

How to Compute Scaling Parameters for Sparse Graph Codes Under Message-Passing Decoding with a Finite Message Alphabet

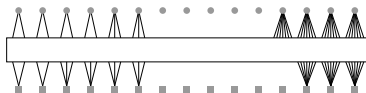
R. Urbanke¹

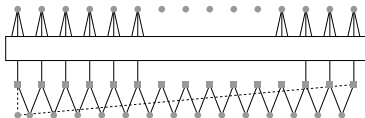
Based on joint work with Jérémie Ezri¹ and Andrea Montanari²

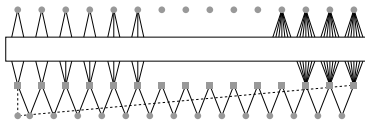
¹EPFL

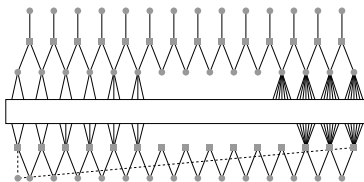
²Stanford University

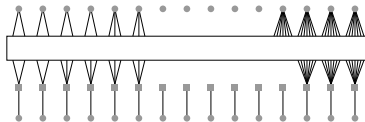
Santa Fe, May 5th 2007

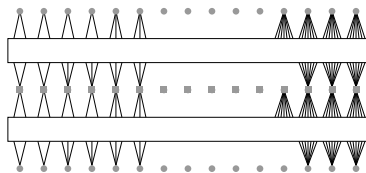




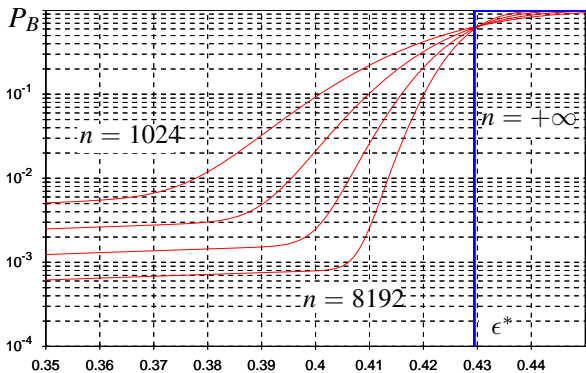








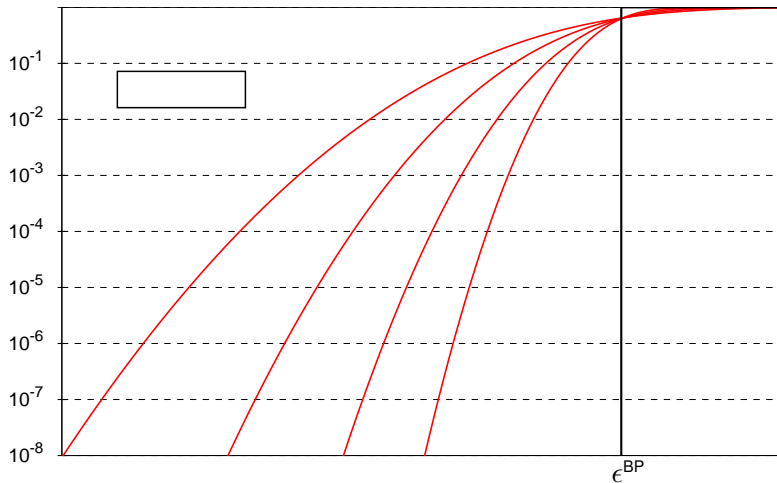
BP, MinSum, LP, Gallager A, Gallager B, Decoder with Erasures, turbo,
time variant, ..., schedules, number of iterations, ...



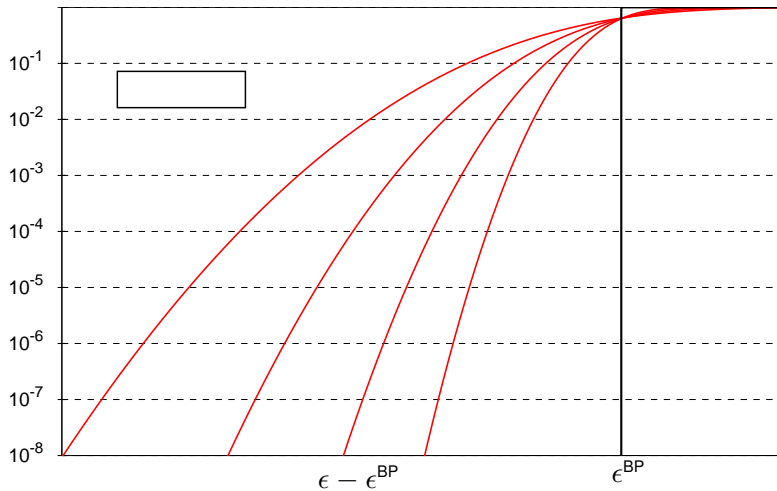
Our Tool: Scaling Law

see talks of David Tse, Eli Ben-Naim and remarks by Christopher Moore

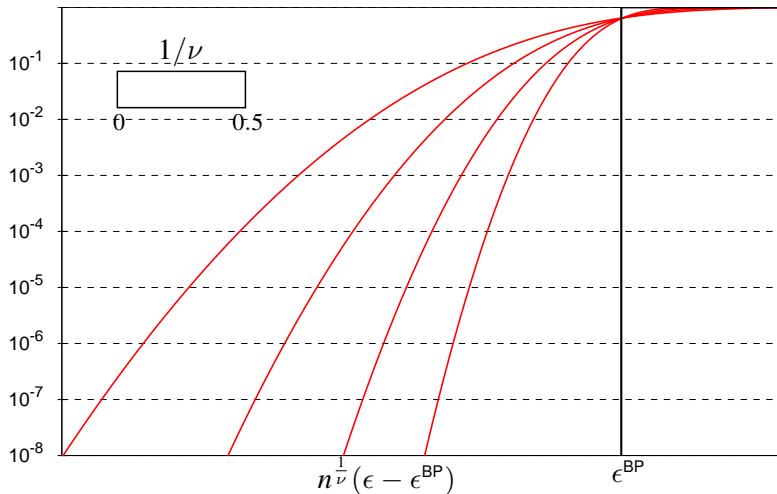
Scaling Around a First Order Phase Transition



