Quantum phase transitions and quantum information

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Overview

- Quantum (and classical) phase transitions
- Critical points, exponents, and universality
- Quantum information perspective
- Ground state fidelity
- Time dependent GSF and decoherence
- Algorithms and experimental implementation with NMR & cold atoms
Quantum phase transitions

In general, a QPT occurs in a quantum many body system when there is competition between two parts of the total Hamiltonian:

\[ \mathcal{H} = \mathcal{H}_0 + \lambda \mathcal{H}_1 \]

\( (T=0) \)

Ground state of \( \mathcal{H}_0 \)

Ground state of \( \mathcal{H}_1 \)
Quantum phase transitions

Example: Ising chain with transverse field

\[ \mathcal{H} = -J \left( \sum_i \sigma_i^z \sigma_{i+1}^z + \lambda \sigma_i^x \right) \]

\[ \begin{align*}
|\uparrow\uparrow\uparrow \ldots \uparrow\rangle & \quad |\rightarrow\rightarrow\rightarrow \ldots \rightarrow\rangle \\
|\downarrow\downarrow\downarrow \ldots \downarrow\rangle
\end{align*} \]
Quantum phase transitions

Example: Ising chain with transverse field

\[ H = -J \left( \sum_i \sigma_i^z \sigma_{i+1}^z + \lambda \sigma_i^x \right) \]

- At the critical point:
  - The gap closes (in thermodynamical limit), equivalent to critical slowing down \((\tau \sim 1/\Delta)\)
  - Quantum correlations diverge with critical exponents \(\xi \sim |\lambda - \lambda_c|^{-\nu}\)
  - Universality (as in classical PT) can be defined and observed
QPTs and quantum information

- Scaling of entanglement at the critical point (not surprising, entanglement $\sim$ correlations)

- Alas, we’ll take another route

QPTs and fidelity

Ground state fidelity:
\[ f_\delta(\lambda) = \langle g(\lambda) | g(\lambda + \delta) \rangle \]

Rationale: two ground states in the same phase are very similar, and orthogonal if in different phases.

Rationale works, and fidelity contains much more information than just the critical point

A first order QPT (a kink in the GS energy) appears as a discontinuous fidelity.

Second order QPTs appear continuous with a diverging derivative (in thermodynamic limit).
Ground state fidelity

The scaling of the second derivative of the fidelity relates to the critical exponents

\[ f_\delta(\lambda) \simeq 1 + \left. \frac{\partial^2 f}{\partial \delta^2} \right|_{\delta=0} \frac{\delta^2}{2} = 1 + S(\lambda) \frac{\delta^2}{2} \]

Susceptibility

Argument using matrix product states QPTs:

\[ |g> = \sum_{i_1,i_2...i_N} Tr(A_{i_1}...A_{i_N})|i_1...i_N> \]

\[ f_\delta(\lambda) = \sum_{k=1}^{D^2} v_k^N \sum_{i=0}^{d-1} A_i^*(\lambda_1) \otimes A_i(\lambda_2) \]

\[ S(\lambda) \simeq N \partial_{\lambda_1} \partial_{\lambda_2} ln v_1(\lambda_1,\lambda_2)|_{\lambda_1=\lambda_2=\lambda} \]

\[ S_{crit}/N = N^{\rho/\nu} Q(N|\lambda - \lambda_c|^\nu) \]

Also see Venuti and Zanardi, PRL 99, 095701 (2007)
Time dependent GSF

Switch to time domain

\[ D_{\lambda}(\omega) = \langle g(\lambda) | \delta(\omega - \mathcal{H}_{\lambda+\delta}) | g(\lambda) \rangle \]

\[ M(t) = \left| \int D_{\lambda}(\omega) e^{i\omega t} d\omega \right|^2 \]

A local density of states

A Loschmidt echo

\[ M(t) = \left| \langle g(\lambda) | e^{i\mathcal{H}_{\lambda} t} e^{-i\mathcal{H}_{\lambda+\delta} t} | g(\lambda) \rangle \right|^2 \]

- Quantum chaos
- Measures sensitivity to perturbations
- Measures fidelity of a quantum simulation
- Measures decoherence
Loschmidt echo and decoherence

\[ \mathcal{H} = \mathcal{H}_1 \otimes I + \lambda \mathcal{H}_2 \otimes I + \delta \mathcal{H}_2 \otimes \sigma_Z \]

