Information propagation in interacting spin systems with disorder

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Introduction

• Quantum spin systems
• Information propagation in quantum spin systems (the Lieb-Robinson bound)
• (At least) 3 regimes:
  • Localisation
  • Diffusion
  • Quantum zeno
• Fault tolerance in quantum computers
Introduction: what is a quantum spin system?

\[ h_j \equiv \mathbb{1}_{1\ldots j-1} \otimes h_j \otimes \mathbb{1}_{j+2\ldots n} \]

- **Hilbert space:** \( \mathcal{H} = \bigotimes_{j=1}^{n} \mathbb{C}^d \)
- **Hamiltonian:** \( H(t) = \sum_{j=1}^{n-2} h_j(t) \)
- **Normalisation:** \( \| h_j(t) \| = O(1) \)
Problem: information propagation

localised disturbance at $A$ at $t = 0$

what is felt here at $B$ after time $t = T$?
How to quantify information propagation?

We typically study information propagation using connected time-dependent correlation functions:

\[ \langle \psi | U(t)^* B U(t) A | \psi \rangle - \langle \psi | U(t)^* B U(t) | \psi \rangle \langle \psi | A | \psi \rangle \]

where $A$ and $B$ are local observables and

\[ U(t) = \mathcal{T} e^{i \int_0^t H(s) ds} \]

(Initial state is usually a product state or ground state.)
Dynamics of correlations

But, for technical reasons it’s more convenient to study the Lieb-Robinson commutator:

\[ C_A(j, t) = \sup_{\|B\|=1} \| [A(t), B] \| \]

where \[ A(t) = U^\dagger(t)AU(t) \] and \[ U(t) = T e^{i \int_0^t H(s) \, ds} \]
and \( B \) is restricted to act nontrivially only on site \( j \). (i.e. \( \text{supp}(B) = \{j\} \))
Dynamics of correlations

Physically the LR commutator measures worst case (coming from the supremum and the norm) influence of an operation on site \( j \) after a time \( t \) as measured by an observable \( A \).
The Lieb-Robinson bound

Proposition 1 (Lieb and Robinson, 1972): For any low dimensional spin system the following bound holds:

$$\| [A(t), B] \| \leq ce^{-\nu d(A,B)+k|t|},$$

where $\nu$, $k$, $c$ are constants, and $A$ and $B$ are local operators.
The LR bound says that two-point dynamical correlations are exponentially suppressed outside of an effective “light cone” with an effective “speed of light” set by $||h||$. 
Discussion

We say that, if the bound is essentially saturated, information propagates *ballistically* through the system.

Can one expect better bounds? For generic translation invariant systems the answer is *no*! (This is the so called “LR wall” encountered in time-dependent DMRG.)

But do there exist non-TI systems with better bounds?
Disorder

We now focus on LR bounds for (statically and dynamically) disordered systems. Intuitively expect better bounds because possibility of:

- *Anderson localisation*; and

- the *quantum Zeno effect*. 
Disorder

In our results we’ll identify 3 types of behaviour:

1. Ballistic (standard LR);

2. Diffusive (new); and

3. Localised regimes (new)
Disorder

Our hamiltonians will be of the form

\[ H(t) = \sum_{j=1}^{n-2} h_j(t) + \sum_{j=1}^{n} \xi_x^j(t)\sigma_j^x + \xi_y^j(t)\sigma_j^y + \xi_z^j(t)\sigma_j^z \]

where \( \xi^j_\alpha(t) \) are either i.i.d. random variables (time independent) or (derivatives of) Wiener processes (time dependent)
2. Time-dependent disorder (diffusive regime)

We consider the disordered $XY$ model

$$H_{XY}(t, \xi(t)) = \sum_{j=1}^{n-2} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sum_{j=1}^{n} \xi_j(t) \sigma_j^z$$

where $\xi_j(t) = dW_j(t)$, and $W_j(t)$ is a brownian motion. We define

$$U(t, \xi(t)) = \mathcal{T} e^{i \int_0^t H_{XY}(s, \xi(s)) ds}$$
2. Time-dependent disorder

\[ \xi_j(t) \]
2. Time-dependent disorder

Averaging over the Wiener processes (the disorder), using Itô’s rule, we obtain the following master equation

\[
\frac{d\rho(t)}{dt} = i \sum_{j=1}^{n-2} [h_j(t), \rho(t)] - 2\gamma \sum_{j=1}^{n} [\sigma_j^z, [\sigma_j^z, \rho(t)]]
\]

where

\[
\rho(t) = \mathbb{E}_{\xi(t)}[U(t, \xi(t))\rho(0)U^\dagger(t, \xi(t))]
\]

is the state of the system after time \( t \) averaged over the disorder.
2. Time-dependent disorder

