Simulation, Estimation, and Control of Gas Pipeline Systems

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Outline

• Motivation: coordination of electricity and gas delivery

• Modeling physics & engineering of gas pipelines

• Some history of developments in gas pipeline analysis

• Transient simulation

• Transient optimization and optimal control

• State and parameter estimation

• Robust optimal control
Motivation: coordination of electricity & gas delivery
Gas pipeline systems

Upstream
- Shale
- Offshore

Midstream
- Pipeline
- Compressor

Downstream
- City Utility
- Power plant

Continental U.S. gas transport pipeline system

Pipes and compressor stations
Emerging issue: variable loads

Traditional paradigm
• Maximize capacity of the pipeline under steady flow

Emerging issue
• Power grid increases variability of gas pipeline flows
Electricity production today

- Electricity production by source in the United States (2019)
  - Gas: 38.4%, coal: 23.5%, nuclear: 19.7%,
  - Renewable 17.5% (wind: 7.3%, solar 1.8%)

U.S. utility-scale electric generating capacity by initial operating year (as of Dec 2016)
gigawatts

- Significant construction of natural gas-fired power plants (Source: US EIA)
Filling the demand curve

• Gas-fired generation is used to fill the demand curve

Hourly electricity generation by fuel in PJM Interconnection (Jul 15-23, 2019) gigawatts

- Natural gas
- Coal
- Nuclear
- Other
- Other renewables
- Hydro

July 19, 6:00 p.m. ET
155 GW

• Requires gas-fired generators to ramp up production quickly
Energy systems are now coupled

- Power & gas transmission infrastructures are coupled through gas generators

- Gas pipeline loads are increasing, and becoming more variable/intermittent

- The coupling is strengthening, as seen in simultaneous price spikes (ISO New England)
Generation Fuel Mix in European Union

- Electricity production by source in European Union (2019)
  - Coal: 14.6%, natural gas: 21.7%, nuclear: 25.5%.
  - Wind: 13.4%, solar: 4.2%, biomass: 6.2%, hydroelectricity: 10.8%.

(Source: Eurostat)
Energy systems are now coupled

- Power & gas transmission infrastructures are coupled through gas generators
- Gas pipeline loads are increasing, and becoming more variable/intermittent

High Voltage Electricity Transmission

High Pressure Natural Gas Transport
Gas pipeline operations

• Natural Gas is traded in regulated markets
  – Bilateral transactions between buyers & sellers for steady ratable flows

• Transmission pipelines sell gas transportation to shippers (buyers and sellers)
  – Marketing and scheduling is time-consuming, not optimized
  – Human operators manage fragmented systems reactively, in real-time
  – Business processes are daily, not hourly
  – Business and operating standards vary by company

• Gas delivery may not adjust in real time
  – Possible disparity between scheduled and actual gas flows and pressures in normal operations
  – Limited ability to react to unplanned contingencies
Modeling physics & engineering of gas pipelines
Pipeline system basics

• Network nodes: physical nodes and custodial meter stations
• Network edges: pipes that connect nodes
• Compressors: machines that boost pressure
• Other elements: valves, regulators, resistors

• Management objectives: operational or economic
  – Operational: minimize cost of operations (energy use of compressors)
  – Economic: maximize profit of gas delivery to buyers minus cost of gas supplied

• Conducted subject to engineering constraints on gas pipeline network
  – Physics of pressure and flow on each pipe
  – Flow balance at nodes
  – Constraints on compressors

• Control parameters
  – Compressor and regulator setpoints
  – Nodal injections or withdrawals
Physics on a pipe

Isothermal Euler equations in one dimension:

- Mass conservation: \( \frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho u) = 0 \)
- Momentum balance: \( \frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2) + \frac{\partial}{\partial x} p = -\lambda \frac{|u|}{2D} - \rho g \sin(\theta) \)
- State equation: \( p = Z(p, T)RT\rho \)

\( \rho \equiv \) density (kg/m\(^3\)), \( p \equiv \) pressure (Pa), \( u \equiv \) velocity (m/s), \( D \equiv \) diameter (m), \( \lambda \equiv \) friction factor, \( \theta \equiv \) pipe angle (deg), \( Z \equiv \) gas compressibility factor, \( R \equiv \) ideal gas constant (J/kg K), \( T \equiv \) Temperature (K), \( g \equiv \) velocity (m\(^2\)/s)

- Compressibility: \( Z(p, T) = \left(1 + \frac{a_1 p 10^{a_2 G}}{T^{a_3}}\right)^{-1} \) (CNGA Formula)

Typical to assume isothermal, simplified flow in a horizontal pipe without shocks:

- \( a = \sqrt{ZRT} \) is constant speed of sound (m/s),
- Neglect advection term \( \frac{\partial}{\partial x} (\rho u^2) \) and set \( \theta = 0 \)
- Define flow rate \( \phi = \rho u \) (kg/m\(^2\)/s)

Simplified dynamic equations:

\[
\begin{align*}
\partial_t \rho + \partial_x \phi &= 0 \\
\partial_t \phi + \alpha^2 \partial_x \rho &= -\frac{\lambda |\phi|}{2D} \frac{\phi}{\rho}
\end{align*}
\]

Steady state solution:

\[
\begin{align*}
\rho_{ij}^2 - \bar{\rho}_{ij}^2 &= \frac{\lambda L}{D a^2} \phi_{ij} |\phi_{ij}| \quad \text{or} \\
p_{ij}^2 - \bar{p}_{ij}^2 &= \beta_{ij} \phi_{ij} |\phi_{ij}|, \text{ where } \beta_{ij} = \frac{\lambda a^2}{D}
\end{align*}
\]
A pipeline system model is an actuated PDE system on a metric graph:

- Set of nodes (junctions) $\mathcal{V}$ and edges (pipes) $\mathcal{E}$
- Edges $(i, j) \in \mathcal{E}$ of length $L_{ij}$, diameter $D_{ij}$, and friction coefficient $\lambda_{ij}$
- Flow $\phi_{ij}(t, x_{ij})$ and pressure $p_{ij}(t, x_{ij})$ on an edge $(i, j) \in \mathcal{E}$ are continuous functions of distance $x$ at all times $t$

Notations and definitions:

- Boundary pressures $p_{ij}(t) = p_{ij}(t, 0)$ and $\bar{p}_{ij}(t) = p_{ij}(t, L)$
- Boundary flows $\underline{\phi}_{ij}(t) = \phi_{ij}(t, 0)$ and $\overline{\phi}_{ij}(t) = \phi_{ij}(t, L)$
- Edges $(i, j) \in \mathcal{E}$ of length $L_{ij}$, diameter $D_{ij}$, and friction coefficient $\lambda_{ij}$
- Defined pressure (slack) nodes $j \in \mathcal{V}_S \subset \mathcal{V}$ with given pressure $p_j$ for $j \in \mathcal{V}_S$
- Defined flow nodes $j \in \mathcal{V}_D \subset \mathcal{V}$ with flow withdrawal (injection) $d_j$ for $j \in \mathcal{V}_D$
- Auxiliary nodal pressure variables $p_j$ for $j \in \mathcal{V}_D$
- Compressors for $(i, j) \in \mathcal{C} \subset \mathcal{E}$ modeled as boost from a node to pipe boundary: $\underline{\rho}_{ij}(t) = \alpha_{ij} \rho_i$ or $\overline{\rho}_{ij}(t) = \alpha_{ij} \rho_j$ as appropriate
Flow balance (Kirchhoff-Neumann conditions):

\[ d_j = \sum_{i \in \partial^+ j} X_{ij} \bar{\phi}_{ij} - \sum_{i \in \partial^- j} X_{ij} \phi_{ij} \quad \forall j \in \mathcal{V} \]

Compressor characteristic diagrams:

Reciprocating:

Model compressor stations as point objects:

