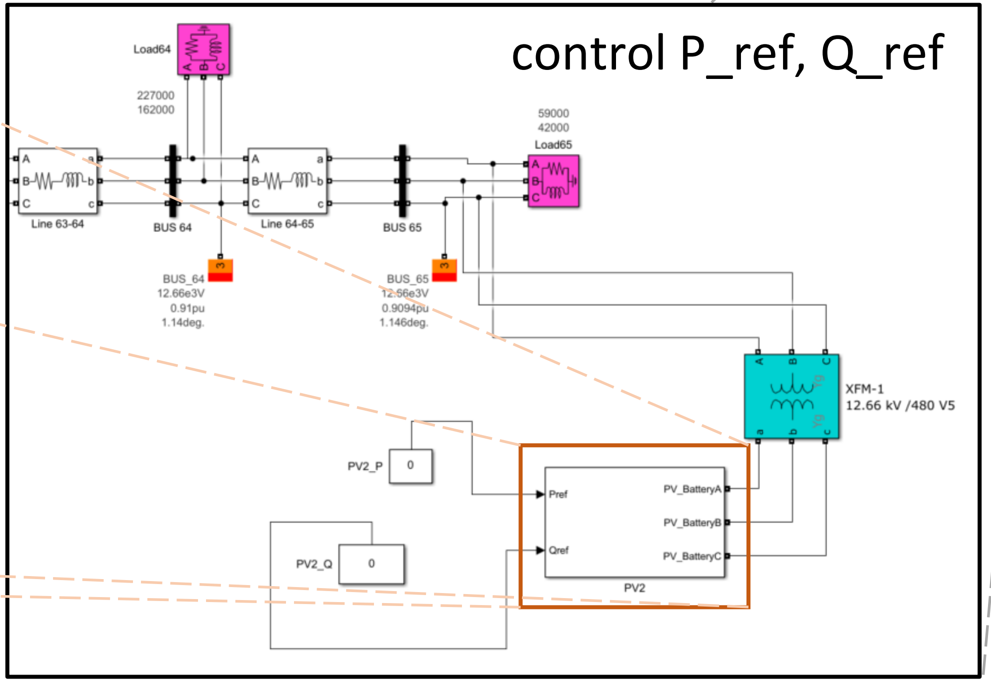
$$||\boldsymbol{z}(t)||_{\hat{\mathcal{A}}} \leq \beta(||\boldsymbol{z}(0)||_{\hat{\mathcal{A}}}, t) + 2\nu, \quad \forall t \geq 0.$$