Product initial state

\[ |\psi(0)\rangle = |\epsilon_0\rangle (a |\uparrow\rangle + b |\downarrow\rangle) \]

\[ |\psi(t)\rangle = a |\uparrow\rangle |\epsilon_\uparrow\rangle + b |\downarrow\rangle |\epsilon_\downarrow\rangle \]

\[ \rho(0) = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} \quad \rightarrow \quad \rho(t) = \begin{pmatrix} |a|^2 & ab^*m(t) \\ a^*bm^*(t) & |b|^2 \end{pmatrix} \]

\[ m(t) = \langle \epsilon_\uparrow | \epsilon_\downarrow \rangle = \langle \epsilon_0 | e^{i\mathcal{H}_\lambda+\delta t} e^{-i\mathcal{H}_\lambda-\delta t} |\epsilon_0\rangle \]

Decoherence from a critical environment

QPTs and the Loschmidt echo

\[ M(t) = \left| \left\langle g(\lambda) | e^{i\mathcal{H}_\lambda t} e^{-i\mathcal{H}_\lambda + \delta t} | g(\lambda) \right\rangle \right|^2 \]

Sensitivity to perturbations is used to detect proximity to critical point: far from \( \lambda_c \), evolutions are similar, near \( \lambda_c \) system undergoes large changes even for small perturbation.

**Long time behavior**

Result from quantum chaos: \( M \) decays to \( 1/\# \) states needed to represent unperturbed states with perturbed Hamiltonian = strong decay only near critical point.

Quan et al PRL 96, 140604 (2006)
\[ M(t) = \left| \langle g(\lambda) | e^{i\mathcal{H}_\lambda t} e^{-i\mathcal{H}_\lambda + \delta t} | g(\lambda) \rangle \right|^2 \]

**Short time behavior**

Perturbation theory gives:

\[ M(t) \cong \exp[-\alpha(\lambda) \delta^2 t^2] \]

Where \( \alpha \) is monotonic with \( \lambda \), first derivative has singularity at critical point. Numerical evidence (no proof) suggests that critical exponents are encoded in higher derivatives of \( \alpha \).

\[ \cong \log |\lambda - \lambda_c| \]
QPTs and the Loschmidt echo

- Decay rate has universality features

\[ H(\lambda) = \sum_k \epsilon_k(\lambda) \left( \gamma_k^\dagger(\lambda) \gamma_k(\lambda) + 1/2 \right) \]

\[ \gamma_k(\lambda_1) = \cos(\theta_k) \gamma_k(\lambda_2) - i \sin(\theta_k) \gamma_k^\dagger(\lambda_2) \]

\[ m(t) = \prod_{k > 0} \cos^2(\theta_k) e^{i\epsilon_k(\lambda_2)t} + \sin^2(\theta_k) e^{-i\epsilon_k(\lambda_2)t} \]

\[ f_d(\lambda) = \prod_{k > 0} \cos(\theta_k) \]

\[ |m(t)|^2 = \exp\left(-\sigma^2 N t^2\right) |\cos(\bar{\epsilon} t)|^N \]
QPTs and the Loschmidt echo

- Coupling to environment (perturbation) can be local

- Temperature is ok (not so high)


Zanardi et al PRA 75, 032109 (2007)
The “Algorithm”

- Prepare initial state (can be T>0)
- Measure decoherence vs λ
- Minimum decoherence signals critical point, derivatives give critical exponents
- An instance of a 1-qubit quantum computer...
- But we can get rid of the other part using a quantum simulator
Time reversal in an optical lattice

\[ M(t) = \left| \langle g(\lambda) | e^{i\mathcal{H}_\lambda t} e^{-i\mathcal{H}_\lambda + \delta t} | g(\lambda) \rangle \right|^2 \]

- The LE can be achieved by changing the sign of \( \mathcal{H} \) (imperfect time reversal)

\[ H = -J \sum_{<i,j>} a_i^{\dagger} a_j + a_j^{\dagger} a_i + U \sum_i n_i(n_i-1) \]

Apply linear ramp potential of slope \( F \) for a short time \( \tau \)

\[ e^{iF\tau x} a_i^{\dagger} a_{i+1} e^{-iF\tau x} \]

\[ \Rightarrow e^{iF\tau} a_i^{\dagger} a_{i+1} \]

\[ F\tau = \pi \equiv J \Rightarrow -J \]

\[ U \propto a_S \int |\psi_w(r)|^2 \, d^3r \]

A Feshbach resonance is used to tune \( a_S \Rightarrow -a_S \)
Time reversal in an optical lattice

Perform time reversal with fixed error and look at decay of $M$ as a function of parameters.

Further, we can make sensors by putting the system near criticality and looking at decay of $M$ as a function of parameters.
Conclusions

- Quantum information brings a fresh perspective to the quantum phase transitions field.
- Fidelity is well suited for certain transitions where study of correlations needs very large systems.
- Fidelity works well with MPS (and PEPS?) classical simulations (can it provide better estimates of critical points, exponents?).
- Study/define non-equilibrium quantum phase transitions
- What is the behavior of classical fidelity in normal PT?