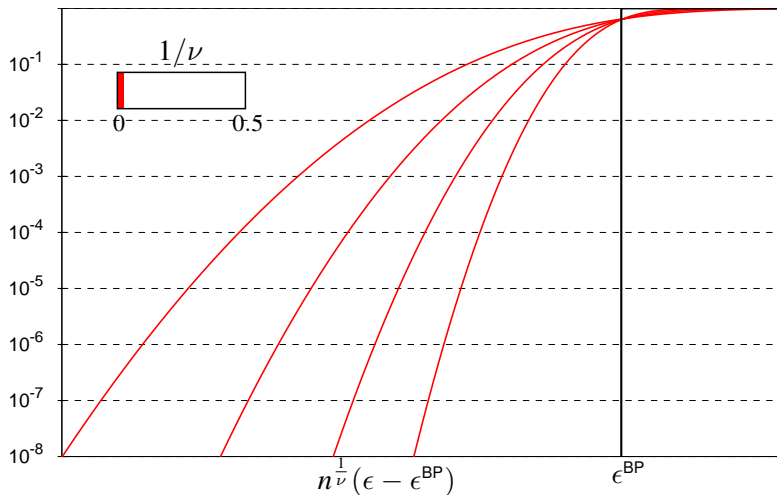
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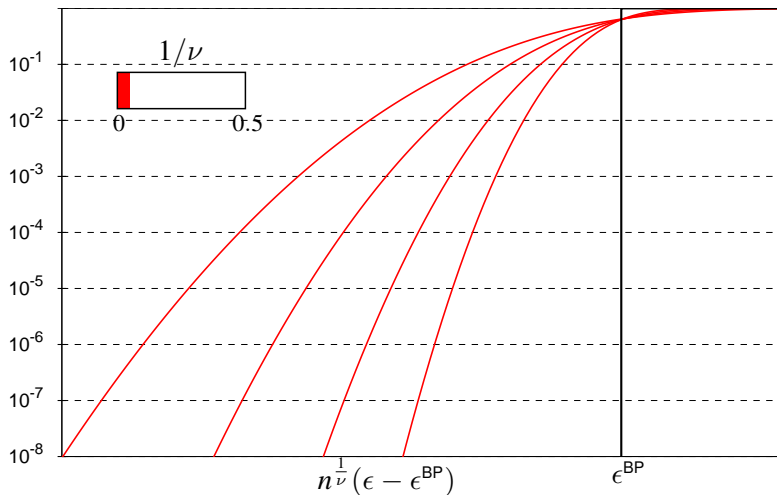
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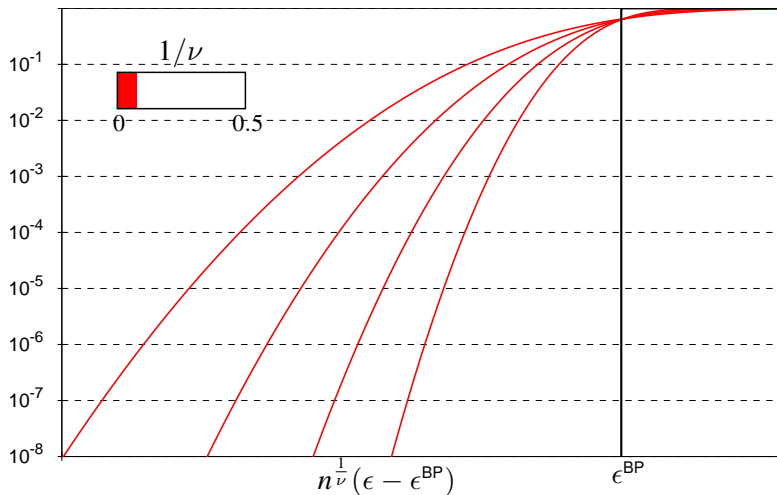
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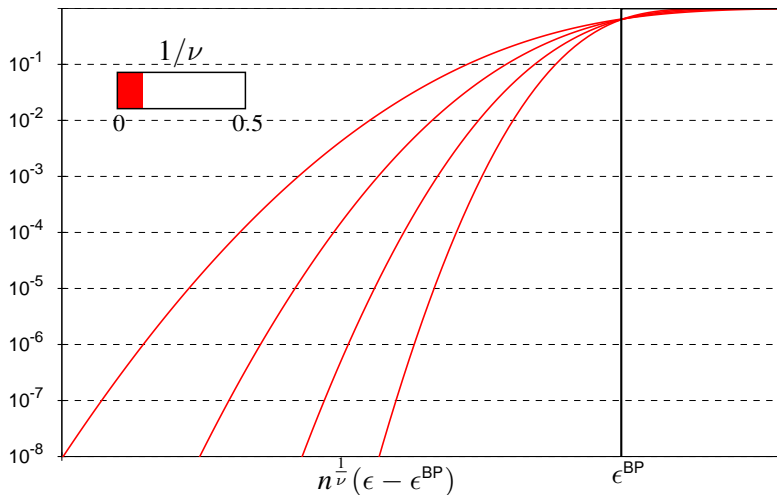
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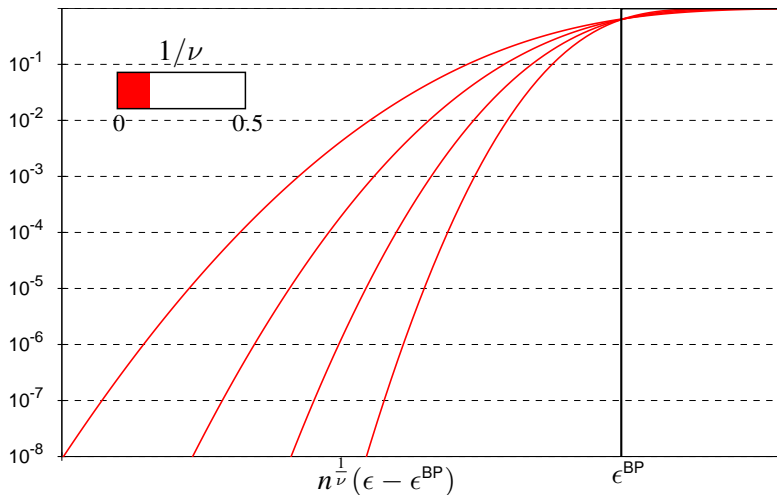
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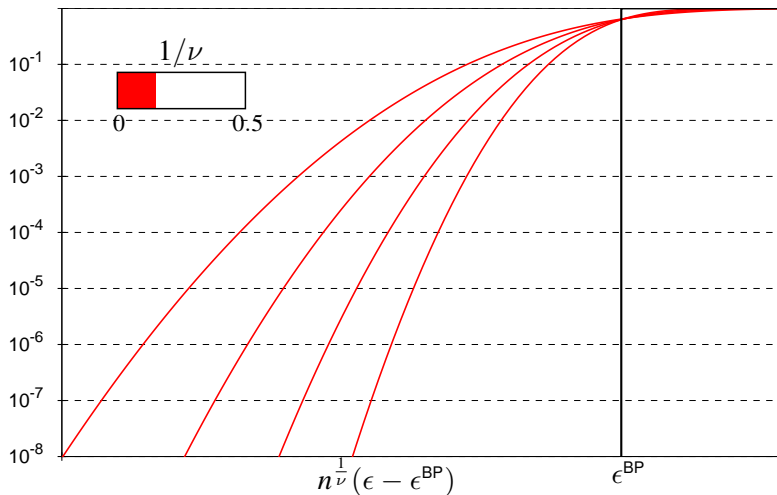
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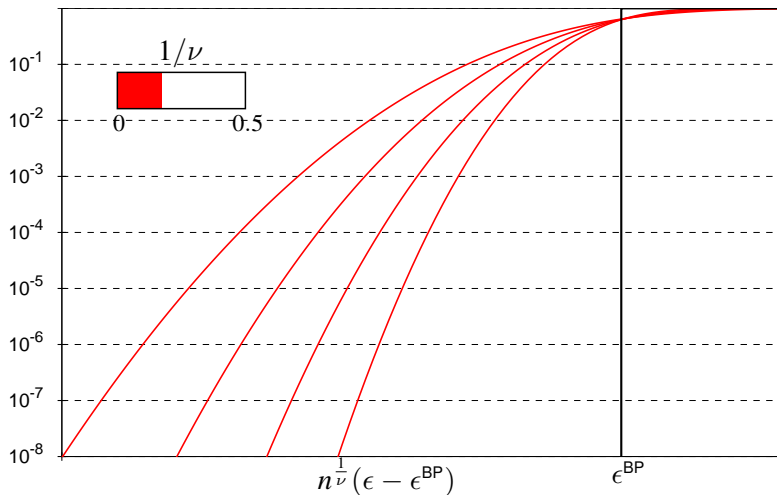
