Nonequilibrium Statistics of Geophysical Flows From Cumulant Expansions and Flow Equations

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Outline

• Climate, Climate Change, and Weather
• Nonequilibrium Statistical Physics:
  • Hopf Functional Approach
  • Cumulant Expansions
  • Flow Equations
• Geophysical Fluid Dynamics:
  • Coriolis Force
  • Stratification
• Test Case: A Point Jet
Variations of the Earth's surface temperature: years 1000 to 2100

Departures in temperature in °C (from the 1990 value)

Observations, Northern Hemisphere, proxy data

Global instrumental observations

IPCC 2001
"More than any other theoretical procedure, numerical integration is also subject to the criticism that it yields little insight into the problem. The computed numbers are not only processed like data but they look like data, and a study of them may be no more enlightening than a study of real meteorological observations. An alternative procedure which does not suffer this disadvantage consists of deriving a new system of equations whose unknowns are the statistics themselves. This procedure can be very effective for problems where the original equations are linear, but, in the case of non-linear equations, the new system will inevitably contain more unknowns than equations, and can therefore not be solved, unless additional postulates are introduced."

Thermodynamics vs. Statistical Mechanics

Equilibrium vs. Out-of-Equilibrium

PV = nRT
Hopf Functional Approach

U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov*

\[
\frac{dx}{dt} = x^2 \quad \Psi(t, u) \equiv e^{iux(t)}
\]

\[
i \frac{\partial}{\partial t} \Psi = u \frac{\partial^2}{\partial u^2} \Psi \quad i \frac{\partial}{\partial t} \overline{\Psi} = u \frac{\partial^2}{\partial u^2} \overline{\Psi}
\]

\[
\hat{H} \overline{\Psi}_0 = 0
\]

\[
\overline{\Psi}_0(u) = \exp\left\{iu\langle x \rangle - \frac{1}{2!}u^2(\langle x^2 \rangle - \langle x \rangle^2) + \ldots\right\}
\]

\[
\langle x \rangle = -i \frac{\partial \overline{\Psi}_0(u)}{\partial u} \bigg|_{u=0}
\]
General Equations of Motion

\[
\frac{dx_i}{dt} = A_{ij} x_j + B_{ijk} x_j x_k
\]

\[
\hat{H} = i A_{ij} u_i \frac{\partial}{\partial u_j} + B_{ijk} u_i \frac{\partial^2}{\partial u_j \partial u_k}
\]
Exact Solution For Orszag-McLaughlin Dynamics

\[ \frac{dx_i}{dt} = x_{i+1}x_{i+2} + x_{i-1}x_{i-2} - 2x_{i+1}x_{i-1} \]

\[ \Psi_0(\vec{u}) = \frac{3\sqrt{\pi}}{4} \left( \frac{R|\vec{u}|}{2} \right)^{-3/2} \times J_{3/2}(R|\vec{u}|) \]

Flow Equation Approach


\[ \hat{H}(s) = \hat{H}_0(s) + \hat{H}'(s) \]

\[ \hat{G}(s) = [\hat{H}_0(s), \hat{H}(s)] \]

\[ \frac{d}{ds} \hat{H}(s) = [\hat{G}(s), \hat{H}(s)] \]

\[ \frac{d}{ds} \hat{O}(s) = [\hat{G}(s), \hat{O}(s)] \]

\[ H(0) = \begin{pmatrix} 1.0 & 0.9 & 0.5 \\ 0.7 & -0.3 & 0.0 \\ 0.2 & -0.3 & -0.4 \end{pmatrix} \]

\[ H_0(s) = \begin{pmatrix} \lambda_1(s) & 0 & 0 \\ 0 & \lambda_2(s) & 0 \\ 0 & 0 & \lambda_3(s) \end{pmatrix} \]
Closure

\[
\begin{bmatrix}
u_i \frac{\partial^2}{\partial u_j \partial u_k}, \\
u_\ell \frac{\partial^2}{\partial u_m \partial u_n}
\end{bmatrix} = \delta_{j \ell} \ u_i \frac{\partial^3}{\partial u_k \partial u_m \partial u_n} + \ldots
\]

more derivatives w.r.t. \( u \)

\[
\hat{\mathcal{A}} = a + a_i \frac{\partial}{\partial u_i} + a_{ij} \frac{\partial^2}{\partial u_i \partial u_j} + a_{ijk} \frac{\partial^3}{\partial u_i \partial u_j \partial u_k} + \ldots
\]

more powers of \( u \)