Pressure boost by compressors:

\[ \rho_{ij}(t) = \alpha_{ij} \rho_i \quad \text{or} \quad \bar{\rho}_{ij}(t) = \bar{\alpha}_{ij} \rho_j \quad \forall (i,j) \in \mathcal{E} \]

Applied power:

\[ HP \propto \left( \frac{T}{m} \right) \left( \frac{1}{a} \right) \cdot \phi \cdot (\alpha^m - 1) \]
Modeling challenges and test networks

- Factors in actual gas pipelines
  - Systems are large, distributed, complex, with many degrees of freedom
  - Pressure, flow, and line pack changes propagate slowly; dynamics are highly nonlinear
  - Boundary flows are always changing; flow never stabilizes to steady-state
  - Thermal effects are highly localized (near compressors)
  - Flow scheduling and compressor operations do not use optimization or model-based engineering
  - Experience-based decisions and labor intensive control by human operators

- Detailed system models
  - Complex, adjustable topology
  - Many control points: Compressors, valves, resistors, regulators
  - Tractable for steady-state modeling
  - Mixed-integer optimization

- Simplified system models
  - Constant topology
  - Few major control points: Compressors
  - Tractable for optimal control
  - Continuous optimization


Some history of developments in gas pipeline analysis
1951: mathematics of fluid flow in pipes

• I. A. Charnyj, 1951
• Gubkin Russian State University of Oil and Gas
• “Unsteady Motion of Real Fluids in Pipes”
  – Covers detailed physics modeling and applied mathematics of 1 dimensional hydraulics
  – Analytical methods that utilize linearization, Fourier series, perturbations, and asymptotic approaches for solutions
  – Extensive analysis of shock and wave effects
  – Based on methods developed by Leonid Leibenzon in 1930’s

Чарный, И. А. "Неустановившееся движение реальной жидкости в трубах." (1951).
1951: graphical computation of gas flow in pipes

- R.H. Olds, Naval Ordnance Test Station
  Inyokern, China Lake, CA
  B. H. Sage, Caltech, 1951
- AIME Petroleum Transactions
- “Transient Flow in Gas Transmission Lines”
  - Study supported by the Stearns-Roger Manufacturing Co., the Southern Counties Gas Co. of California, and Southern California Gas Company
  - Method of graphical operations
  - Approximates compressible gas flow in long pipes with varying boundary conditions
  - Applicable to looped/branched lines

1953: numerical scheme for gas flow in one dimension

- Douglas Hartree, Harvard University
- Subcontract Sc-7 of Contract W-7405-eng-36, Los Alamos Scientific Laboratory
- Report LA-HU-1 1953
  - Compares three methods of using characteristics for the evaluation of solutions of equations of non-steady isentropic compressible flow in one space variable
  - One method involves a grid of characteristics in terms of Eulerian variables
  - Two methods use properties of characteristics to relate the flow at the beginning and end of a given time-interval (using Eulerian and Lagrangian variables, respectively)
  - Early computations done on ENIAC, University of Pennsylvania

1962: validated simulation of gas flow in pipelines

- T. D. Taylor, N. E. Wood, J. E. Powers
- University of Oklahoma, 1962
- Society of Petroleum Engineers
  - The U. of Oklahoma Computer Laboratory, Data Processing Center of Texas Engineering Experiment Station
  - Data supplied by Lone Star Gas Co.
  - Mass and momentum equations
  - Single pipe with constant inlet pressure and varying outlet flow
  - Compares orders of magnitude in terms of momentum balance equation
  - Characteristic scheme with finite difference, including stability criteria
  - Programmed on IBM 650 & IBM 709
  - Example of 44.9 miles pipe of 8.15 inch diameter with gas of specific gravity $G=0.675$ at 50 degrees F
  - Compare simulation with field data

1968: pipeline optimization by dynamic programming

- Peter Wong and Robert Larson
- Information & Control Lab at Stanford Research Institute
  - Single pipeline with specified flow and multiple compressors
  - Minimize energy used for compressors (adiabatic energy equation)
  - Subject to steady-state flow equation, maximum compressor ratio, pressure constraints on each section of pipe

1982: pipeline state estimation study

- M. Chapman, R. Jones, A. Pritchard
- 1982 IFAC Symposium
- Control Theory Centre, University of Warwick and British Gas Corporation
  - Finite dimensional linear dynamic observer for reconstructing pressure profile in a gas pipeline.
  - Can be used to detect unmonitored outflows
  - Linearized about steady-state flow solution through a pipe
  - Finite dimensional state space model is developed based on Galerkin approximation
  - Dynamic observer is applied (Kwaakernak & Sivan, 1972)
  - Tracks pressure along the pipe and in time
  - Error decreases with time

1984: simulation of transients in large gas networks

- Andrzej Osiadacz, 1984
- Department of Control Eng., Warsaw Institute of Oil & Gas
- "Simulation of Transient Gas Flows in Networks"
  - Method of lines solution to mass & momentum equations
  - Ideal gas equation of state
  - Large networks: 50 nodes, 70 edges, 24 hours in <20 sec
  - Numerical comparison of relative order of magnitude of terms in momentum equation indicates validity of approximations

- Flow Equations:
  \[-\frac{\partial p}{\partial x} = \frac{\partial (\rho w)}{\partial t} + \frac{\lambda}{2D} \rho w^2 + \frac{\partial (\rho w^2)}{\partial x}\]
  \[-\frac{\partial p}{\partial t} = c^2 \frac{\partial (\rho w)}{\partial x}\]

- Let \( \Delta P = \int_0^L \frac{\partial (\rho w)}{\partial t} dx + \int_0^L \frac{\lambda w^2}{D} \rho dx + (\rho w^2)_{x=L} - (\rho w^2)_{x=0} \)
- \( L=50 \text{ km}, p_0=5 \text{ MPa}, D=0.7 \text{ m}, p_L=4.7 \text{ MPa}, Q_v^* = 90.28 \text{ m}^3/\text{s} \)
- Terms \( \delta_1 = \frac{100}{\Delta P} \int_0^L \frac{\partial (\rho w)}{\partial t} dx \), \( \delta_2 = \frac{100}{\Delta P} \int_0^L \frac{\lambda w^2}{D} \rho dx \), \( \delta_3 = \frac{100}{\Delta P} (\rho w^2)_{x=L} - (\rho w^2)_{x=0} \)
  - Values: \( \delta_1 = 0.513\% \), \( \delta_2 = 162.3\% \), \( \delta_3 = 0.021\% \)

1985: gas pipeline transient optimization formulated

- V. Mantri, L. Preston, C. S. Pringle, 1985
- Scientific Software-Intercomp
- Pipeline Simulation Group Annual Meeting
  - Minimize
    \[
    F = W_1 \int_{T_2}^{T_1} \sum_{j=1}^{J} CHP_j(t) \, dt + W_2 \sum_{j=1}^{J} \sum_{i=1}^{I} CSC_{j,i}
    \]
    Where:
    - \(W_1, W_2\) = objective term weights
    - \(F\) = objective function
    - \(T_0, T_1\) = initial and terminal time
    - \(J\) = # of compressor stations
    - \(I\) = # of compressor units per station
    - \(CSC\) = cost of compressor unit status change
    - \(CHP\) = cost of applied horsepower
  - Subject to:
    - designated flow rate schedules at receipt and delivery points;
    - feasible operating ranges at stations;
    - physical behavior of gas in pipelines

1988: model predictive control for gas pipelines

- D. Marques and M. Morari
- Automatica, 1988
  - Simulator-based model-predictive optimal control for pipeline systems
  - Simulation results obtained with the simulator GANESI developed at TU Munich
  - Moving horizon optimizer with hierarchically decomposed control
  - At each time window, repeated executions of simulator are used to compute objective and dynamics, and successive quadratic programming (SQP) searches for an optimal policy
  - Inequality constraints reflect compressor operating envelope