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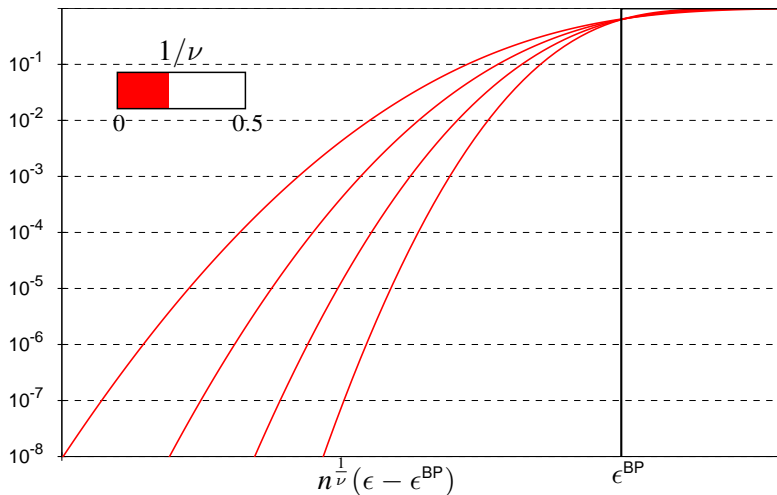
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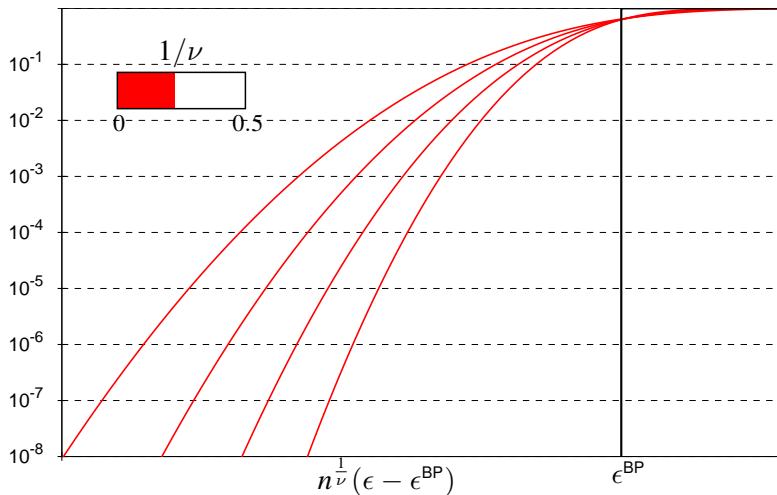
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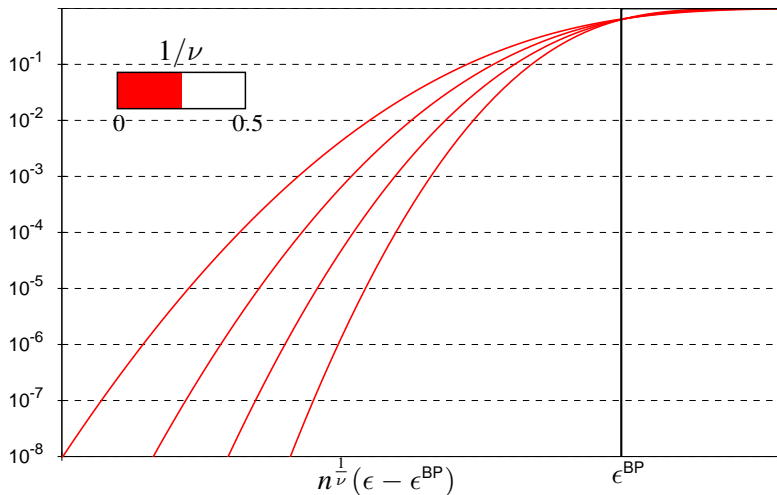
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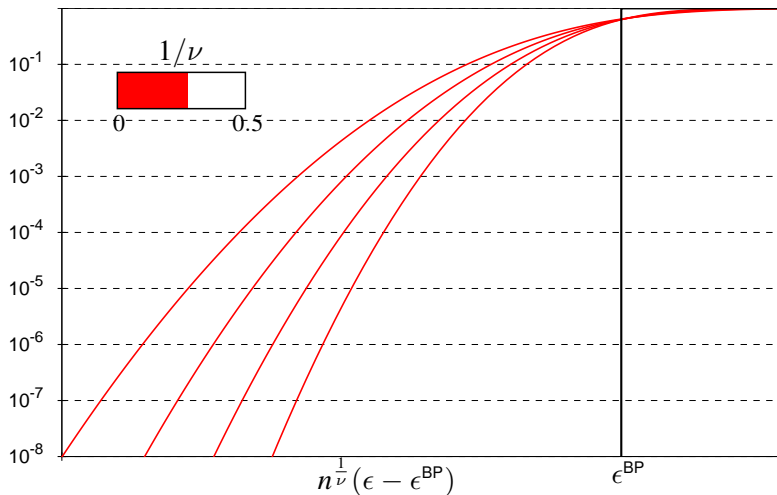
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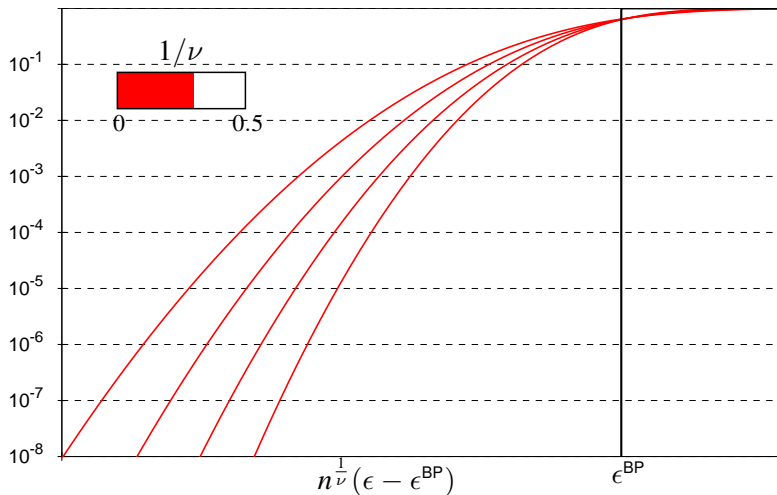
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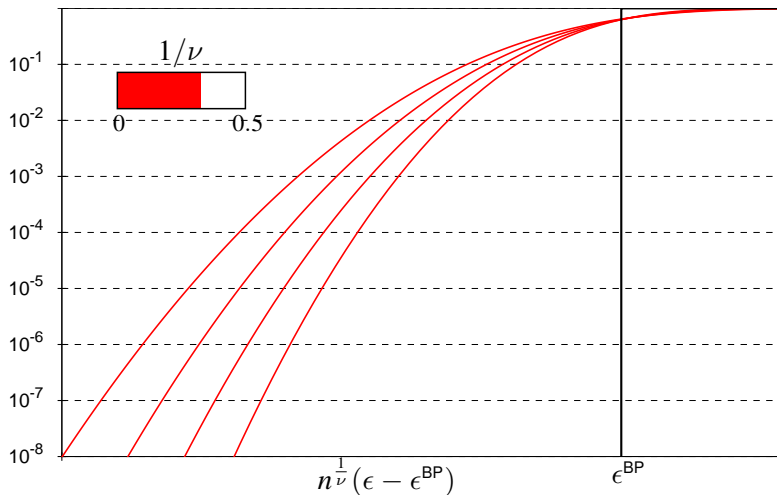
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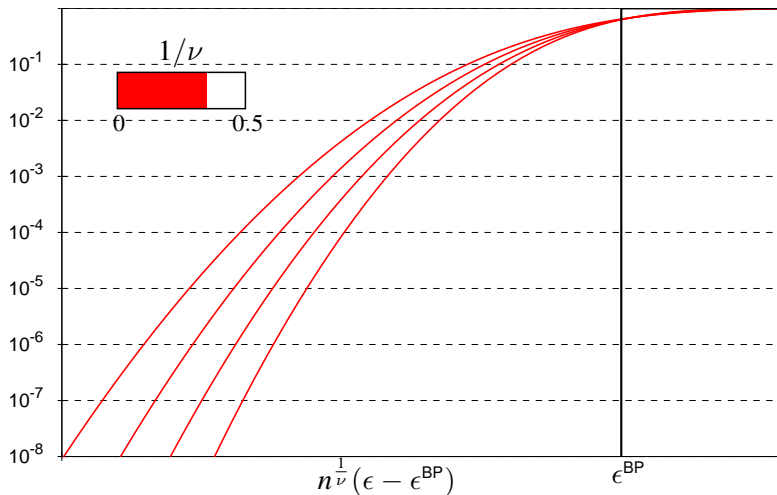