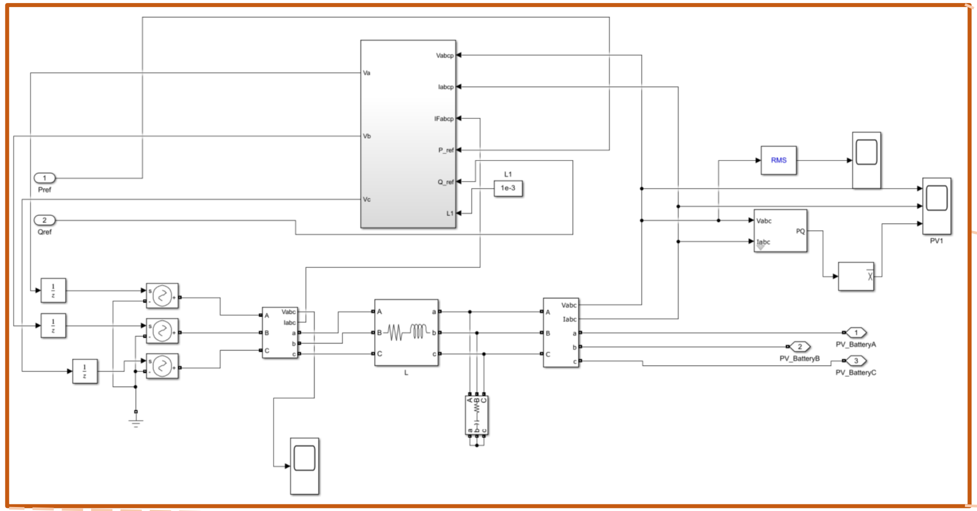
Simulation



3-Phase IEEE 69-Bus Distribution System



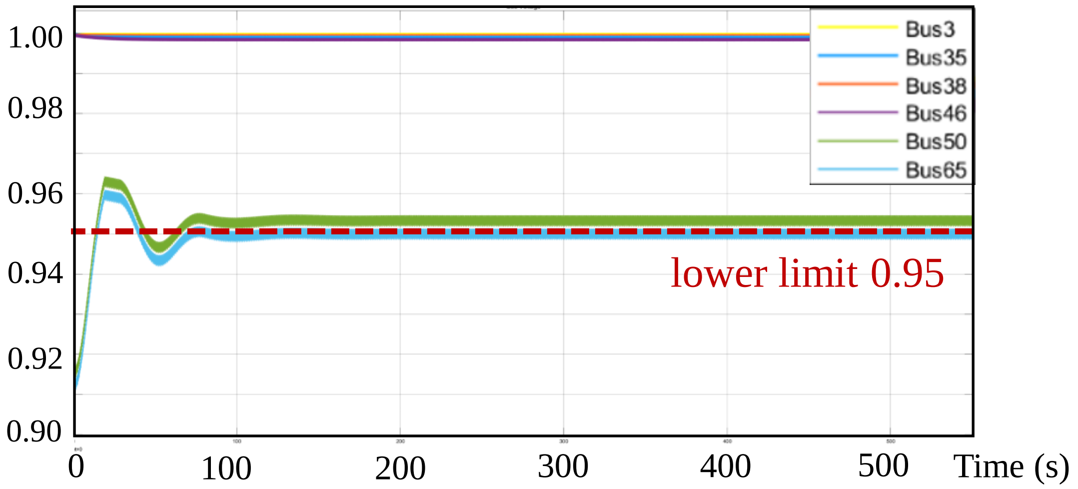




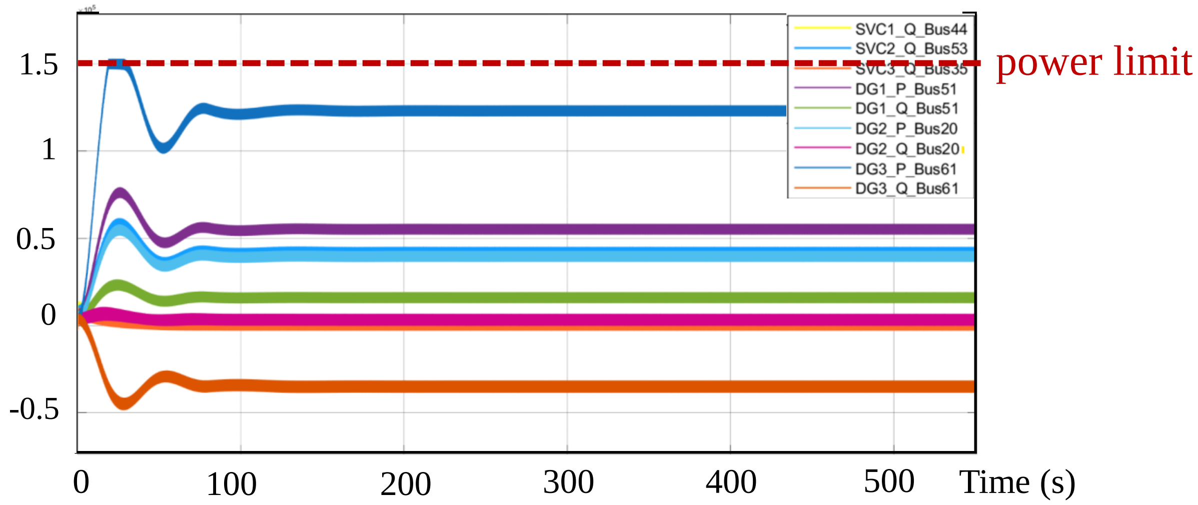
high-fidelity EMT inverter model

Case 1. Step Power Disturbance

Bus Voltage Magnitude (p.u.)

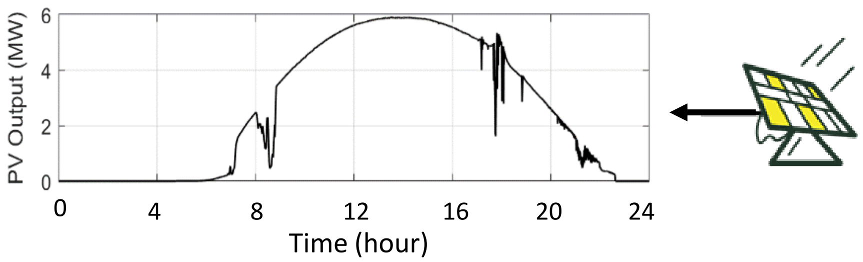


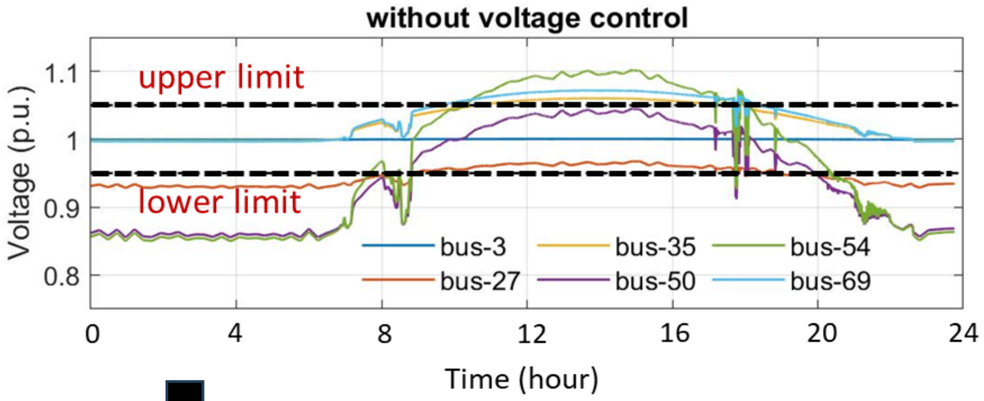
Flexible DER Power Outputs (MW/Mvar)

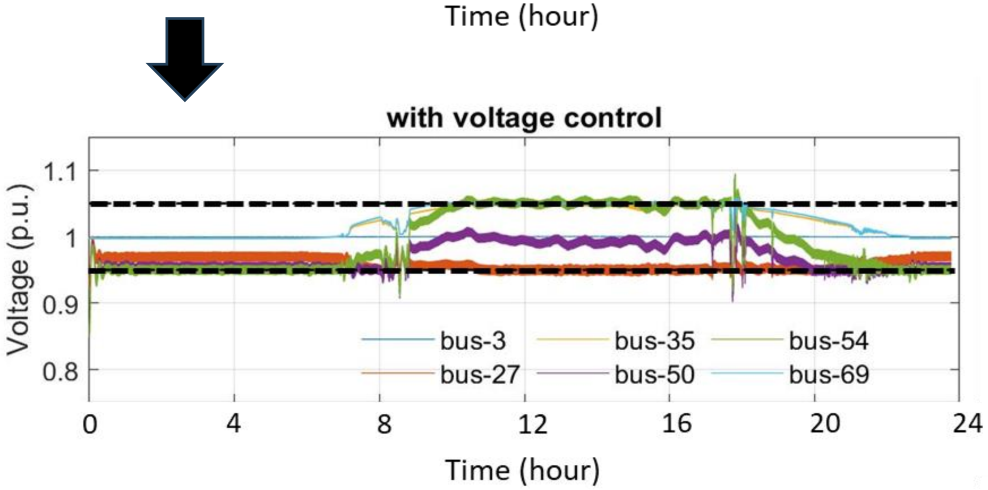


Case 2. Continuous Power Disturbance

Real time-varying solar generation







Projected Primal-Dual Zeroth-Order Method

Feedback Optimization **Zeroth-order Gradient Estimation**

(Physical System Feedback + Control Law = Optimization Algorithm)