1994: implicit transient gas flow simulation

- Tatsuhiko Kiuchi, Toyo Engineering Corporation, 1994
- “An implicit method for transient gas flows in pipe networks”
  - Applied to networks using an iterative fixed point scheme for junctions
  - Stability guarantees
  - Comparison with Crank-Nicolson method, method of characteristics, Lax-Wendroff method
  - Method exhibits much less oscillatory behavior than previous explicit methods
  - Used as a basis of comparison in many subsequent simulation studies

2000: simulator based transient optimization

- Henry Rachford, Richard Carter, 2000
- Stoner Associates, Inc.
- Pipeline Simulation Interest Group
- “Optimizing Pipeline Control in Transient Gas Flow”
  - Presents a solution to the transient gas pipeline optimization problem
  - Minimize total compressor energy expended
  - Transitions system from initial to target state
  - Only controls available to pipeline operator are optimized (i.e. compressor stations)
  - Pressure constraints and compressor horsepower limitations
  - Linepack is managed to meet required deliveries
  - Proposes comprehensive pipeline management system

2003: transient optimization by nonlinear program

- Erhardt and Steinbach, 2003
- Zuse Institute Berlin
- “Nonlinear Optimization in Gas Networks”
  - Operative planning problem in natural gas pipelines
  - Optimizes governing PDE equations with suitable discretization for transient optimization
  - Formulates an NLP based on KKT system
  - Explicit definition of gradient and Jacobians

- Minimize \[ \text{Cost} = \sum_{a \in A} c_s \int_0^{t_e} q z(p_{in}, T_{in}) \frac{\kappa}{\kappa - 1} \left[ \frac{p_{out}}{p_{in}} \right]^{\kappa - 1} - 1 \] \[ dt \]

- Subject to \[ \partial_t \rho + \partial_x q = 0, \]
  Euler Eq.: \[ \partial_t q + \partial_x p + \partial_x (\rho u^2) + g \rho \partial_x h = -\frac{\lambda(q)}{2D} \rho v |v|, \] \forall j \in P
  \[ p = \gamma(T)z(p, T) \rho \]
  State Eq.: \[ z(p, T) = 1 + 0.257(p/p_c) - 0.533(p/p_c)/(T/T_c) \]
  Flow balance, pressure limits, compressor power limits
  Terminal: \[ m_{\text{min}} \leq \sum_{j \in P} \int_0^{l_a} \rho_a(x, t_e) \] \[ dx \]

2005: simulator based transient optimization

- Virtual Pipeline System Testbed to Optimize the U.S. Natural Gas Transmission Pipeline System
  - U.S. Department of Energy Strategic Center for Natural Gas Award Number DE-FC26-01NT41322

- M. Abbaspour, Gregg Engineering & Kinder Morgan
- P. Krishnaswami & K. Chapman, Kansas State U
- Transactions of ASME, (published 2007)
  - Formulates an NLP on a small number of variables that determine setpoints of a high-fidelity simulation
  - Detailed compressor and non-isothermal physical flow modeling
  - Sequential unconstrained minimization technique (SUMT)
  - Minimizes compressor energy used to steer linepack to a target state

• After 2005: Significant growth in academic studies on modeling, model reduction, simulation, optimization, estimation, and optimal control for pipeline systems

• After 2010: Many academic studies on coordination and joint optimization of electricity and natural gas delivery networks
Transient simulation
Pipeline simulation: predictive analytics

- **Inputs**
  - Initial conditions (pressure and flow)
  - Either flow or pressure at each node over a time interval $T$

- **Simulation**
  - Initial value problem with unique solution

- **Outputs**
  - Flows and pressures throughout the system

\[
\begin{align*}
\phi(t, 0) &= s(t) \\
\phi(t, x) &= p(t, x) \\
\phi(t, L) &= d(t)
\end{align*}
\]
Explicit staggered grid approach

- Natural gas is highly non-ideal at high pressures (above 2 Mpa)
- New solver for nonlinear hyperbolic PDE on graphs
  - Developed by Gyrya et al. 2017-2020
  - Guaranteed stable when Courant-Friedrichs-Levy condition holds
  - Guaranteed mass conservation up to machine precision
  - Second order numerical accuracy
  - Can be used for large networks with arbitrary topology
  - Can be used for non-ideal gas physics
  - Explicit, parallelizable
- Nonlinear transformations between pressure and density
  - Apply nonlinear map to density variables in reduced model equations
  \[
  \rho = \frac{p(b_1 + b_2 p)}{R_g T} \\
  p = \frac{-b_1 + \sqrt{b_1^2 + 4b_2 R_g T \rho}}{2b_2}
  \]
- Compared with several other solvers
  - Kiuchi 1994 (implicit trapezoid method)
  - Dyachenko et al. 2017 (operator splitting)
  - Zlotnik et al. 2015 (lumped elements)
Explicit staggered grid approach

- New solver for actuated hyperbolic PDE on graphs
  - Fast
  - Provably stable
  - 2nd order accurate
  - Explicit
  - Handles compressibility

- Discretization
  - Staggered grid
  - Nodal and compressor conditions preserved exactly

\[
\begin{align*}
\frac{\partial_t \rho + \partial_x \phi}{\rho} &= 0 \\
\frac{\partial_t \phi + a^2 \partial_x \rho}{\phi} &= -\frac{\lambda}{2D} \frac{\phi \mid \phi}{\phi} \\
\rho_i^{n+1} - \rho_i^n + \frac{\phi_j^m - \phi_{j-1}^n}{\Delta x} &= 0. \\
\phi_j^{n+1} - \phi_j^m + \frac{p_i^{n+1} - p_i^n}{\Delta x} &= -\beta \frac{\phi \mid \phi}{\rho_{i+1}^{n+1}}, \quad \beta = \lambda / (2D) \\
\frac{\phi_j^{n+1} - \phi_j^n}{\rho_j^{n+1} + \rho_{i+1}^{n+1}} &= \frac{\phi_j^m - \phi_j^{n-1}}{\rho_i^{n+1} + \rho_{i+1}^{n+1}} = \phi_j^m - \Delta t \frac{p_i^{n+1} - p_i^{n-1}}{\rho_{i+1}^{n+1} + \rho_{i+1}^{n+1}} - \beta \Delta t \frac{\phi \mid \phi}{\rho_{i+1}^{n+1} + \rho_{i+1}^{n+1}}
\end{align*}
\]

Explicit staggered grid approach

• Single pipe comparison of ideal and non-ideal gas modeling

Explicit staggered grid approach

- **Test network simulation**
  - Test network with 5 nodes, 5 pipes, 3 compressors
  - Mass balance preserved to machine precision

Simulation of rapid depressurization

Boundary Conditions for Damage
• Change boundary condition at location node $j$ from time $t_d$ of depressurization to $p_j(t) = p_{atm}$ for $t \geq t_d$

Boundary Conditions for Containment
• Set flow at upstream and downstream pipe endpoints to $\phi_{ij}(t) = 0$ and $\phi_{jk}(t) = 0$ for $t \geq t_c$, where $t_c = t_d + t_\Delta$ is the valve closing time and $t_\Delta$ is the time elapsed until operators take action

Pipeline flow in rapid depressurization

Simulation of pressure, first 50 seconds after rupture

Pressure vs Distance, m

Pressure, Pa

0 1000000 2000000 3000000 4000000 5000000 6000000

0 2000 4000 6000 8000 10000 12000 14000 16000 18000 20000

A B

sec -1 Pressure sec 0 Pressure sec 1 Pressure sec 2 Pressure sec 3 Pressure sec 5 Pressure sec 6 Pressure sec 7 Pressure sec 8 Pressure sec 9 Pressure sec 10 Pressure sec 11 Pressure sec 20 Pressure sec 30 Pressure sec 40 Pressure sec 50 Pressure
Pipeline flow in rapid depressurization