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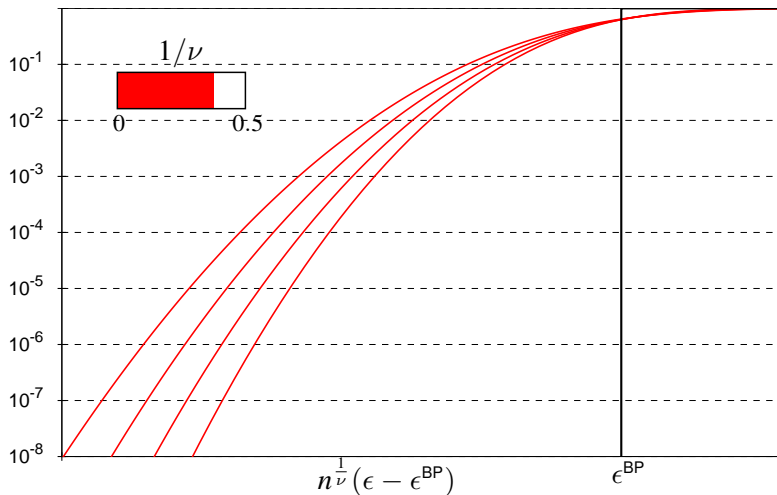
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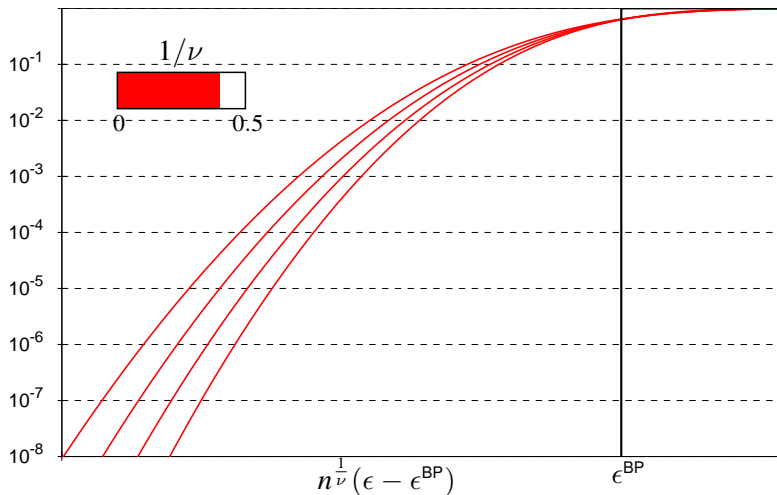
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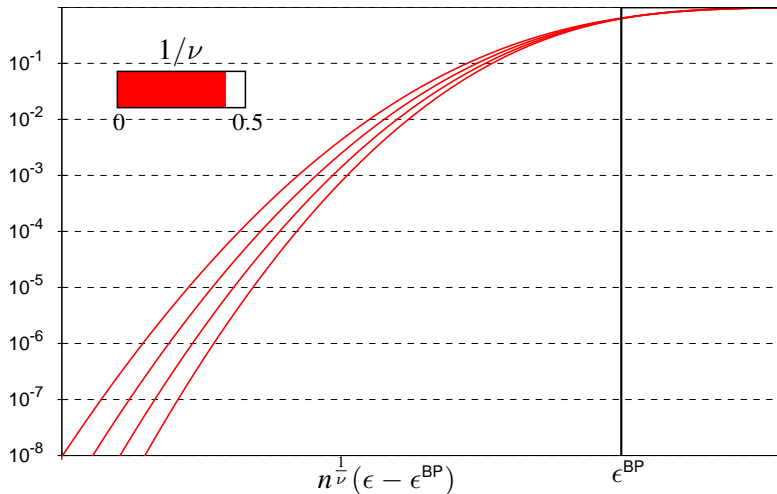
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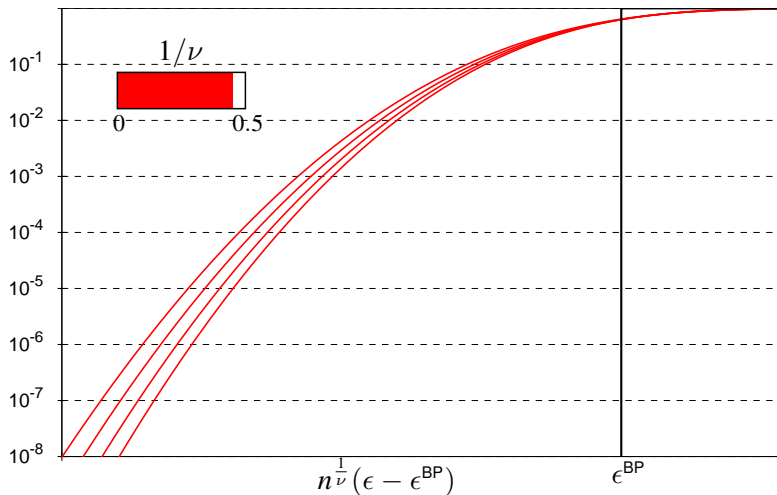
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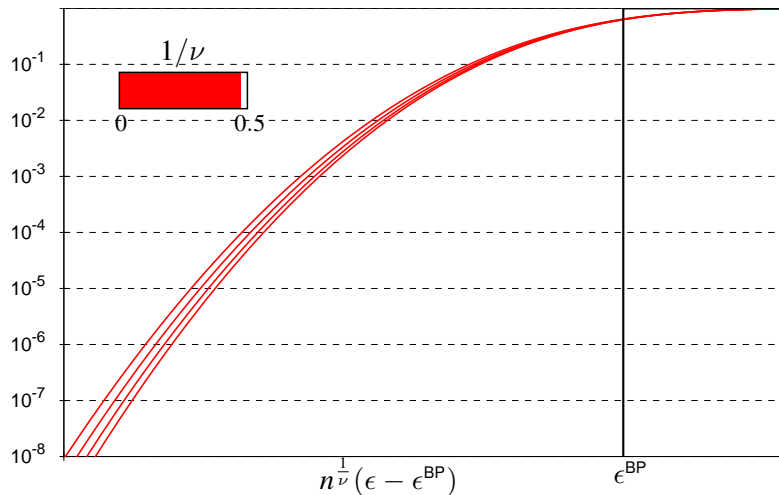
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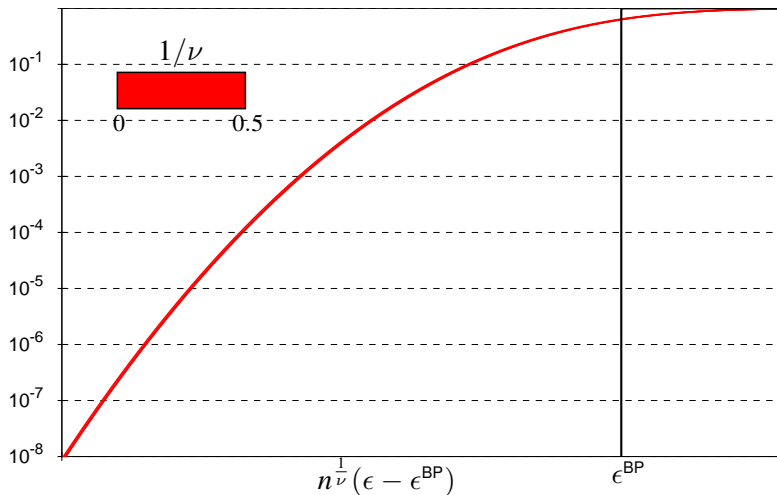
Scaling Around a First Order Phase Transition