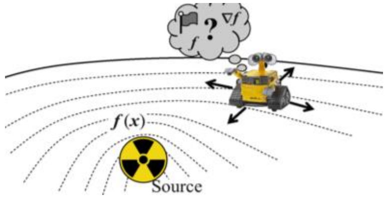
➢ **Real-time model-free optimal control** of complex (multi-agent) physical systems

- ✓ Unknown system model
- ✓ Scalability ✓ Safety
- ✓ Performance guarantees

Building Energy Control

Wind Farm Control

Source Seeking



Resource Allocation

[1] **X. Chen**, J. I. Poveda, N. Li, "Continuous-Time Zeroth-Order Dynamics with Projection Maps: Model-Free Feedback Optimization with Safety Guarantees", 2024. (accepted by IEEE Trans. Automatic Control)

This Talk

I. Model-free Optimal Voltage Control in Power Distribution Systems

II. Improve Single-point Zeroth-order Optimization Using Filters

Zeroth-Order Optimization (ZO)

$$\min_{\boldsymbol{x}\in\mathbb{R}^d} f(\boldsymbol{x})$$

Single-Point ZO

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \eta \underbrace{\frac{d}{r} f(\boldsymbol{x}_k + r\boldsymbol{u}_k)\boldsymbol{u}_k}_{\text{single-point gradient estimator}}$$

- ✔ one function evaluation → online problems
- ✘ large variance and slow convergence.

Two-Point ZO

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \eta \underbrace{\frac{d}{r} \frac{f(\boldsymbol{x}_k + r\boldsymbol{u}_k) - f(\boldsymbol{x}_k - r\boldsymbol{u}_k)}{2}\boldsymbol{u}_k}_{\text{two-point gradient estimator}}$$

- ✘ impractical for online or dynamic settings.

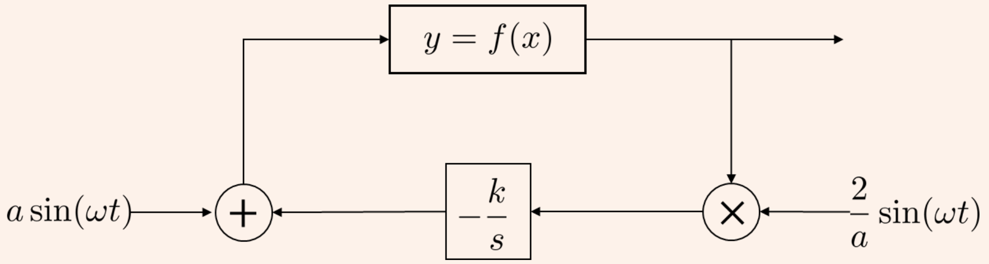
How to Improve Single-point ZO?

Close connection between ***ES control*** *and* ***Single-point ZO***

Extremum Seeking Control (ESC)

$$\dot{\boldsymbol{x}} = -k \cdot \frac{2}{a} f(\boldsymbol{x} + a\sin(\boldsymbol{\omega} t))\sin(\boldsymbol{\omega} t)$$