Simulation of mass flow, first 50 seconds after rupture
Pipeline flow in rapid depressurization

Simulation of temperature, first 50 seconds after rupture

Temperature, degrees K

Distance, m

Temperature

sec -1 Temperature
sec 2 Temperature
sec 6 Temperature
sec 9 Temperature
sec 20 Temperature
sec 50 Temperature
sec 0 Temperature
sec 3 Temperature
sec 7 Temperature
sec 10 Temperature
sec 30 Temperature
sec 1 Temperature
sec 5 Temperature
sec 8 Temperature
sec 11 Temperature
sec 40 Temperature

Los Alamos National Laboratory
Model verification and validation

- Reduced model of subsystem used for capacity planning for a real pipeline
  - 78 nodes, 91 pipes, 4 compressors, 31 custody transfer meters at 24 locations (labelled A to X)
  - Hourly SCADA time-series of pressure and flow at meters for a month during “polar vortex” conditions

- Verification test
  - Specify pressure at A and flows leaving system elsewhere
  - Simulate to obtain incoming flow at A and pressures elsewhere (isothermal Euler equations with CNGA gas compressibility formula)
  - Compare measurements of pressure from simulation with pressure from sensor data
Verification and validation

- Comparing relative distance (%) of SCADA vs. simulation
- Pressure at B to X mean relative error: 2.47%
  - Mass inflow at node A mean relative error 0.317%
- Model is validated with less than 2.5% mean relative error

Transient optimization or optimal control
Transient optimization: decision analytics

- **Inputs**
  - Desired outlet flow
  - Objective (minimize compressor power)

- **Optimization**
  - A decision among many possibilities for the best solution

- **Outputs**
  - Control of pressure by compressors

- **Results**
  - Guarantee feasibility for inequality constraints
  - Optimal solution
Transient optimization: decision analytics

= **dynamic constraints** that specify system behavior

= **inequality constraints** that bound variables

Pipe properties

= Compressible fluid flow, represented by Euler equations in 1 dimension (momentum balance, mass conservation, equation of state)
= Maximum operating pressure

Node properties

= Mass balance of incoming and outgoing flows
= Minimum pressure
= Minimum and maximum supply and demand

Compressors

= Interface between nodes and pipes
= Maximum power output
Pipeline as conservation laws on directed metric graph

Junctions $j \in \mathcal{V} = \mathcal{V}_S \cup \mathcal{V}_D$ with given density $\sigma_j$ for $j \in \mathcal{V}_S$ and given flow (kg/s) withdrawal $d_j$ for $j \in \mathcal{V}_D$, auxiliary “nodal” density $\rho_j$ for $j \in \mathcal{V}$

Pipes $(i, j) \in \mathcal{E}$ of length $L_{ij}$, diameter $D_{ij}$, x-section area $X_{ij}$, friction coeff. $\lambda_{ij}$

Mass flux $\phi_{ij}(t, x_{ij})$ and density $\rho_{ij}(t, x_{ij})$ with

mass conservation: \[ \partial_t \rho_{ij} + \partial_x \phi_{ij} = 0 \]

momentum balance: \[ \partial_t \phi_{ij} + a^2 \partial_x \rho_{ij} = -\frac{\lambda}{2D} \frac{\rho_{ij} \phi_{ij}}{\rho_{ij}} \]

- Define boundary values
  \[ \rho_{ij}(t) = \rho_{ij}(t, 0), \quad \rho_{ij}(t) = \rho_{ij}(t, L_{ij}), \]
  \[ \phi_{ij}(t) = \phi_{ij}(t, 0), \quad \phi_{ij}(t) = \phi_{ij}(t, L_{ij}), \]

- Compressor action for $\forall (i, j) \in \mathcal{E}$:
  \[ \rho_{ij}(t) = \alpha_{ij}(t) \rho_i(t), \quad \rho_{ij}(t) = \alpha_{ij}(t) \rho_j(t) \]

- Flow balance for $\forall j \in \mathcal{V}$:
  \[ d_j(t) = \sum_{i \in \partial_+ j} X_{ij} \phi_{ij}(t) - \sum_{k \in \partial_- j} X_{jk} \phi_{jk}(t) \]
Approximation with nodal boundary conditions

Pipeline network model with pipelines $\mathcal{E}$ and junctions $\mathcal{V} = \mathcal{V}_S \cup \mathcal{V}_D$

Input and output flows $\phi_{ij}, \overline{\phi}_{ij}$ and densities $\rho_{ij}, \overline{\rho}_{ij} > 0$ for $(i, j) \in \mathcal{E}$

Density $\sigma_j$ specified at ‘slack’ nodes $j \in \mathcal{V}_S$ where $\rho_j \equiv \sigma_j$

Mass flow withdrawals (injections) $d_j$ given at junctions $j \in \mathcal{V}_D$

Compression $\alpha_{ij} \geq 1$ at junction $i \in \mathcal{V}$ compressing through pipe $(i, j) \in \mathcal{E}$

\[
d_j = \sum_{i \in \partial_+ j} X_{ij} \overline{\phi}_{ij} - \sum_{k \in \partial_- j} X_{jk} \phi_{jk} \quad \forall j \in \mathcal{V}_D
\]

\[
\rho_j = \sigma_j \quad \forall j \in \mathcal{V}_S
\]

\[
\partial_t \frac{\overline{\rho}_{ij} + \rho_{ij}}{2} = -\frac{\overline{\phi}_{ij} - \phi_{ij}}{L_{ij}} \quad \forall (i, j) \in \mathcal{E}
\]

\[
\partial_t \frac{\phi_{ij} + \overline{\phi}_{ij}}{2} = -\frac{\overline{\rho}_{ij} - \rho_{ij}}{L_{ij}} - \frac{L_{ij}}{4D_{ij}} \left( \frac{(\overline{\phi}_{ij} + \phi_{ij})(\overline{\phi}_{ij} + \phi_{ij})}{\overline{\rho}_{ij} + \rho_{ij}} \right) \quad \forall (i, j) \in \mathcal{E}
\]

\[
\rho_{ij} = \alpha_{ij} \rho_i, \quad \overline{\rho}_{ij} = \overline{\alpha}_{ij} \rho_j \quad \forall (i, j) \in \mathcal{E}
\]
Graph representation of flow balance

- Let $D = |\mathcal{V}_D|$, $E = |\mathcal{E}|$, and $S = |\mathcal{V}_S|$.
- Let $\pi_e : \mathcal{E} \to \{1, \ldots, E\}$ enumerate the edges, so $\phi_k = \phi_{ij}$, $\overline{\phi}_k = \phi_{ij}$, when $k = \pi_e(ij)$.
- Let $\phi, \overline{\phi} \in \mathbb{R}^E$ be boundary flows $\forall (i, j) \in \mathcal{E}$.
- Let $d \in \mathbb{R}^D$ be withdrawals $\forall j \in \mathcal{V}_D$.
- Incidence matrix $A$:
  $A_{ik} = \begin{cases} 
 1 & \text{edge } k \text{ enters node } i, \\
 -1 & \text{edge } k \text{ leaves node } i, \\
 0 & \text{else}
  \end{cases}$

- Let $A$ and $\overline{A}$ be negative and positive parts of $A_d$.
- Let $X = \text{diag}(X_k)$, with $X_k = X_{ij}$ for $k = \pi_e(ij)$.
- Then flow balance constraint is given as
  $$AX\phi + \overline{A}X\overline{\phi} = d.$$
Graph representation of density gradients