In the Heisenberg picture we therefore have that the dynamics of *averaged* observables satisfy

\[
\frac{dA(t)}{dt} = i \sum_{j=1}^{n-2} [A(t), h_j(t)] - 2\gamma \sum_{j=1}^{n} [[A(t), \sigma_j^z], \sigma_j^z]
\]

**Intuition:** the disorder acts as a continuous weak measurement hence the Zeno effect should suppress information propagation.
2. Time-independent disorder

**Proposition 2** (Burrell, Eisert, and Osborne, (2008)). Let $A$ be a local observable. Then for $H_{XY}(t, \xi(t))$ the LR commutator of the averaged observable satisfies the bound

$$\| [ \langle \xi(t) \rangle [A(t)], B ] \| \leq c \frac{|t|}{d(A, B)^2}$$

This is diffusive behaviour.
2. Time-dependent disorder

**Conjecture 1.** Consider the hamiltonian

\[ H(t) = \sum_{i=1}^{n-2} h_j + \sum_{i=1}^{n} \xi^j(t) \sigma^j_z \]

where \( \xi^j(t) = dW^j(t) \), and \( W^j(t) \) is now a continuous-time martingale, and \( h_j \) is general. Then

\[ \| [\mathbb{E}_{\xi(t)}[A(t)], B] \| \leq c \frac{|t|}{d(A, B)^2} \]
2.1. Interlude: time-independent disorder is much stronger

We consider the disordered XY model

\[ H = \sum_{j=1}^{n-2} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sum_{j=1}^{n} \xi_j \sigma_j^z \]

Where \( \xi_j \) are chosen according to, eg., Gaussian distribution. Quenched disorder.
2.1. Time-independent disorder

Gaussian distributed field

\[ \xi_j \]

site
2.1. Time-independent disorder

Proposition 3 (Burrell and Osborne, (2007)). Let $A$ be a local operator, then for almost all $\xi_j$ the disordered $XY$ model satisfies “information localisation”

$$\| [A(t), B] \| \leq c |t| e^{-kd(A,B)}$$

“logarithmic light cone”
2.1. Time-independent disorder

Conjecture 2. Consider the Hamiltonian

\[ H = \sum_{j=1}^{n-2} h_j + \sum_{j=1}^{n} \xi^j \sigma_j^z \]

where \( \xi_j \) are i.i.d. RV’s, then for almost all \( \xi_j \)

\[ \| [A(t), B] \| \leq c |t| e^{-k d(A, B)} \]

[Maybe need the Hamiltonian to be]

\[ H = \sum_{j=1}^{n-2} h_j + \sum_{j=1}^{n} \xi^j \sigma_j^x + \xi^j \sigma_j^y + \xi^j \sigma_j^z \]
3. Time-dependent disorder: ballistic to localised crossover

We consider now a general Hamiltonian + noise

\[ H(t) = \sum_{j=1}^{n-2} h_j(t) + \gamma \sum_{j=1}^{n} \xi_x^j(t) \sigma_j^x + \xi_y^j(t) \sigma_j^y + \xi_z^j(t) \sigma_j^z \] (*)

Averaging over the Wiener processes (the disorder), using Itô’s rule, we obtain the following master equation

\[
\frac{d\rho(t)}{dt} = i \sum_{j=1}^{n-2} [h_j(t), \rho(t)] - 2\gamma \sum_{j=1}^{n} \sum_{\alpha=x,y,z} [\sigma_j^\alpha, [\sigma_j^\alpha, \rho(t)]]
\]
3. Time-dependent disorder

In the Heisenberg picture we therefore have

\[
\frac{dA(t)}{dt} = i \sum_{j=1}^{n-2} [A(t), h_j(t)] - 2\gamma \sum_{j=1}^{n} \sum_{\alpha=x,y,z} [[A(t), \sigma_j^\alpha], \sigma_j^\alpha]
\]

**Again intuition:** the disorder acts as a continuous weak measurement hence the Zeno effect should suppress information propagation.
3. LR for time-dep. disorder

**Proposition 4:** for the model \((*)\) if \(\| h_j(t) \| < O(\gamma)\) then

\[
\| [\mathbb{E}_{\xi(t)}[A(t)], B] \| \leq c e^{-kd(A,B)}
\]

where \(A\) is a local operator with \(O(1)\) support. If \(\| h_j(t) \| \geq O(\gamma)\) then information can, in principle, propagate ballistically.