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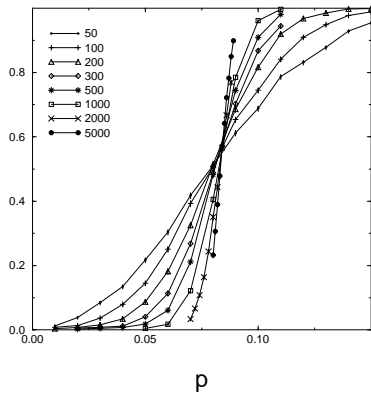


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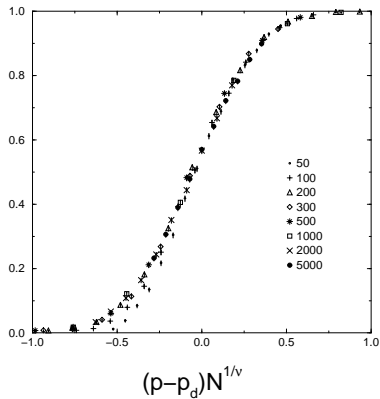


► Play it Again!

Block Error Probability



Block Error Probability



[A. Montanari 02]

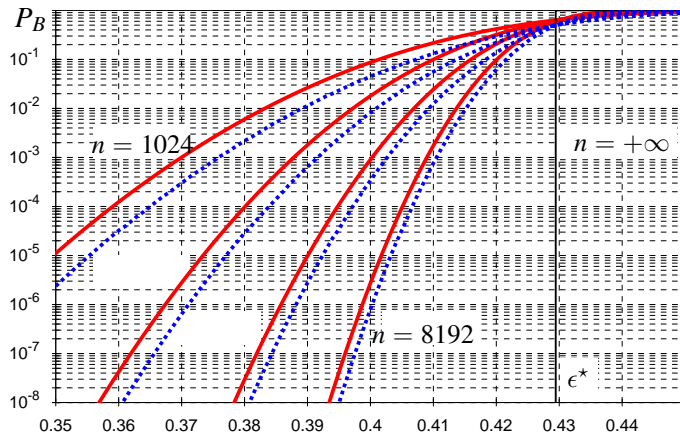
The Simplest Case ... Transmission over the BEC

Theorem (Basic Scaling Law – Amraoui, Montanari, Richardson, Urbanke)

As n tends to infinity with argument of $Q(\cdot)$ kept fixed

$$P_B(n, \lambda, \rho, \epsilon) = Q\left(\frac{\sqrt{n}(\epsilon^{BP} - \epsilon)}{\alpha}\right) (1 + o(1))$$

Waterfall Approximation



Refined Scaling Law For LDPC Codes

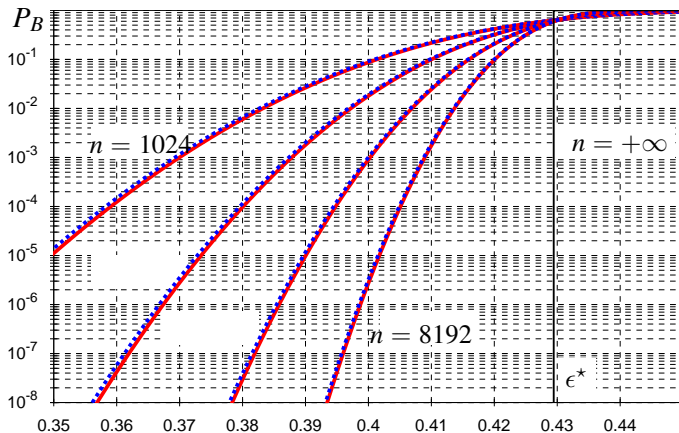
Conjecture (Refined Scaling Law – Amraoui, Montanari, Richardson, Urbanke – recently proved by Dembo and Montanari for Poisson ensembles.)

As n tends to infinity with argument of $Q(\cdot)$ kept fixed

$$P_B(n, \lambda, \rho, \epsilon) = Q\left(\frac{\sqrt{n}(\epsilon^{BP} - \beta n^{-\frac{2}{3}} - \epsilon)}{\alpha}\right) (1 + O(n^{-1/3}))$$

where $\alpha = \alpha(\lambda, \rho)$ and $\beta = \beta(\lambda, \rho)$.

Refined Waterfall Approximation



Computation Leads to ...

$$\alpha = \left(\frac{\rho(\bar{x})^2 - \rho(\bar{x}^2) + \rho'(\bar{x})(1 - 2x\rho(\bar{x})) - \bar{x}^2\rho'(\bar{x}^2)}{L'(1)\lambda(y)^2\rho'(\bar{x})^2} + \frac{\epsilon^2\lambda(y)^2 - \epsilon^2\lambda(y^2) - y^2\epsilon^2\lambda'(y^2)}{L'(1)\lambda(y)^2} \right)^{1/2},$$

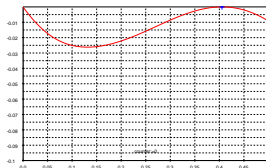
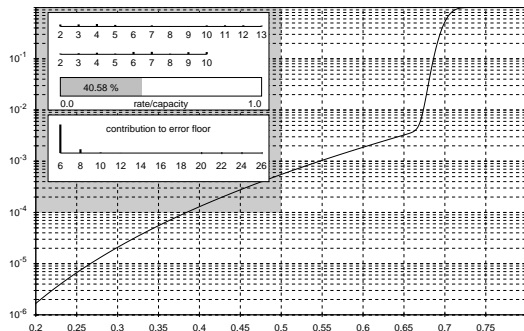
$$\beta = \left(\frac{\epsilon^4 r_2^2 (\epsilon \lambda'(y)^2 r_2 - x(\lambda''(y)r_2 + \lambda'(y)x))^2}{L'(1)^2 \rho'(\bar{x})^3 x^{10} (2\epsilon \lambda'(y)^2 r_3 - \lambda''(y)r_2 x)} \right)^{1/3},$$

$$r_i = \sum_{m \geq j \geq i} (-1)^{i+j} \binom{j-1}{i-1} \binom{m-1}{j-1} \rho_m(\epsilon \lambda(y))^j,$$

with ϵ the channel erasure probability at the critical point,
 x and y the erasures probabilities in the decoder at that point,
with $\bar{x} = 1 - x$ and $y = 1 - \rho(1 - x)$.

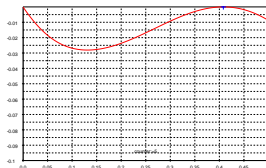
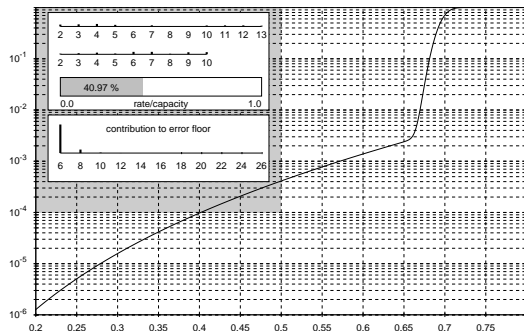
Optimization For BEC

- Complete approximation for the BEC (waterfall + error floor)
- Fix ϵ , n and a target error probability P_{targ}
→ degree distribution optimization using LP