- Let $\pi_e : \mathcal{E} \to \{1, \ldots, E\}$ enumerate the edges, so $\rho_k = \rho_{ij}$, $\bar{\rho}_k = \bar{\rho}_{ij}$, when $k = \pi_e(i,j)$.
- Let $\bar{\rho}, \rho \in \mathbb{R}^E$ be boundary densities $\forall (i, j) \in \mathcal{E}$.
- Let $\sigma \in \mathbb{R}^S$ be slack pressures $\forall j \in \mathcal{V}_S$.
- Weighted incidence matrix $B$:

  $B_{ik} = \begin{cases} 
  \bar{\alpha}_{ij} & \text{edge } k = \pi_e(i,j) \text{ enters node } i, \\
  -\alpha_{ij} & \text{edge } k = \pi_e(i,j) \text{ leaves node } i, \\
  0 & \text{else}
  \end{cases}$

- Nodal and boundary densities are related by

  $|B_s^T|\sigma + |B_d^T|\rho = \bar{\rho} + \rho,

  B_s^T\sigma + B_d^T\rho = \bar{\rho} - \rho.
Matrix differential algebraic equations

Let $\Lambda, K, X \in \mathbb{R}^{E \times E}$ by $\Lambda = \text{diag}(L_k)$, $K = \text{diag}\left(\frac{\ell \lambda_k}{D_k}\right)$, $X = \text{diag}(X_k)$ for $k = \pi_e(i,j)$

Define $g : \mathbb{R}^E \times \mathbb{R}^E \to \mathbb{R}^E$ by pointwise action $g_j(x, y) = \frac{x_j |x_j|}{y_i}$

\[
\begin{align*}
    d &= AX\phi + AX\phi, \\
    |B_s^T|\dot{\phi} + |B_d^T|\dot{\rho} &= -4\Lambda^{-1}\frac{1}{2}(\phi - \bar{\phi}), \\
    \frac{1}{2}(\phi + \bar{\phi}) &= -\Lambda^{-1}(B_s^T\sigma + B_d^T\rho) - Kg\left(\frac{1}{2}(\phi + \bar{\phi}), |B_s^T|\sigma + |B_d^T|\rho\right)
\end{align*}
\]
Matrix differential algebraic equations

Let $\Lambda, K, X \in \mathbb{R}^{E \times E}$ by $\Lambda = \text{diag}(L_k)$, $K = \text{diag}\left(\frac{\ell \lambda_k}{D_k}\right)$, $X = \text{diag}(X_k)$ for $k = \pi_e(i,j)$

Define $g : \mathbb{R}^E \times \mathbb{R}^E \rightarrow \mathbb{R}^E$ by pointwise action $g_j(x, y) = \frac{x_j|y_j|}{y_i}$

\[
\begin{align*}
    d &= \bar{A}X\bar{\phi} + \underline{A}X\underline{\phi}, \\
    |B_s^T|\dot{\bar{\sigma}} + |B_d^T|\dot{\bar{\rho}} &= -4\Lambda^{-\frac{1}{2}}(\bar{\phi} - \underline{\phi}), \\
    \frac{1}{2}(\bar{\phi} + \underline{\phi}) &= -\Lambda^{-1}(B_s^T\bar{\sigma} + B_d^T\underline{\rho}) - Kg\left(\frac{1}{2}(\bar{\phi} + \underline{\phi}), |B_s^T|\bar{\sigma} + |B_d^T|\underline{\rho}\right)
\end{align*}
\]

Define $\phi = \frac{1}{2}(\bar{\phi} + \underline{\phi})$, $\phi_- = \frac{1}{2}(\bar{\phi} - \underline{\phi})$

Then $0 = \bar{A}X\bar{\phi} + \underline{A}X\underline{\phi} - d$ is equivalent to $0 = A_dX\phi + |A_d|X\phi_- - d$

Multiply second equation by $|A_d|X\Lambda$ and substitute first equation
Matrix differential algebraic equations

Let \( \Lambda, K, X \in \mathbb{R}^{E \times E} \) be \( \Lambda = \text{diag}(L_k), \ K = \text{diag}\left(\frac{\ell \lambda_k}{D_k}\right), \ X = \text{diag}(X_k) \) for \( k = \pi_e(i,j) \)

Define \( g : \mathbb{R}^E \times \mathbb{R}^E \rightarrow \mathbb{R}^E \) by pointwise action \( g_j(x, y) = \frac{x_j |x_j|}{y_i} \)

\[
\begin{align*}
\dot{d} & = \overline{A} X \overline{\phi} + A X \phi, \\
|B_s^T|\dot{\phi} + |B_d^T|\dot{\rho} & = -4\Lambda^{-1}\frac{1}{2}(\phi - \overline{\phi}), \\
\frac{1}{2} (\overline{\phi} + \phi) & = -\Lambda^{-1}(B_s^T \sigma + B_d^T \rho) - Kg\left(\frac{1}{2}(\phi + \overline{\phi}), |B_s^T|\sigma + |B_d^T|\rho\right)
\end{align*}
\]

Define \( \phi = \frac{1}{2}(\overline{\phi} + \phi), \ \phi_\omega = \frac{1}{2}(\overline{\phi} - \phi) \)

Then \( 0 = \overline{A} X \overline{\phi} + A X \phi - d \) is equivalent to \( 0 = A_d X \phi + |A_d| X \phi_\omega - d \)

Multiply second equation by \( |A_d| X \Lambda \) and substitute first equation

\[
\begin{align*}
|A_d|\Lambda|B_d^T|\dot{\rho} & = 4(A_d \phi - d) - |A_d|\Lambda|B_s^T|\dot{\sigma} \\
\dot{\phi} & = -\Lambda^{-1}(B_s^T \sigma + B_d^T \rho) - Kg(\phi, |B_s^T|\sigma + |B_d^T|\rho)
\end{align*}
\]
Modeling gas pipelines for control

- **Model-predictive optimal control of gas pipelines**
  - Old paradigm: Given predicted flow profiles, how to operate compressors such that pressure remains within set limits (if possible)?
  - Example system
  - 5 compressors, 8 loads, 1 source
  - 300 miles of pipes

Comparing discretization schemes

• **Alternative discretizations**
  – In space: trapezoidal rule (TZ) and lumped elements (LU)
  – In time: pseudospectral approximation (PS) and trapezoidal rule (TZ)
  – Tested by two-stage scheme

• **First stage**
  – minimizes compressor energy
  \[ C_1 \approx \sum_{P_i \in \mathcal{P}} \sum_{m=0}^{M} U_m S_{ij}^m. \]

• **Second stage**
  – minimizes solution variation
  \[ C_2 = \sum_{P_i \in \mathcal{P}} \sum_{m=0}^{M} \left( \frac{\partial^2 R_{ij}^m}{\partial t^2} \right)^2 \]
  \[ C_1 \leq (1 + r) \ell, \text{ where } 0 \leq r \leq 1 \]

Comparing discretization schemes

<table>
<thead>
<tr>
<th>Max relative difference (%) in pressure simulation and optimization: 24 Pipe</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Space-Time Schemes</strong></td>
</tr>
<tr>
<td>TZ-TZ</td>
</tr>
<tr>
<td>--------</td>
</tr>
<tr>
<td><strong>TZ-TZ</strong></td>
</tr>
<tr>
<td><strong>LU-TZ</strong></td>
</tr>
<tr>
<td><strong>TZ-PS</strong></td>
</tr>
<tr>
<td><strong>LU-PS</strong></td>
</tr>
</tbody>
</table>

Motivation

• Model-predictive optimal control of gas pipelines
  
  – **Old paradigm**: Given predicted flow profiles, how to operate compressors such that pressure remains within set limits (if possible)?
  