continuous-time dynamics

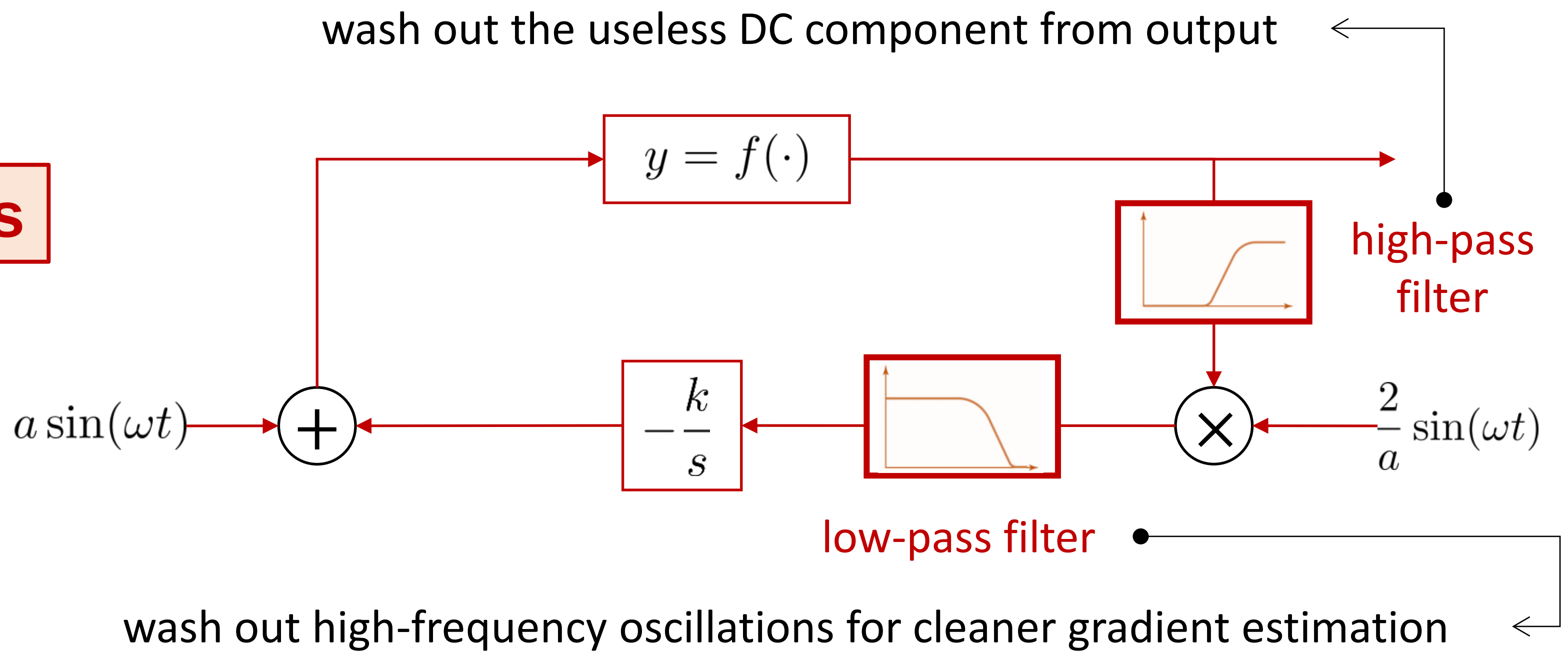


$$\min_x f(x)$$

Extremum Seeking Control	VS	Single-Point ZO
$\dot{x} = -k \cdot \frac{2}{a} f(x + a\sin(\omega t))\sin(\omega t)$		$x_{k+1} = x_k - \eta \cdot \frac{d}{r} f(x_k + r z_k) z_k$
continuous-time	$\longleftrightarrow$	discrete time
deterministic probing signal $\sin(\omega t)$	$\longleftrightarrow$	random perturbation direction z
probing magnitude a	$\longleftrightarrow$	smoothing radius r
control gain k	$\longleftrightarrow$	step size η

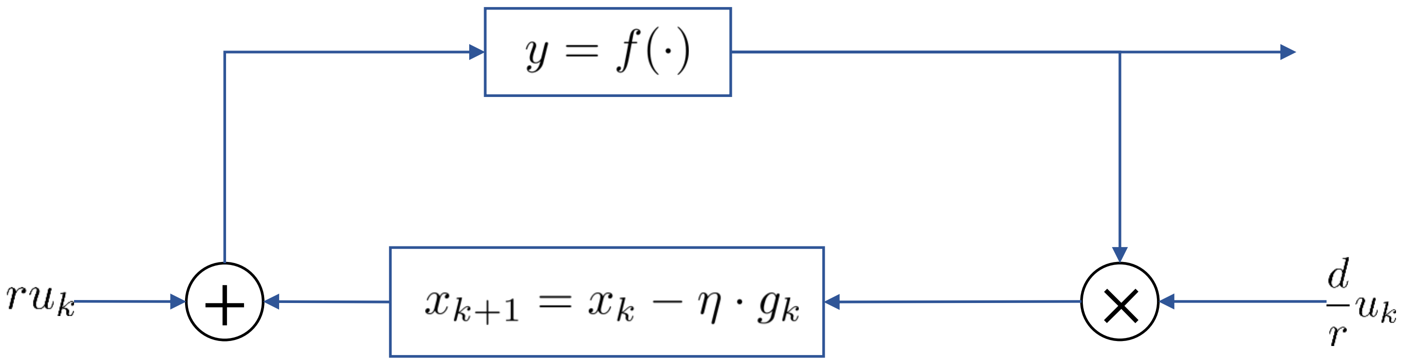
ES Control + Filters

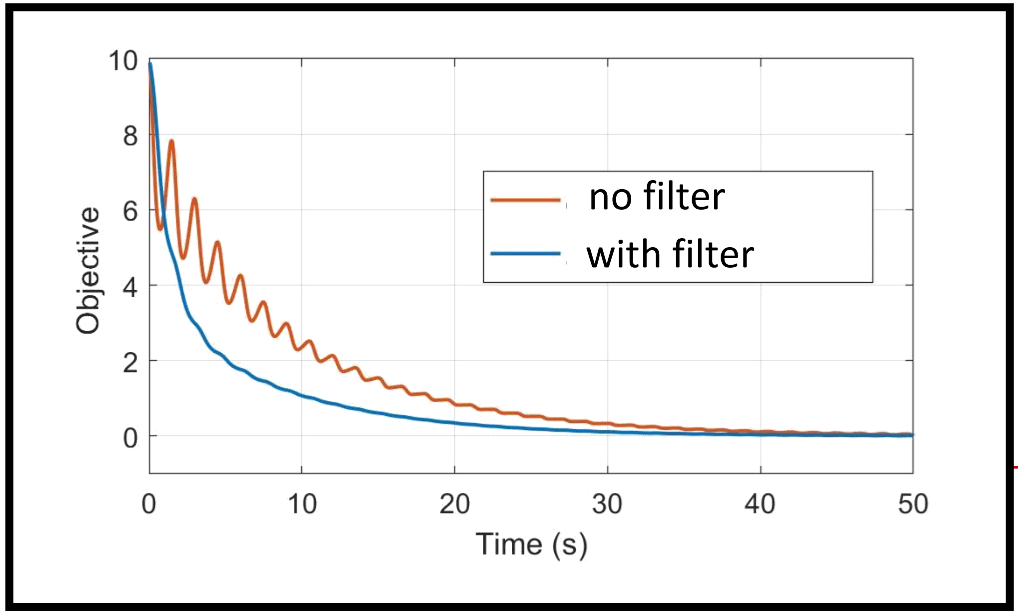
$$\dot{\boldsymbol{x}} = -k \cdot \frac{2}{a} f(\boldsymbol{x} + a\sin(\boldsymbol{\omega} t))\sin(\boldsymbol{\omega} t)$$