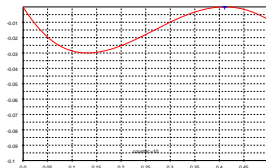
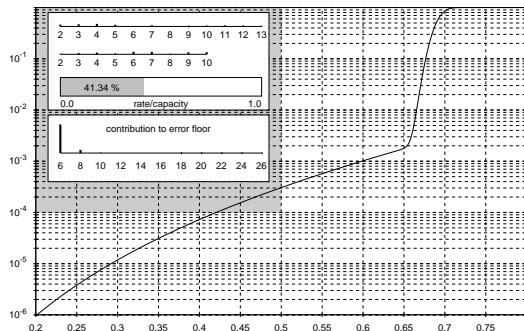
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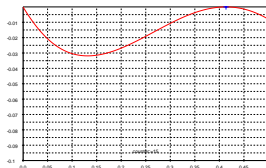
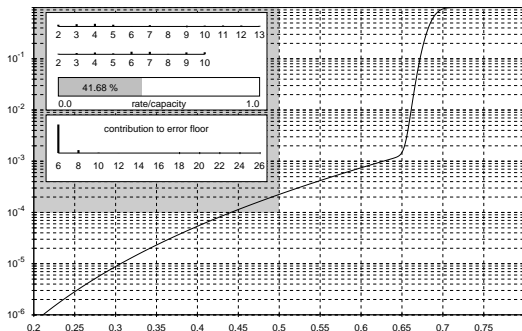
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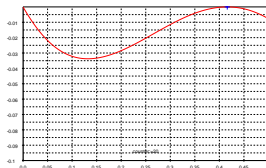
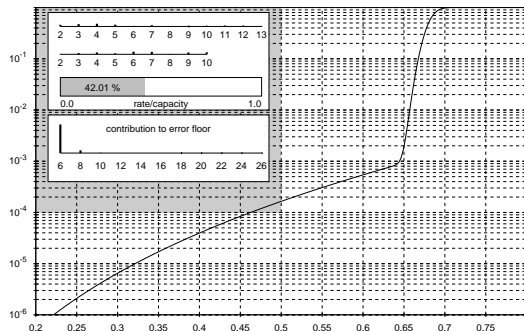
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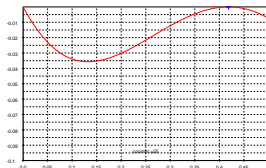
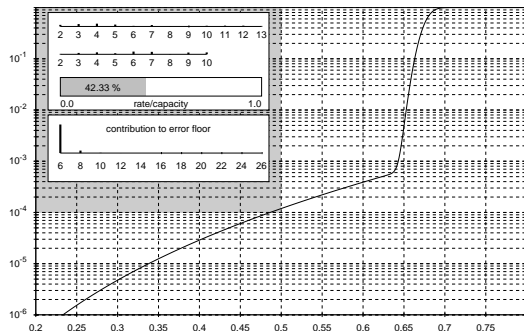
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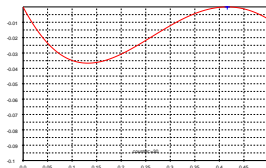
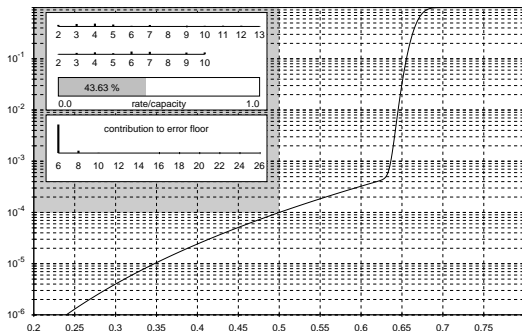
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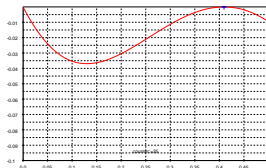
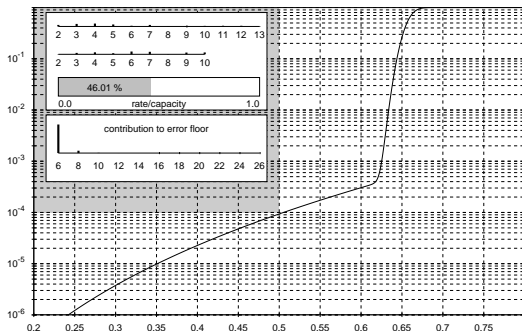
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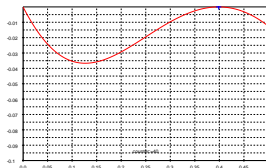
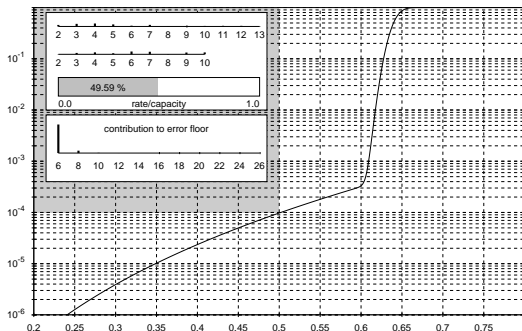
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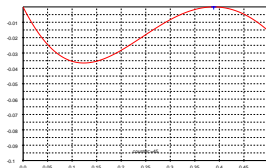
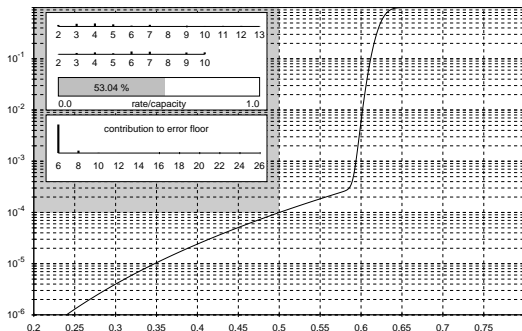
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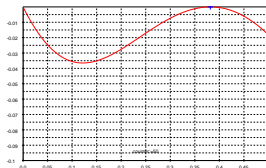
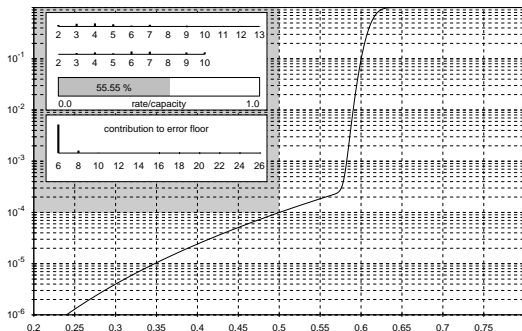
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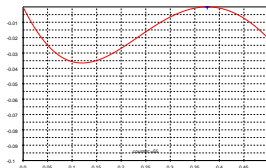
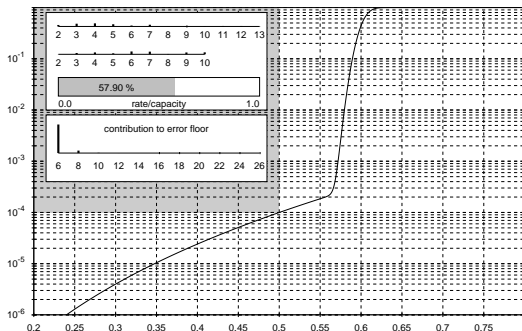
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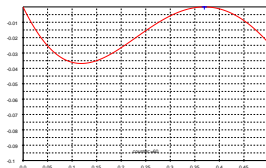
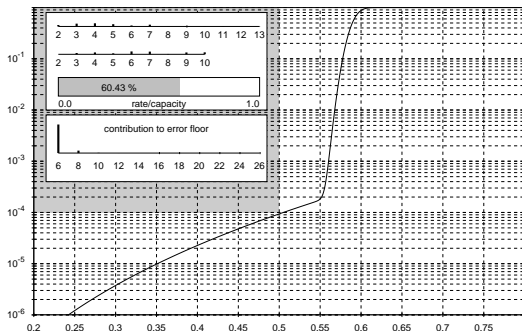
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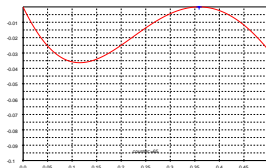
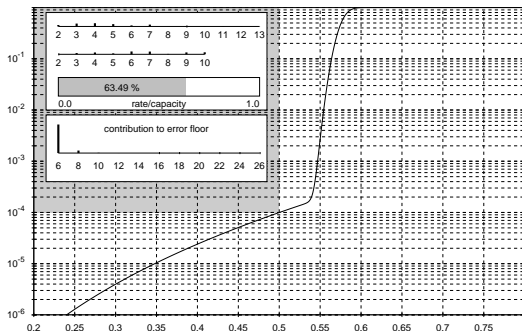
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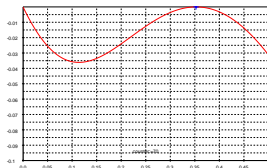
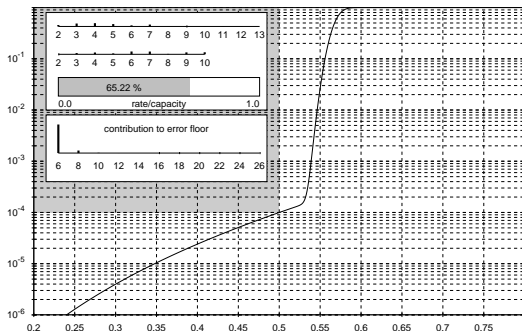
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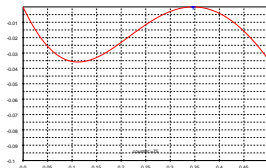
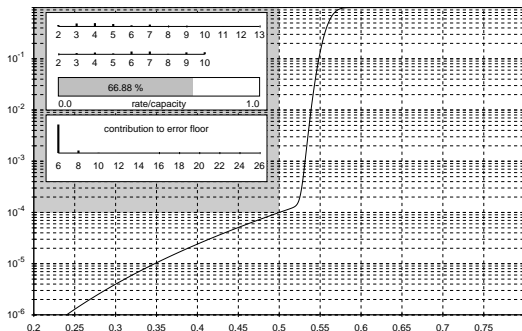
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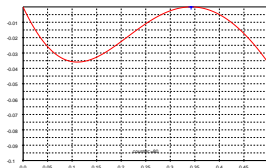
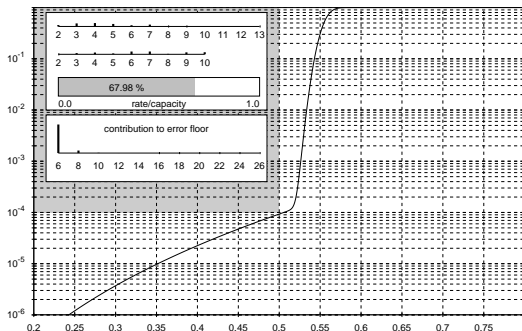
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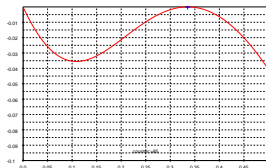
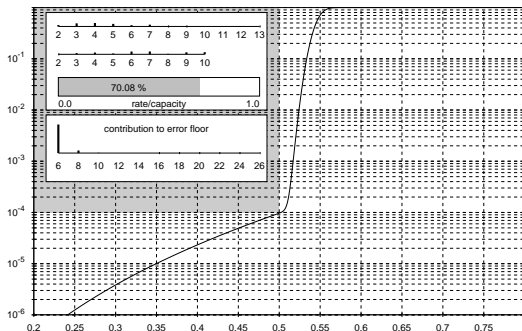