  – **New paradigm**: Given price/quantity bids of shippers, what is the best allocation of flows, and feasible compressor control, so pressure remains within limits (guaranteed)?
Gas balancing market (GBM)

- **Two-sided auction market**
  - Over the entire pipeline network
  - Suppliers and offtakers submit hourly price/quantity (P/Q) offers to sell/buy gas

- **Auctioneer’s objective function**
  - To maximize market surplus: payments for delivery minus costs of supply over the optimization horizon

- **Enforce physical feasibility of solution**
  - Dynamic constraints & Inequality constraints

- **Shadow prices in time and space**
  - Locational trade values (LTVs) of gas
  - Price separation occurs when pipe is at capacity

Gas storage modeling

- The reservoir is modelled as
  1) a domain with constant volume $V_s^R$
  2) gas density $\rho_s^r$ is homogeneous in domain
  3) temperature changes caused by injection $f_{s}^{bh}$ at bottom hole are small

- Reservoir density: $V_s^R \cdot \dot{\rho}_s^R(t) = f_{s}^{bh}(t)$

- Well flow:
  \[
  \partial_t \rho + \partial_x \varphi = 0 \\
  a^2 \partial_x \rho = \rho g - \frac{\lambda}{2D \rho} |\varphi| = 0
  \]

- Reservoir capacity: $m_s^{\text{min}} \leq \rho_s(t) \cdot V_s^R \leq m_s^{\text{max}}$

- Wellhead compressor:
  \[
  \rho_i(t) = \alpha_s \cdot \rho_s^{wh}(t) \\
  (\alpha_s^{\text{max}})^{-1} \leq \alpha_s(t) \leq \alpha_s^{\text{max}}
  \]

Transient optimization scalability study with storage

- **Questar system**
  - 506 nodes, 20 compressors, 4 gas storage facilities, 196 metered receipt/delivery points, 3490 km of pipelines
  - Solved transient optimization problem over 24 hours with 1 hour time discretization in under 3 minutes

- **Gas storage modeling**

  - well-head
  - well (discretized into sub-pipes)
  - bottom hole
  - reservoir density: $\rho_s^r(t)$
  - reservoir with fixed volume: $V_s^R$

- **Kanth Hari, Saikrishna and Kaarthik Sundar, Shriram Srinivasan, Anatoly Zlotnik, and Russell Bent.** "Operation of Natural Gas Pipeline Networks with Storage under Transient Flow Conditions." Submitted to IEEE Transactions on Control System Technology
State and parameter estimation
Representing uncertainty for estimation model

- Account for uncertainty using a noise process $\eta$
  - simplification of physical modeling
  - uncertainty in model parameters
  - process and measurement noise

$$|A_d| X \Lambda \left( |B_d^T| \dot{\rho} + |B_s^T| \dot{s} \right) + 4(- A_d X \Phi + \tilde{d}) + \eta = 0$$

- Minimize estimation error using least squares objective

$$\mathcal{L}(d, \tilde{d}, \rho, \tilde{\rho}) \equiv \int_0^T (d - \tilde{d})^T W_1 (d - \tilde{d}) + (\rho - \tilde{\rho})^T W_2 (\rho - \tilde{\rho}) dt$$
State estimation problem

- Estimate friction factors $\lambda_{ij}$ and state $\rho_{ij}(t, x_{ij})$ and $\phi_{ij}(t, x_{ij})$
- Given measurements of compressor boost $\hat{\alpha}_{ij}(t)$, $\bar{\alpha}_{ij}(t)$ and metered withdrawals $\hat{d}_j(t)$ and nodal densities $\hat{\rho}_j(t)$ at time points $\{t_k\}$ in $[0, T]$ for $k = 1, \ldots, K$

**Objective (1):**
$$\min J_E \Delta \sum_{j \in \mathcal{D}} \sum_{k=1}^K \left[ ||d_j(t_k) - \hat{d}_j(t_k)|| + ||\rho_j(t_k) - \hat{\rho}_j(t_k)|| \right]$$

**Mass conservation (2):**
$$\partial_t \rho_{ij} + \partial_x \phi_{ij} = 0 \quad \forall (i, j) \in \mathcal{E}$$

**Momentum conservation (3):**
$$\partial_t \phi_{ij} + \partial_x \rho_{ij} = -\frac{\lambda_{ij}}{2D_{ij}} \frac{\phi_{ij}}{\rho_{ij}} \quad \forall (i, j) \in \mathcal{E}$$

**Nodal controllers (4):**
$$\rho_{ij}^{\text{min}} \leq \rho_{ij}(t, x) \leq \rho_{ij}^{\text{max}} \quad \forall (i, j) \in \mathcal{E}$$

**Flow balance (6):**
$$\sum_{i \in \partial_+ j} X_{ij} \bar{\phi}_{ij}(t) - \sum_{k \in \partial_- j} X_{jk} \phi_{jk}(t) = d_j(t) \quad \forall (i, j) \in \mathcal{E}$$

**Controller constraints (7):**
$$1 \leq \alpha_{ij}(t), \quad \eta_{ij} |\phi_{ij}| (\alpha_{ij}^{(\gamma-1)/\gamma} - 1) \leq P_{ij}^{\text{max}}, \quad \forall (i, j) \in \mathcal{C}$$

**Flow limits (8):**
$$\hat{d}_j(t) < \bar{\rho}_j(t) < d_j^{\text{max}}(t) \quad \text{and} \quad s_j^{\text{min}}(t) < s_j(t) < s_j^{\text{max}}(t), \quad \forall j \in \mathcal{V}$$

**System mass balance (9):**
$$\sum_{(i, j) \in \mathcal{E}} \int_0^L (P_{ij}(0, x) - \rho_{ij}(T, x)) dx = 0$$
Estimation for synthetic data

\[
\min_{\rho, \Phi, d, K} \mathcal{L}(d, \tilde{d}, \rho, \tilde{\rho}) \equiv \int_0^T (d - \tilde{d})^T W_1 (d - \tilde{d}) + (\rho - \tilde{\rho})^T W_2 (\rho - \tilde{\rho}) \, dt
\]

subject to:

\[
|A_d| X \Lambda |B_d^\top| \tilde{\rho} = 4 (A_d X \Phi - d) - |A_d| X \Lambda |B_s^\top| \tilde{s},
\]

\[
\Lambda K \Phi \odot \Phi = -B^\top \rho^N \odot |B^\top| \rho^N,
\]

\[
\rho_{\text{min}} \leq \rho \leq \rho_{\text{max}},
\]

\[
\rho(0) = \rho(T), \quad \Phi(0) = \Phi(T), \quad \text{and} \quad d(0) = d(T).
\]

Good model identification for synthetic data:

<table>
<thead>
<tr>
<th>$e^d_{\text{avg}}$</th>
<th>$e^P_{\text{avg}}$</th>
<th>$e^\Phi_{\text{avg}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.17</td>
<td>0.26</td>
<td>4.84</td>
</tr>
<tr>
<td>2.55</td>
<td>0.18</td>
<td>3.16</td>
</tr>
<tr>
<td>1.24</td>
<td>0.09</td>
<td>1.43</td>
</tr>
</tbody>
</table>

Joint state and parameter estimation for real data

- Reduced model of subsystem used for capacity planning for a real pipeline
  - 78 nodes, 91 pipes, 4 compressors 31 custody transfer meters at 24 locations (labelled A to X)
  - Hourly SCADA time-series of pressure and flow at meters for a month during congested conditions

Robust optimal control
Physical flow network as a directed metric graph

- Network - graph topology with discrete connectivity structure
- Metric - a mapping of edges to, e.g., length & dissipation parameter
- Flows - distributed dynamic relations (PDEs) on edges
- Control - nodal actuators that control density at node-edge interface
Physical flow network as a directed metric graph

- Metric graph $\Gamma = (\mathcal{V}, \mathcal{E}, \lambda)$
- $\mathcal{V}$ is a set of vertices (nodes)
- $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is set of directed edges $(i, j) \in \mathcal{E}$ that connect the vertices $i, j \in \mathcal{V}$
- $\lambda: \mathcal{E} \to \mathbb{R}_+$ is a metric on the edges - length $L_{ij} = \lambda(i, j) > 0$
- Incoming and outgoing neighborhoods of $j \in \mathcal{V}$
  \[ \partial_+ j = \{ i \in \mathcal{V} \mid (i, j) \in \mathcal{E} \} \]
  \[ \partial_- j = \{ k \in \mathcal{V} \mid (j, k) \in \mathcal{E} \} \]
Distributed dynamics on edges