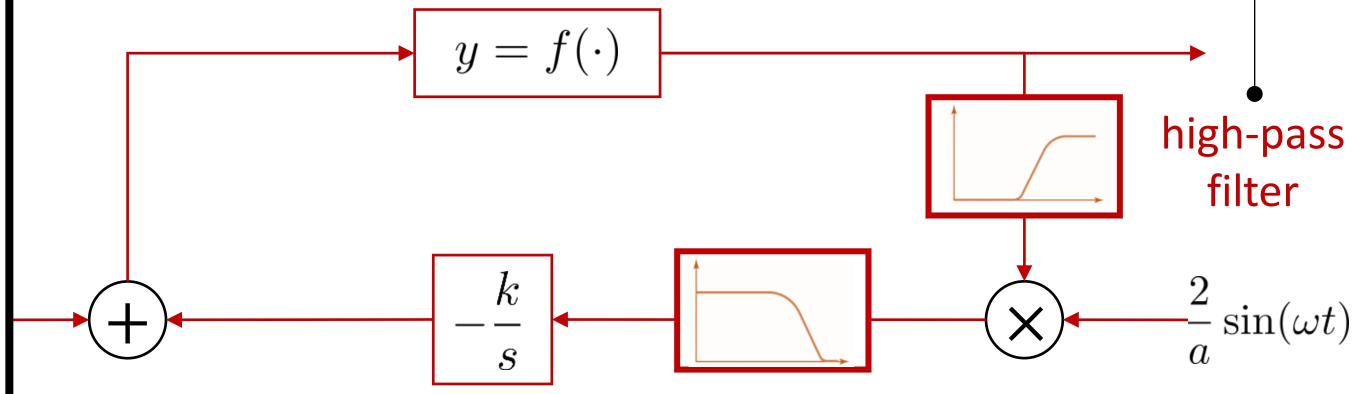
Single-point ZO (SZO)

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \eta \frac{d}{r} f(\boldsymbol{x}_k + r\boldsymbol{u}_k)\boldsymbol{u}_k$$





wash out the useless DC component from output

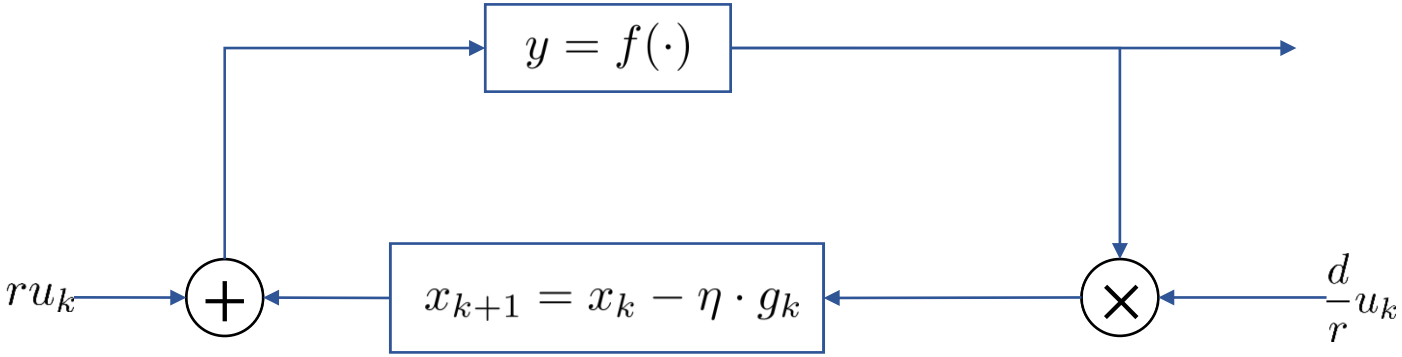


low-pass filter

wash out high-frequency oscillations for cleaner gradient estimation

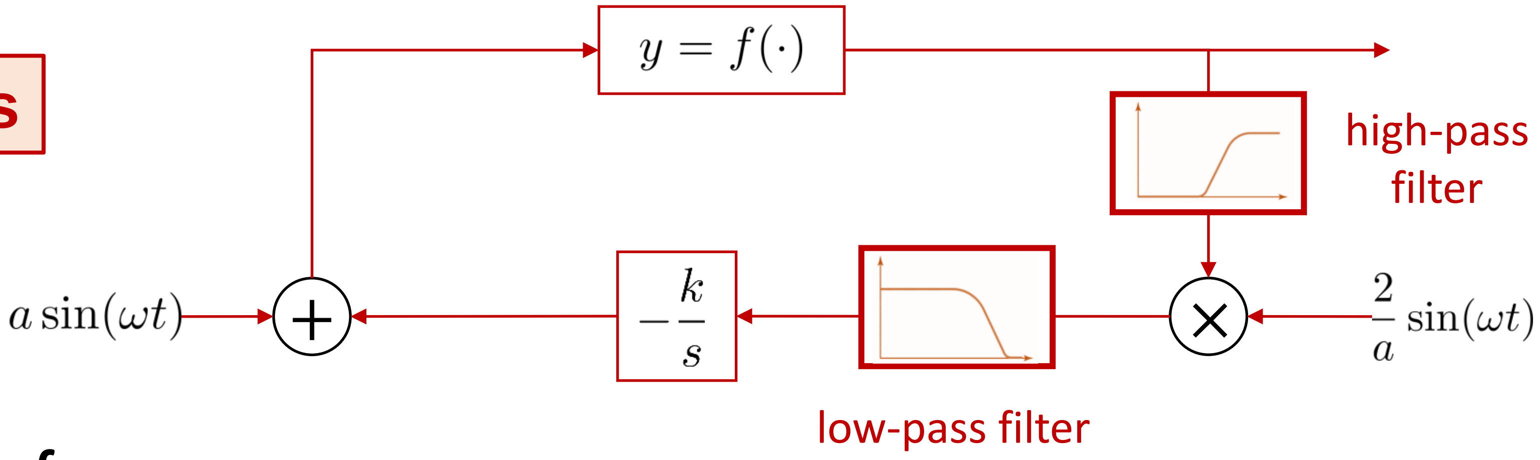
Single-point ZO (SZO)

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \eta \frac{d}{r} f(\boldsymbol{x}_k + r\boldsymbol{u}_k)\boldsymbol{u}_k$$