\[
+ a^{(1)}_i u_i + a^{(1)}_{ij} u_i \frac{\partial}{\partial u_j} + a^{(1)}_{ijk} u_i \frac{\partial^2}{\partial u_j \partial u_k} + \ldots
\]

\[
+ a^{(2)}_{ij} u_i u_j + a^{(2)}_{ijk} u_i u_j \frac{\partial}{\partial u_k} + \ldots
\]

\[
+ a^{(3)}_{ijk} u_i u_j u_k + \ldots
\]

\[
+ \ldots
\]
# Direct Numerical Simulations vs. Statistical Approaches

<table>
<thead>
<tr>
<th>Observable</th>
<th>DNS</th>
<th>Cumulant</th>
<th>Hopf / Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt;z&gt;$</td>
<td>22.7</td>
<td>23.4</td>
<td>22.4</td>
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<tr>
<td>$&lt;y^2&gt;$</td>
<td>42.6</td>
<td>32.9</td>
<td>40.2</td>
</tr>
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</table>
Atmospheric Dynamics

from Climate Change 1995: The Science of Climate Change
Single Layer Models

\[ \frac{D\omega}{Dt} = 0 \]

\[ \omega = \hat{r} \cdot (\nabla \times \vec{v}) \]

\[ \frac{\partial \omega}{\partial t} + \vec{v} \cdot \nabla \omega = 0 \]

\[ \vec{v} = \hat{r} \times \nabla \psi \]

\[ \nabla \cdot \vec{v} = 0 \]

\[ \omega = \nabla^2 \psi \]

\[ J(\psi, \omega) \equiv \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} \]
Freely Decaying Turbulence on Sphere
Coriolis Force

\[ \frac{\partial q}{\partial t} + J(\psi, q) = 0 \]

Relative vorticity

\[ q = \omega + f \]

Absolute vorticity

\[ = \nabla^2 \psi + f \]

Coriolis term

\[ f = 2\Omega \sin(\phi) \]
Coriolis Force
Stratification

\[ q = \nabla^2 \psi + f - \frac{\psi}{\ell_R^2} \]

\[ \ell_R^2 = \frac{gh}{f^2} \]

\[ \ell_R = \mathcal{O}(1,000 \text{ km}) \]
Stratification Sets Synoptic Length Scale
Test Case: A Point Jet

\[
\frac{\partial q}{\partial t} + J[\psi, q] = \frac{q_{\text{jet}} - q}{\tau}
\]
Forced-Damped Equatorial Barotropic Jet

\[ \frac{\partial \omega}{\partial t} + \vec{v} \cdot \vec{\nabla}(\omega + f) = \frac{(\omega_0 - \omega)}{\tau} \]

\[ \omega_0 = -0.6 \Omega \text{ sgn}(\phi) \]

Limiting Cases With No Fluctuations

\[
\frac{\partial \omega}{\partial t} + \vec{v} \cdot \vec{\nabla} (\omega + f) = \frac{(\omega_0 - \omega)}{\tau}
\]

\(\tau \rightarrow 0: \quad \omega \rightarrow \omega_0\)

\(\tau \rightarrow \infty: \quad \omega \rightarrow \text{equilibrium jet}\)

[Turkington et al., PNAS99, 12346 (2001); Weichman, PRE73, 036313 (2006)]

Largest fluctuations at intermediate relaxation times
Cumulant Expansion

\[ \langle \omega(\phi, \lambda) \rangle = c_1(\phi) \quad \text{Azimuthal symmetry} \]

\[ \langle \omega(\phi, \lambda) \, \omega(\phi', \lambda') \rangle_C = c_2(\phi, \phi', \lambda - \lambda') \]

\[ \langle \omega \, \omega' \rangle_C \equiv \langle \omega \, \omega' \rangle - \langle \omega \rangle \langle \omega' \rangle \]

\[ \langle \omega \, \omega' \, \omega'' \rangle_C = 0, \text{ etc.} \quad \text{Closure} \]

\[ \langle \omega \omega \rangle_C \geq 0 \quad \text{Positivity} \]
Direct Numerical Simulation of Jet

jet relaxation time = 25 days

2nd Cumulant = 2-point Correlation Function
25 days
Figure (a) shows the eddy diffusivity plots for different models: DNS, CE, Sqrt(<ψψ> - <ψ><ψ>), and <ψ> / 100. The plots are displayed over the range of latitude from -80° to 80°.