- State on each edge \((i, j) \in \mathcal{E}\) defined on space & time domains \([0, L_{ij}]\) and \([0, T]\).
- Flow \(\phi_{ij} : [0, T] \times [0, L_{ij}] \to \mathbb{R}\) and density \(\rho_{ij} : [0, T] \times [0, L_{ij}] \to \mathbb{R}_+\).
- **Dissipative relations** for density and flow dynamics on the edge \((i, j) \in \mathcal{E}\)
  \[
  \partial_t \rho_{ij}(t, x_{ij}) + \partial_x \phi_{ij}(t, x_{ij}) = 0 \quad \text{(Continuity)}
  
  \phi_{ij}(t, x_{ij}) + f_{ij}(t, \rho_{ij}(t, x_{ij}), \partial_x \rho_{ij}(t, x_{ij})) = 0 \quad \text{(Momentum dissipation)}
  
  \]

- \(f_{ij}\) characterizes dissipative properties for each edge
Nodal compatibility conditions

- Every vertex $i \in \mathcal{V}$ has internal nodal density $\rho_i(t) : [0, T] \rightarrow \mathbb{R}_+$
- Boundary condition is time-dependent flow injection (withdrawal) $q_i : [0, T] \rightarrow \mathbb{R}$
- State values at boundaries of edge domains:
  \[
  \begin{align*}
  \underline{\rho}_{ij}(t) & \triangleq \rho_{ij}(t, 0), & \overline{\rho}_{ij}(t) & \triangleq \rho_{ij}(t, L_{ij}) \\
  \underline{\phi}_{ij}(t) & \triangleq \phi_{ij}(t, 0), & \overline{\phi}_{ij}(t) & \triangleq \phi_{ij}(t, L_{ij})
  \end{align*}
  \]

- **Kirchhoff-Neumann** flow conservation
  \[
  q_j(t) + \sum_{i \in \partial_+ j} \overline{\phi}_{ij} - \sum_{k \in \partial_- j} \underline{\phi}_{jk} = 0, \quad \forall j \in \mathcal{V}.
  \]

- Compatibility of nodal density actuators:
  \[
  \underline{\rho}_{ij}(t) = \underline{\alpha}_{ij}(t, \rho_i(t)), \quad \overline{\rho}_{ij}(t) = \overline{\alpha}_{ij}(t, \rho_j(t)), \quad \forall (i, j) \in \mathcal{E}
  \]

- $\underline{\alpha}_{ij}(t, \rho)$ and $\overline{\alpha}_{ij}(t, \rho)$ monotonically increasing in $\rho$ $\forall t \in [0, T]$ $\&$ $\rho > 0$. 
Well-posedness & regularity assumptions

- Instantaneous state of the system at $t = 0$
  \[
  \rho_{ij}(0, x) = \rho_{ij}^0(x), \quad \phi_{ij}(0, x) = \phi_{ij}^0(x), \quad \forall (i, j) \in \mathcal{E}
  \]

1. Well-posedness & regularity of initial conditions:
   \[\exists\text{ integer } k \geq 2 \text{ s.t.} \quad \rho_{ij}^0, \phi_{ij}^0 \in C^k([0, L_{ij}]) \quad \forall \ (i, j) \in \mathcal{E}.\]
   Kirchoff-Neumann flow balance and pressure compatibility hold at $t = 0$.

2. Continuity of boundary conditions (inputs and controls):
   \[\alpha_{ij}, \overline{\alpha}_{ij} \in C^k_+([0, T] \times \mathbb{R}_+) \text{ for all } (i, j) \in \mathcal{E},\]
   \[q_i \in C^k([0, T]) \quad \forall \ i \in \mathcal{V}.
   \]

3. Well-posedness of coupled network dynamics:
   The initial value problem has a unique classical solution that is twice continuously differentiable, given by $\rho_{ij}(t, x_{ij})$ and $\phi_{ij}(t, x_{ij})$ for $(i, j) \in \mathcal{E}$. 
4 Stability under small perturbations: Let \( \rho_{ij, \epsilon}(t, x_{ij}) \) and \( \phi_{ij, \epsilon}(t, x_{ij}) \) for \( \forall (i, j) \in \mathcal{E} \) be a solution to the perturbed system

\[
\begin{align*}
\partial_t \rho_{ij, \epsilon}(t, x_{ij}) + \partial_x \phi_{ij, \epsilon}(t, x_{ij}) - \epsilon &= 0 \\
\phi_{ij, \epsilon}(t, x_{ij}) + f_{ij}(t, \rho_{ij, \epsilon}(t, x_{ij}), \partial_x \rho_{ij, \epsilon}(t, x_{ij})) &= 0,
\end{align*}
\]

with perturbed initial conditions

\[
\rho_{ij, \epsilon}(0, x) = \rho^0_{ij}(x) + \epsilon, \quad \phi_{ij, \epsilon}(0, x) = \phi^0_{ij}(x)
\]

Then as \( \epsilon \to 0 \), the perturbed solution converges point-wise to the original solution, i.e., for all \( (i, j) \in \mathcal{E}, \ x_{ij} \in [0, L_{ij}] \) and \( t \in [0, T] \), we have

\[
\lim_{\epsilon \to 0} \rho_{ij, \epsilon}(t, x_{ij}) = \rho_{ij}(t, x_{ij}).
\]
Theorem: monotone order propagation

Theorem (Main Result)

Suppose that

- The initial value problem satisfies the assumptions.
- Flow strictly increases with the pressure gradient, so that for \( \forall (i, j) \in \mathcal{E} \)
  \[
  \frac{\partial}{\partial u} f_{ij}(t, u, v) > 0
  \]

Next,

- Let \( \rho^{(1)}_{ij}(0, x_{ij}) \) and \( \rho^{(2)}_{ij}(0, x_{ij}) \) be two initial conditions that satisfy
  \[
  \rho^{(1)}_{ij}(0, x_{ij}) \geq \rho^{(2)}_{ij}(0, x_{ij}) \text{ for all } (i, j) \in \mathcal{E}, x_{ij} \in [0, L_{ij}].
  \]
- Let \( S \subseteq \mathcal{V} \) be an arbitrary subset of \( \mathcal{V} \).
- Let \( t_0 \in [0, T] \) and suppose that \( \forall i \in S \) we have that \( q^{(1)}_i(t) \geq q^{(2)}_i(t) \) for \( \forall t \in [0, t_0] \) and for \( \forall i \in \mathcal{V} \setminus S \) we have that \( \rho^{(1)}_i(t) \geq \rho^{(2)}_i(t) \) for all \( t \in [0, t_0] \).

Then the densities in the system satisfy \( \rho^{(1)}_{ij}(t, x_{ij}) \geq \rho^{(2)}_{ij}(t, x_{ij}) \) for all \( (i, j) \in \mathcal{E}, x_{ij} \in [0, L_{ij}] \) and \( t \in [0, t_0] \).
**Definition (Parameterized control system)**

Consider a control system

\[ \dot{x} = g(x, u, p), \quad x(0) = y \quad (1) \]

with state \( x(t) \in \mathcal{X} \), control vector \( u(t) \in \mathcal{U} \subset \mathbb{R}^m \), and parameter vector \( p(t) \in \mathcal{P} \subset \mathbb{R}^p \) where \( \mathcal{U}, \) and \( \mathcal{P} \) are closed and convex.