ES Control + Filters

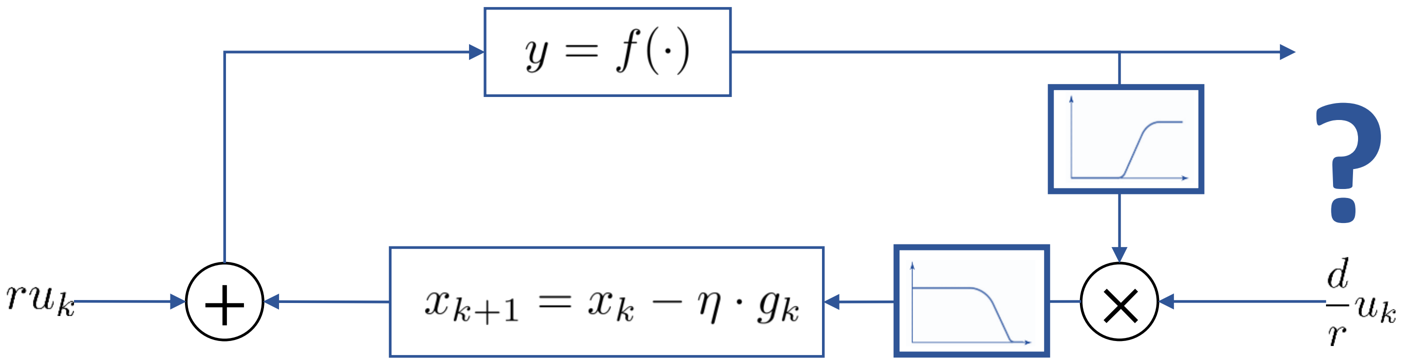
$$\dot{\boldsymbol{x}} = -k \cdot \frac{2}{a} f(\boldsymbol{x} + a\sin(\boldsymbol{\omega} t))\sin(\boldsymbol{\omega} t)$$



Can we borrow the idea of high-pass and low-pass filters to improve SZO?

Single-point ZO (SZO)

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \eta \frac{d}{r} f(\boldsymbol{x}_k + r\boldsymbol{u}_k)\boldsymbol{u}_k$$



“Can we borrow the idea of high-pass and low-pass filters to improve SZO?”

YES!

Vanilla **SZO**

$$x_{k+1} = x_k - \eta \frac{d}{r} f(x_k + r u_k) u_k$$

$+$

High-pass **F**ilter

$$\frac{s}{s+\omega_H}$$

&

Low-pass **F**ilter

$$\frac{\omega_L}{s+\omega_L}$$

recycled from last iteration

$=$

Our proposed **HLF-SZO** [1]

$$x_{k+1} = x_k - \eta \cdot \frac{d}{r}\Big(f(x_k + r u_k) - f(x_{k-1} + r u_{k-1})\Big) u_k + \alpha(x_k - x_{k-1})$$

residual feedback [2]

“momentum”

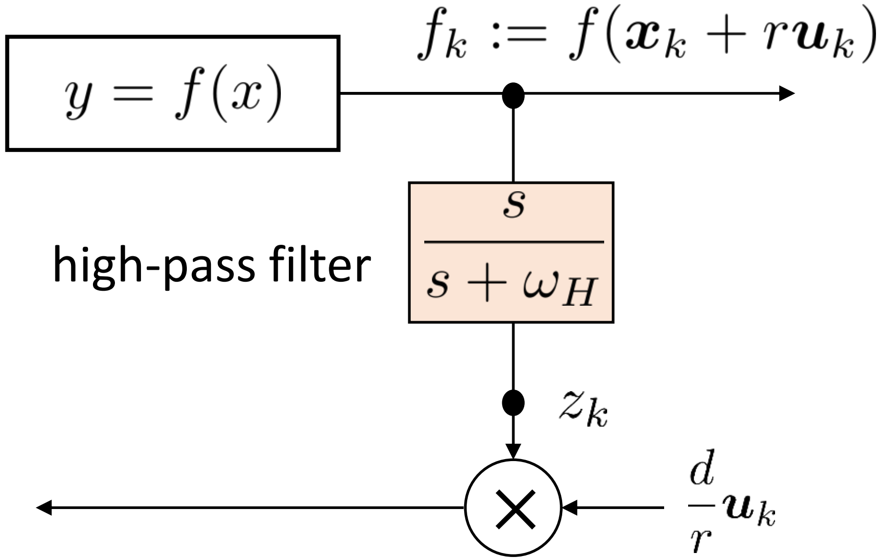
[1] **X. Chen**, Y. Tang, N. Li, “Improve Single-Point Zeroth-Order Optimization Using High-Pass and Low-Pass Filters”, ICML, 2022.
[2] Yan Zhang, et al. A new one-point residual-feedback oracle for black-box learning and control. Automatica, 2021.

Derivation Process

residual-feedback SZO ← $\beta^* = 1$ ← **HF-SZO** → $\beta = 0$ → vanilla SZO

Vanilla SZO + **High-pass Filter** = **HF-SZO**

$$\boxed{\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \eta \frac{d}{r} f(\boldsymbol{x}_k + r\boldsymbol{u}_k)\boldsymbol{u}_k} + \boxed{\frac{s}{s+\omega_H}} = \begin{cases} z_k = (1-\beta) z_{k-1} + f(\boldsymbol{x}_k + r\boldsymbol{u}_k) - f(\boldsymbol{x}_{k-1} + r\boldsymbol{u}_{k-1}) \\ \boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \eta \cdot \frac{d}{r} z_k \boldsymbol{u}_k \end{cases}$$



$$\mathcal{L}\{z\} = \frac{s}{s+\omega_H}\mathcal{L}\{f\} \iff \dot{z} + \omega_H z = \dot{f}$$

↓ time discretization

$$\frac{z_k - z_{k-1}}{\delta} + \omega_H z_{k-1} = \frac{f_k - f_{k-1}}{\delta}$$

$$\implies z_k = (1 - \underbrace{\delta\omega_H}_{\beta}) z_{k-1} + f_k - f_{k-1}$$

Derivation Process

residual-feedback SZO ($\beta^* = 1$) ← **HF-SZO** → vanilla SZO ($\beta = 0$)

Vanilla SZO

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \eta \frac{d}{r} f(\boldsymbol{x}_k + r\boldsymbol{u}_k)\boldsymbol{u}_k$$