(We drop the expectation symbol from now on.)
Proof

Proof: set up LR commutator (assume $A(0)$ lives on site 1 and $B$ lives on site $j$):

$$C_A(j, t) = \sup_{\|B\|=1} \| [A(t), B] \|$$

Now, we work out $C_A(j, t)$ a little time later:

$$C_A(j, t + \epsilon) \leq (1 - 8\epsilon \gamma n) C_A(j, t) + \epsilon \| [A(t), [H_0, B]] \| + \epsilon \gamma \| [8nA(t) - F(A(t)), B] \|$$

where

$$F(M) = \sum_{j=1}^{n} \sum_{\alpha=x,y,z} [[M, \sigma^\alpha_j], \sigma^\alpha_j]$$

and

$$H_0 = \sum_{j=1}^{n-2} h_j(t)$$
**Proof**

The superoperator $\mathcal{F}(M)$ is a sum of projections onto the linear space of traceless operators:

$$8nM - \mathcal{F}(M) = 2 \sum_{j=1}^{n} \sum_{\alpha=0}^{3} \sigma_{j}^{\alpha} M \sigma_{j}^{\alpha}$$

so that the last term on LHS becomes

$$\epsilon\gamma \|[8nA(t) - \mathcal{F}(A(t)), B]\| \leq 2\epsilon\gamma \sum_{k\neq j} \sum_{\alpha_{k}=0}^{3} \|[A(t), \sigma_{k}^{\alpha} B \sigma_{k}^{\alpha}]\|$$

$$= 8\epsilon\gamma(n - 1)C_{A}(j, t)$$

(Our upper bound will hold only a.e. but this is irrelevant for integrating the ODE.)
Proof continued

Putting this together with

\[ \| [A(t), [H_0, B]] \| \leq \kappa (C_A(j-1, t) + C_A(j, t) + C_A(j+1, t)) \]

Gives us

\[ \frac{C_A(j, t + \epsilon) - C_A(j, t)}{\epsilon} \leq -8\gamma C_A(j, t) + \kappa (C_A(j-1, t) + C_A(j, t) + C_A(j+1, t)) \]

Taking limsup gives us:

\[ \frac{D C_A(j, t)}{D t} \leq -8\gamma \Delta_j(t) + \kappa (C_A(j-1, t) + C_A(j, t) + C_A(j+1, t)) \]
Proof continued

Write as a matrix equation

\[
\frac{DC(t)}{Dt} \leq MC(t)
\]

where

\[
M = \begin{pmatrix}
\kappa - 8\gamma & \kappa & \cdots & 0 \\
\kappa & \kappa - 8\gamma & \kappa & \cdots \\
0 & \kappa & \kappa - 8\gamma & \kappa \\
\vdots & \vdots & \vdots & \vdots
\end{pmatrix} = (\kappa - 8\gamma) \mathbf{I} + \kappa \mathcal{T}
\]
Proof continued

Integrating the equality case (or iterating the integral equation):

\[ C(t) = e^{(\kappa-8\gamma)t} e^{t\kappa T} C(0) \]

where

\[ C_A(j, 0) \leq \delta_{1,j} \| A(0) \| \]

After tedious algebra to bound taylor series we find that if \( \gamma \geq \kappa + O(1) \) then

\[ C_A(j, t) \leq c e^{-\omega j} \]
Proof continued

If this condition isn’t satisfied then information can, in principle, propagate ballistically: the evolution of a 1D quantum computer can maintain coherence for long time scales under threshold.

Our results can be seen as a continuous-time analogue of D. Aharonov, Phys. Rev. A 62, 062311 (2000) on fault tolerance in quantum computers. Our bound applies to all local observables.
Summary

• Disorder can improve bounds on information propagation in quantum spin chains

• Time-independent case exploits Anderson localisation

• Time-dependent case exploits quantum Zeno effect.

• Ballistic, diffusive, and localised regimes can occur.