**Definition (Monotone parameterized control system)**

The control system (1) is **monotone parameterized** with respect to \( p(t) \) if, for all \( t \geq 0 \)

\[ y_1, y_2 \in \mathcal{X}, \quad u(t) : (0, \infty) \rightarrow \mathcal{U}, \quad \text{and piecewise-continuous functions} \]

\[ p_1(t), p_2(t) : (0, \infty) \rightarrow \mathcal{P}, \]

\[ y_1 \leq y_2, \quad p_1(s) \leq p_2(s), \quad \forall \ s \in [0, t] \Rightarrow x_1(t) \leq x_2(t), \quad (2) \]

where inequalities for are meant componentwise for vectors (i.e., \( a \leq b \) means that \( a_i \leq b_i \) for all \( i = 1, \ldots, n \)), pointwise for \( x, y \in \mathcal{X} \), and \( x_j(t) \), for \( j = 1, 2 \), stands for the solution to \( \dot{x} = g(x, u, p_j) \) with initial condition \( x(0) = y_j \).
Application: friction-dominated models

• Friction dominated modeling
  – Omit flux derivative term, approximate hyperbolic system by parabolic one

\[
\begin{align*}
\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\phi &= 0 \\
\frac{\partial}{\partial t}\phi + a^2 \frac{\partial}{\partial x}\rho &= -\frac{\lambda}{2D} \frac{\phi|\phi|}{\rho}
\end{align*}
\]

  – Proposed to simplify mathematical modeling for simulation and optimization

• Monotonicity property for gas pipelines assumes friction-dominated flow

\[
a^2 \frac{\partial}{\partial x}\rho = -\frac{\lambda}{2D} \frac{\phi|\phi|}{\rho} \quad \rightarrow \quad \phi_{ij}(t, x_{ij}) + f_{ij}(t, \rho_{ij}(t, x_{ij}), \frac{\partial}{\partial x}\rho_{ij}(t, x_{ij})) = 0
\]

  – Under what conditions are these assumptions valid?
Application: friction-dominated models

- **Example**: \( L=20 \text{ km}, \ D=0.9144 \text{ m} \) pipe
- **Left: Fast Transients**
  - Flow in a single pipe with sinusoidal variation (3 cycles over 1 hour) in outlet flow with maximum magnitudes of 120, 300, 400, and 600 kg/s.
  - The monotonicity theorem does not apply for the fast transient regime
- **Right: Slow Transients**
  - Flow in a single pipe with slow sinusoidal variation (3 cycles in 24 hours) in outlet flow with max magnitudes of 120, 300, 400, & 600 kg/s.
  - The monotonicity theorem holds in the slow transient regime
- **Guidance**: friction-dominated modeling should not be used to represent fast transients

Application: real data validation

- Testing the monotonicity property in the normal operating regime of gas pipelines
  - Top left: Baseline withdrawals (kg/s) custody transfer stations.
  - Top right: Increase of withdrawals above baseline by 5%.
  - Bottom left: Simulated pressure (PSI) solutions given baseline withdrawals.
  - Bottom right: Simulated pressure solutions given increased withdrawals.

Application: robust optimal control

Consider a monotone parameterized control system \( \dot{x} = g(x, p, u), \ x(0) = y \) where \( x(t) \in \mathcal{X} \) is the state, \( p(t) \in \mathcal{P} \subset \mathbb{R}^m \) is an uncertain (time-varying) parameter vector, and \( u(t) \in \mathcal{U} \subset \mathbb{R}^k \) is a (decision variable) control vector.

Suppose that \( \mathcal{P} \) is defined by \( p_{\min}(t) \leq p(t) \leq p_{\max}(t) \), inequality pointwise.

Call \( x(t, p, u, y) \) the solution for a given \( p(t), u(t), \) and \( y \).

Dynamic constraint for an optimal control problem, where for \( \forall t \in [0, T] \)

\[
\begin{align*}
\min_{u} & \quad \int_{0}^{T} \mathcal{L}(x(t, p_{\text{nom}}, u, y), u(t))dt \\
\text{s.t.} & \quad \dot{x}(t) = g(t, x(t), p_{\text{nom}}, u(t)) , \\
& \quad e(x(t, p_{\text{nom}}, u, y), u(t)) = 0 , \ \forall p \in \mathcal{P} \\
& \quad h(x(t, p_{\text{nom}}, u, y), u(t)) \leq 0 , \ \forall p \in \mathcal{P}
\end{align*}
\]

\[
\begin{align*}
\min_{u} & \quad \int_{0}^{T} \mathcal{L}(x(t, p_{\text{nom}}, u, y), u(t))dt \\
\text{s.t.} & \quad \dot{x}(t) = g(t, x(t), p_{\text{nom}}, u(t)) , \\
& \quad \dot{x}(t) = g(t, x(t), p_{\text{min}}, u(t)) , \\
& \quad \dot{x}(t) = g(t, x(t), p_{\text{max}}, u(t)) , \\
& \quad e(x(t, p_{\text{nom}}, u, y), u(t)) = 0 , \\
& \quad e(x(t, p_{\text{min}}, u, y), u(t)) = 0 , \\
& \quad e(x(t, p_{\text{max}}, u, y), u(t)) = 0 , \\
& \quad h(x(t, p_{\text{min}}, u, y), u(t)) \leq 0 \\
& \quad h(x(t, p_{\text{max}}, u, y), u(t)) \leq 0
\end{align*}
\]

Key Idea: Evaluating constraints at extremal parameter functions guarantees satisfaction for all parameters.

Optimization solution is robust to model parameter uncertainty.
• Economic optimal control formulation
  - Decision variables:
    Compressor ratios: $\alpha$
    Gas withdrawals: $d$
    Gas supply: $s$
    Densities: $\rho$
    Flows: $\phi$
  - Deterministic gas pipeline constraints:
    $\Gamma(\alpha, d, s, \rho, \phi) =$
    - Gas flow dynamics
    - Mass flow balance
    - Slack node density
    - Compressor action
    - Density limits
    - Compressor constraints
    - Withdrawal limits
    - Time periodicity
  - Economic and compressor efficiency objectives:
    $J_E(d, s) = \sum_{j \in V} \int_0^T \left( c_j^d(t) d_j(t) + c_j^s(t) s_j(t) \right) dt$
    $J_C(\alpha) = \sum_{(i, j) \in C} \int_0^T \eta_{ij} |\phi_{ij}(t)| \left( (\alpha_{ij}(t))^m - 1 \right) dt$

• Deterministic formulation (1)
  - Given nominal schedule $d(t)$:
    $\max_{\alpha, d, s, \rho, \phi} J_E(d, s) - J_C(\alpha)$
    s. t.: $\Gamma(\alpha, d, s, \rho, \phi)$

• Robust formulation (2)
  - Given minimum and maximum withdrawals $d_1(t)$ and $d_2(t)$:
    $\max_{\alpha, d, s, \rho, \phi} J_E(d, s) - J_C(\alpha)$
    s. t.: $\Gamma(\alpha, d_1, s, \rho, \phi)$
    $\Gamma(\alpha, d_2, s, \rho, \phi)$
    $d_1(t) \leq d(t) \leq d_2(t)$

Many open problems!
Transition to practice

Coordinated Operation of Electric And Natural Gas Supply Networks: Optimization Processes And Market Design

Commercial ENELYTIX system
Power System Optimizer (PSO) by Polaris (CPLEX).
Gas System Optimizer (GSO) by LANL (IPOPT).
Scalable and flexible cloud-based architecture.

Fuel Reliability for Electric Energy Delivery by Optimized Management of Gas-pipeline Automation Systems (FREEDOM GAS)
Software development, system integration, and pilot study

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- **GRAIL: Gas Reliability Analysis Integrated Library**
  - Open source LANL-developed prototype algorithms
    - [https://github.com/lanl-ansi/grail](https://github.com/lanl-ansi/grail) (OSTI ID 18546)
Questions?
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