\+ **High-pass Filter** $\frac{s}{s+\omega_H}$ =

$$\begin{cases} z_k = (1-\beta) z_{k-1} + f(\boldsymbol{x}_k + r\boldsymbol{u}_k) - f(\boldsymbol{x}_{k-1} + r\boldsymbol{u}_{k-1}) \\ \boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \eta \cdot \frac{d}{r} z_k \boldsymbol{u}_k \end{cases}$$

\+ **Low-pass Filter** $\frac{\omega_L}{s+\omega_L}$ =

LF-SZO

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \eta \frac{d}{r} f(\boldsymbol{x}_k + r\boldsymbol{u}_k)\boldsymbol{u}_k + \underline{\alpha(\boldsymbol{x}_k - \boldsymbol{x}_{k-1})}$$

(momentum term)

Vanilla SZO + **High-pass Filter** + **Low-pass Filter** = **HLF-SZO**

High-pass Filter: **residual-feedback** (variance reduction)

Low-pass Filter: **momentum** (acceleration)

Performance Comparison

under the Lipchitz and smooth conditions

$$\min_{\boldsymbol{x}\in\mathbb{R}^d} f(\boldsymbol{x})$$

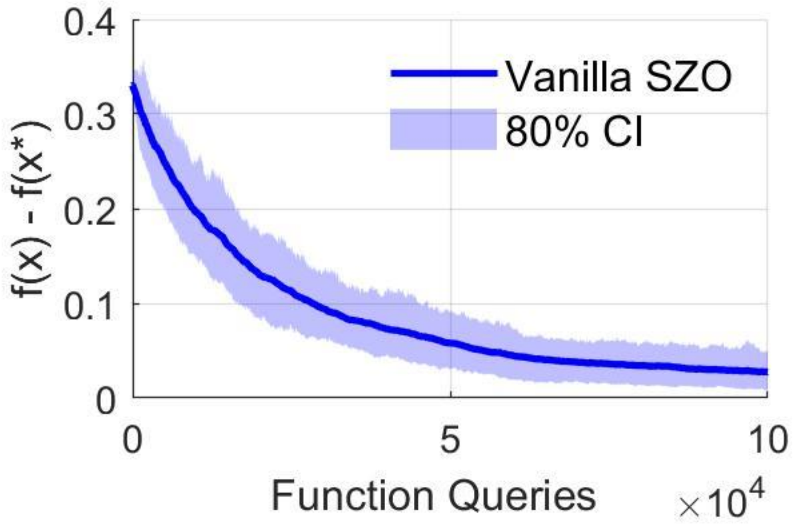
		Convex $\mathbb{E}(f(\overline{\boldsymbol{x}}_T)) - f(\boldsymbol{x}^*) \leq \epsilon$	Nonconvex $\frac{1}{T}\sum_{k=1}^{T}\mathbb{E}[\|\nabla f(\boldsymbol{x}_k)\|^2] \leq \epsilon$
Vanilla SZO	Gasnikov, Krymova, et al. (2017)	$\mathcal{O}(d^2/\epsilon^3)$	/
Residual-Feedback SZO	Zhang, Zhou, et al (2021)	$\mathcal{O}(d^2/\epsilon^{\frac{3}{2}})$	$\mathcal{O}(d^2/\epsilon^{\frac{3}{2}})$
HLF-SZO	Chen, Tang, Li (2022)	$\mathcal{O}(d^{\frac{3}{2}}/\epsilon^{\frac{3}{2}})$	$\mathcal{O}(d^{\frac{3}{2}}/\epsilon^{\frac{3}{2}})$
Two-Point ZO	Nesterov, Spokoiny (2017)	$\mathcal{O}(d/\epsilon)$	$\mathcal{O}(d/\epsilon)$

[2] **X. Chen**, Y. Tang, N. Li, "Improve Single-Point Zeroth-Order Optimization Using High-Pass and Low-Pass Filters", ICML, 2022.

Numerical Tests

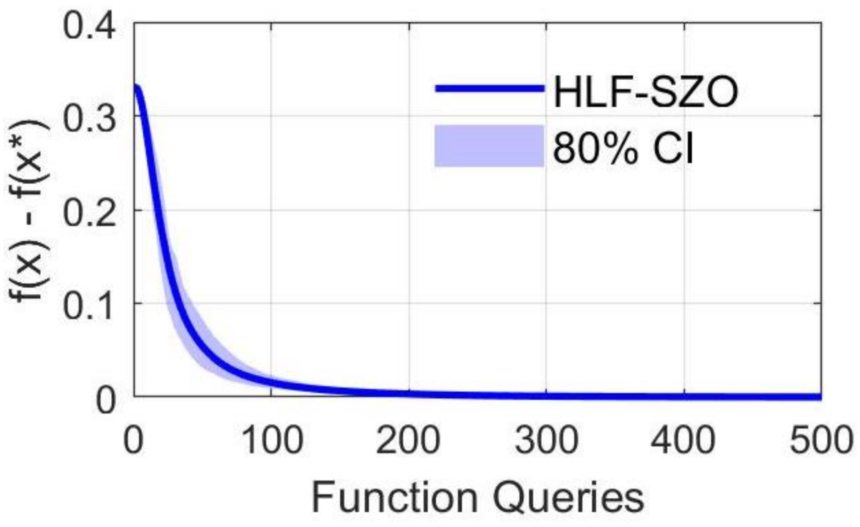
- **Logistic Regression:** (d=2, N=200) $\min_{\boldsymbol{x}\in\mathbb{R}^d} f(\boldsymbol{x}) = \frac{1}{N}\sum_{i=1}^{N}\log\left(1+\exp(-y_i\cdot A_i^\top \boldsymbol{x})\right)$