Figure (b) follows the same format as (a), comparing DNS, CE, Sqrt(<ψψ> - <ψ><ψ>), and <ψ> / 100, but the scale is different with a focus on a more detailed view of the diffusivity values.
Figure 1. Typical instantaneous vorticity fields ($10^{-5}$ s$^{-1}$) in (a) the full simulation and (b) the simulation without eddy-eddy interactions. The horizontal surface shown is in the mid-troposphere at $\sigma = 0.5$. The fields are shown at times after the simulations have reached statistically steady states.
shallowing of the spectrum is reminiscent of the \( C_0^{5/3} \) mesoscale range that is seen in observational data near the tropopause \( \text{[Nastrom and Gage, 1985]} \), but here it is more likely an indication that eddy-eddy interactions prevent a build-up of energy at these relatively small length scales. \( \text{[16]} \) That the shape of the eddy energy spectrum is recovered without eddy-eddy interactions over a wide wavenumber range helps resolve the paradox that the atmospheric energy spectrum has the power-law decay that would result from a enstrophy cascade in an inertial range even though there are significant terms in the atmospheric spectral energy budget other than those related to eddy-eddy interactions. Our results show that these other terms would by themselves lead to a similar energy spectrum. We also found similar results for other parameter settings in the GCM and using an idealized GCM \( \text{[Held and Suarez, 1994]} \) with different radiation and boundary layer schemes and without a convection scheme. Therefore, our results for the energy spectrum do not appear to be artifacts of specific parameter settings or parameterizations. The approximate \( C_0^{3/3} \) spectrum in the simulation without eddy-eddy interactions may be explained by an unconventional enstrophy cascade in which the zonal-mean component plays a central role \( \text{[cf. Bartello and Warn, 1988]} \), but this is difficult to assess given the number of important terms in the spectral energy budget. Although we have only considered the atmosphere here, similar considerations may apply to the energy spectrum in the ocean at length scales smaller than the oceanic Rossby deformation radius. \( \text{[17]} \) The eddy energy spectrum of the simulation without eddy-eddy interactions (Figure 2) is more jagged than the spectrum of the full simulation. One effect of eddy-eddy interactions, then, is to smooth the spectrum by transferring energy between wavenumbers. Eddy-eddy interactions also tend to isotropize the eddies in the horizontal. The two-dimensional spectral energy distribution reveals isotropization of the eddy energy in the full simulation for length scales smaller than the eddy length scale but anisotropy at all length scales in the simulation without eddy-eddy interactions.

4.3. Mean Circulations

\( \text{[18]} \) The general circulation of the simulation without eddy-eddy interactions shares many features with that of the full simulation, but there are also significant differences. The mean zonal wind in both simulations exhibits upper-level westerly jets, which illustrates that extratropical jets can form as a result of eddy-mean flow interactions alone, without nonlinear eddy-eddy interactions (Figure 3). However, the simulation without eddy-eddy interactions has a second jet in each hemisphere, and the Eulerian-mean circulation has corresponding extra eddy-driven cells. The mean circulation of the simulation without eddy-eddy interactions is compressed in the meridional direction relative to that of the full simulation. A possible explanation is that although most eddy energy resides at the energy-containing eddy length scale in the full simulation, and there is no general cascade of energy to larger scales \( \text{[Rhines, 1975; Vallis and Maltrud, 1993]} \), leading to eddy

**Figure 3.** Mean eastward wind (m s\(^{-1}\)) in the meridional plane in (a) the full simulation and (b) the simulation without eddy-eddy interactions. The mean is a zonal, time, and interhemispheric average with mass weighting. The thick solid lines are the zero-wind lines.
Advantages of Statistical Approach

• Deeper understanding possible.

• Possibility to treat all processes statistically, not just the subgridscale ones.

• Inhomogeneous geophysical flows with mean shear flows are less nonlinear than isotropic turbulence -- progress possible.

• Faster: Time-independent fixed point.

• Faster: Statistics vary slowly in space.
Lake Mead
What Can Theoretical Thinking Contribute?

Aspen Center for Physics

Summer 2005 Workshop
*Novel Approaches To Climate*
Funding: NSF, BP Research, & ICAM

John Harte’s long-term ecosystem heating experiment at the Rocky Mountain Biological Laboratory near Aspen.

Kavli Institute for Theoretical Physics

*Physics of Climate Change*
April 28 -- July 11, 2008
*Frontiers of Climate Science & Engineering the Earth*
May 6 -- 10, 2008

Co-organizers:
J. Carlson, G. Falkovich, J. Harte, J. B. Marston, and R. Pierrehumbert