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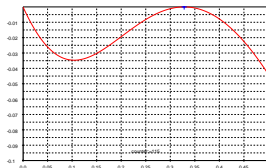
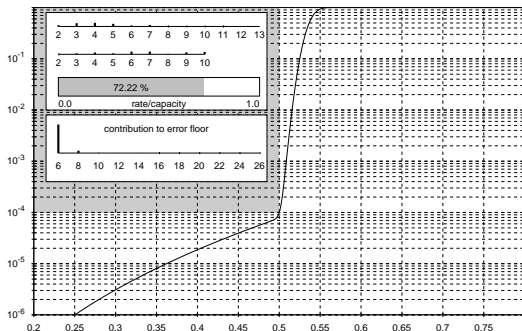
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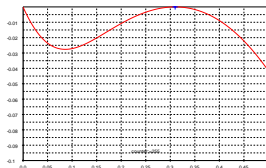
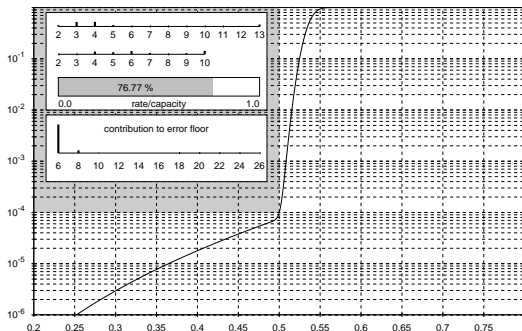
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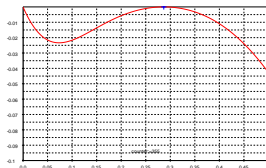
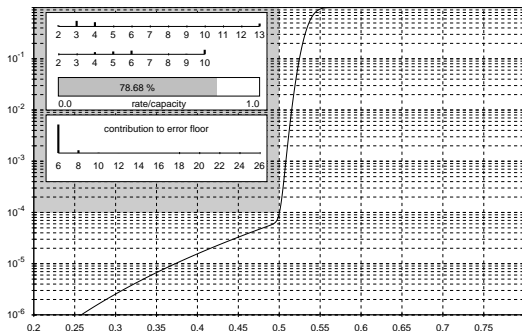
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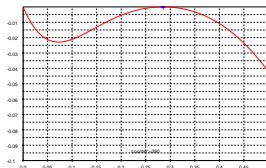
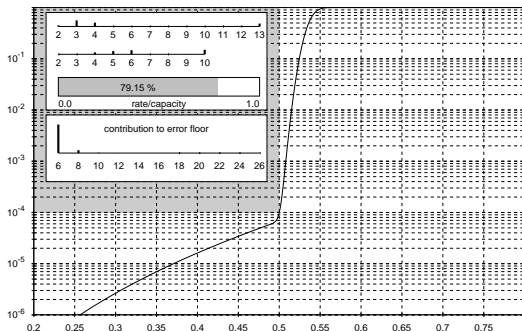
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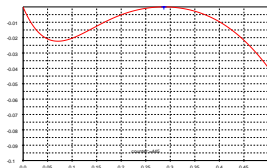
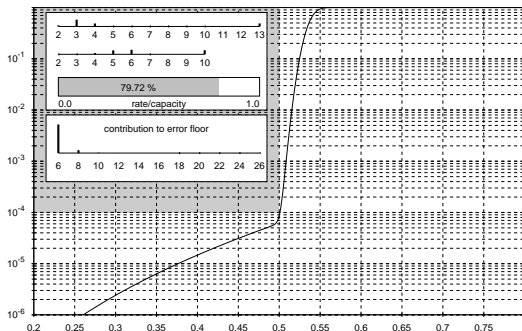
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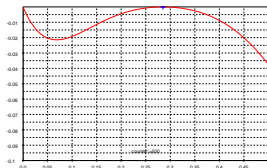
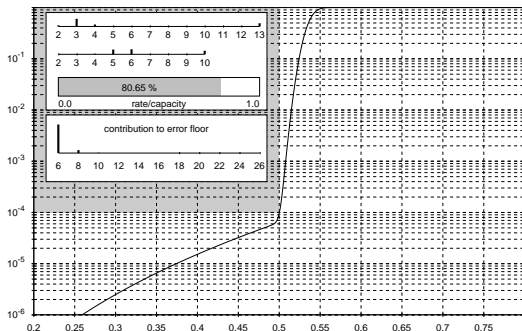
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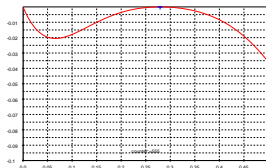
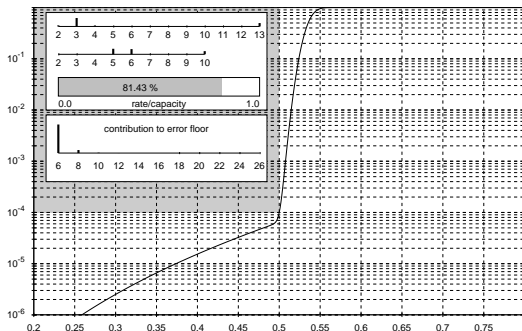
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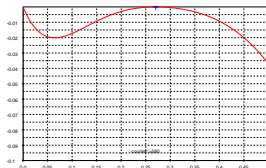
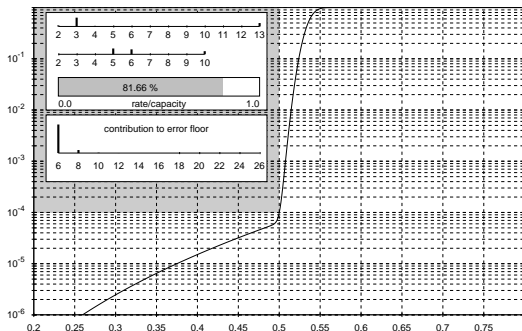
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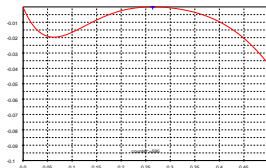
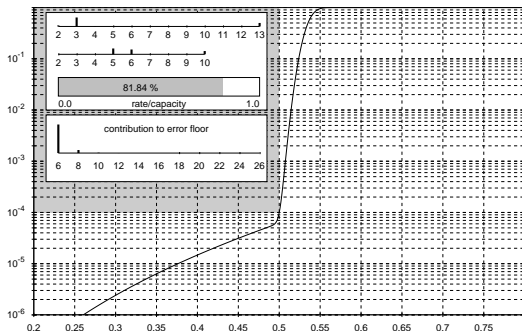
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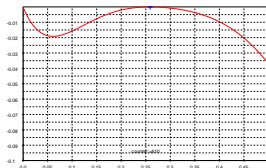
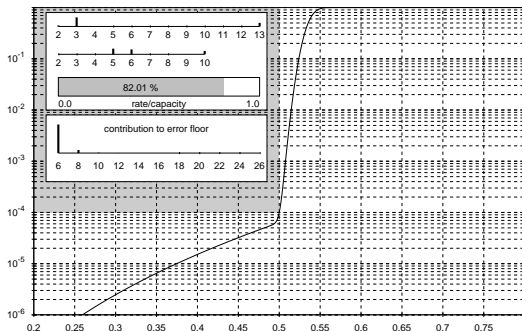
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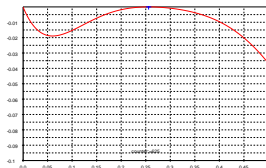
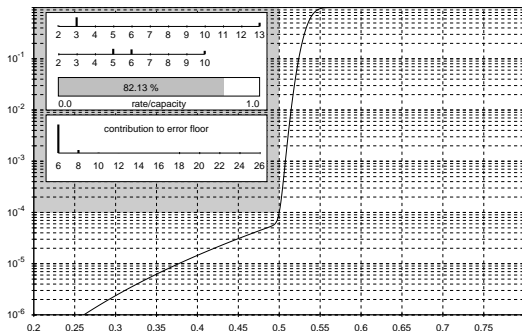
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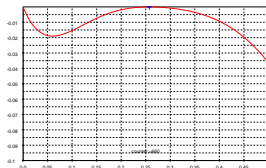
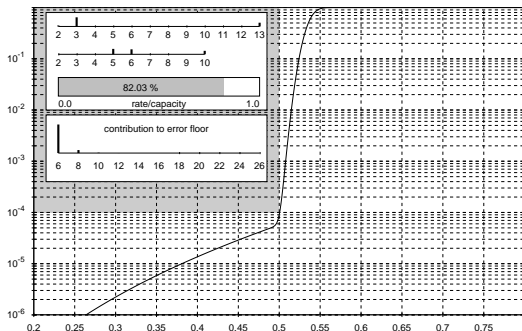
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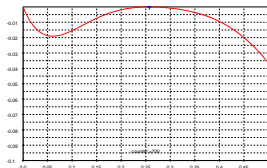
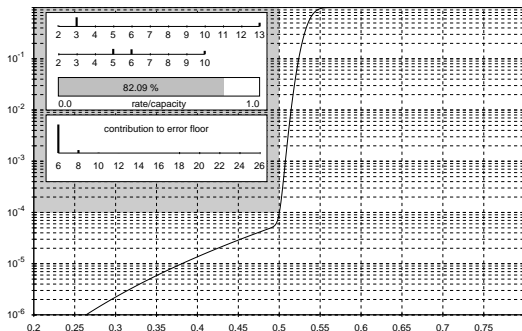
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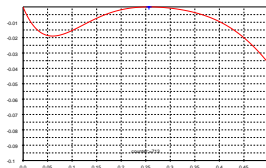
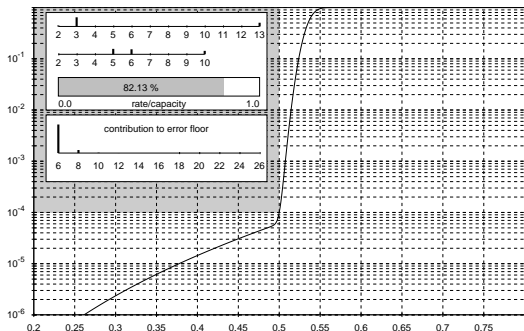
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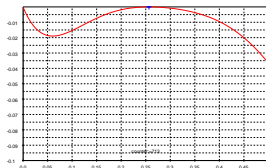
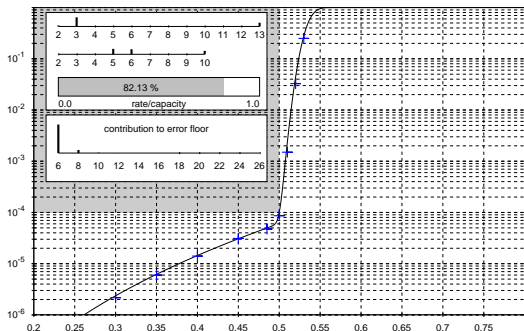
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$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12}$$