Vanilla SZO



+ Filters

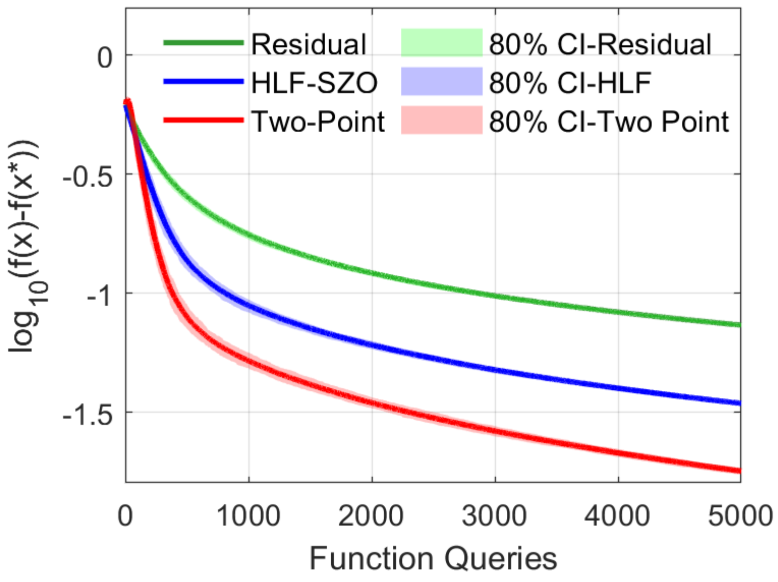
HLF-SZO



Case Studies (Residual SZO, HLF-SZO, Two-Point ZO)

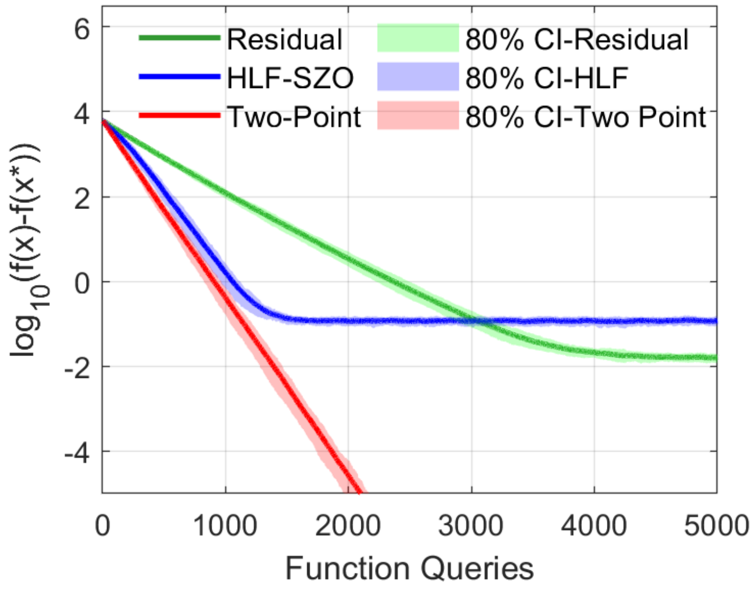
- **Logistic Regression (d=50, N=1000)**

$$\min_{\boldsymbol{x}\in\mathbb{R}^d} f(\boldsymbol{x}) = \frac{1}{N}\sum_{i=1}^{N}\log\left(1+\exp(-y_i\cdot A_i^\top \boldsymbol{x})\right)$$



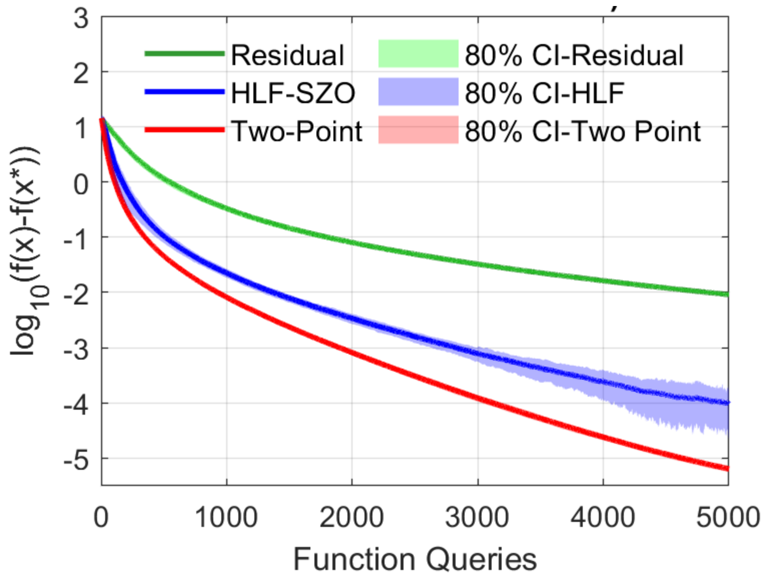
- **Ridge Regression (d=50, N=1000)**

$$\min_{\boldsymbol{x}\in\mathbb{R}^d} f(\boldsymbol{x}) = \frac{1}{2}||\boldsymbol{b}-H\boldsymbol{x}||_2^2 + \frac{1}{2}c||\boldsymbol{x}||_2^2$$



- **Minimize Beale function**

$$f(\boldsymbol{x}) = (1.5-x_1+x_1x_2)^2 + (2.25-x_1+x_1x_2^2)^2 + (2.625-x_1+x_1x_2^3)^2.$$



Takeaways

- Close connection between **Single-point Zeroth-order Optimization (SZO)** and **Extremum Seeking Control (ESC)**

- Borrow the idea of **high-pass and low-pass filters** from **ESC**, which significantly improves the convergence of **SZO**

Vanilla SZO: $\mathcal{O}(d^2/\epsilon^3)$ $\longrightarrow$ **HLF-SZO**: $\mathcal{O}(d^{\frac{3}{2}}/\epsilon^{\frac{3}{2}})$

Control $\longleftrightarrow$ **Optimization**

[2] **X. Chen**, Y. Tang, N. Li, “Improve Single-Point Zeroth-Order Optimization Using High-Pass and Low-Pass Filters”, ICML, 2022.