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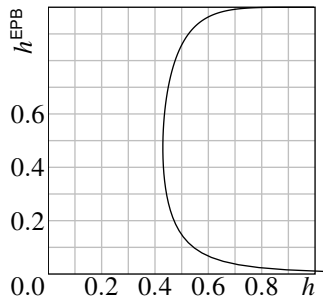


$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

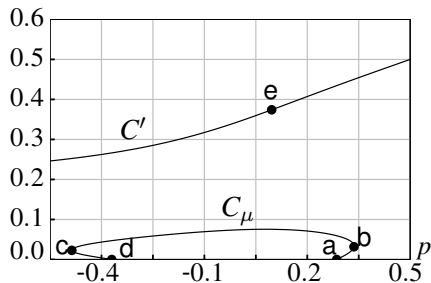
Scaling for General Message-Passing Decoders

computation of scaling parameter

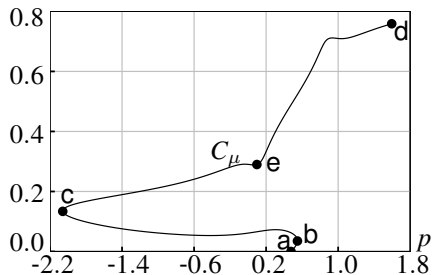
EXIT LIKE CURVES



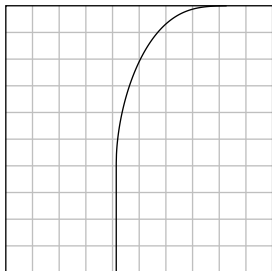
Do Such Curves Exist?



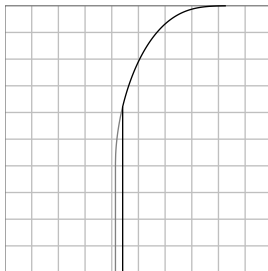
[Rathi, U 07]



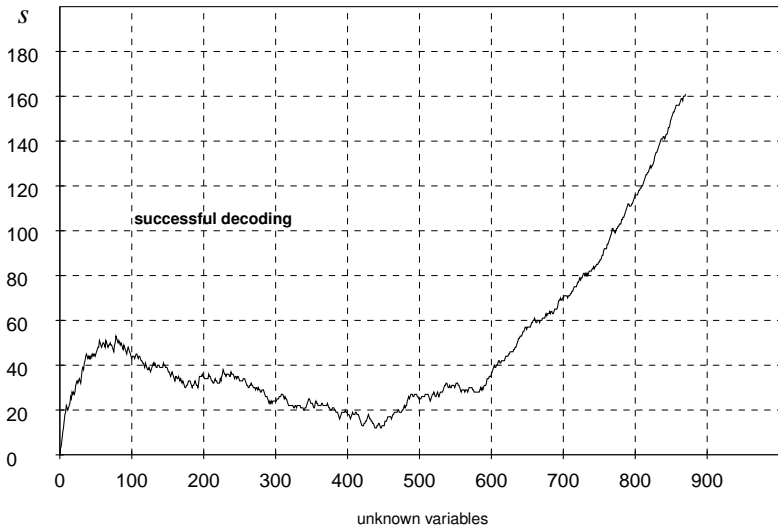
Admissible EXIT Curves



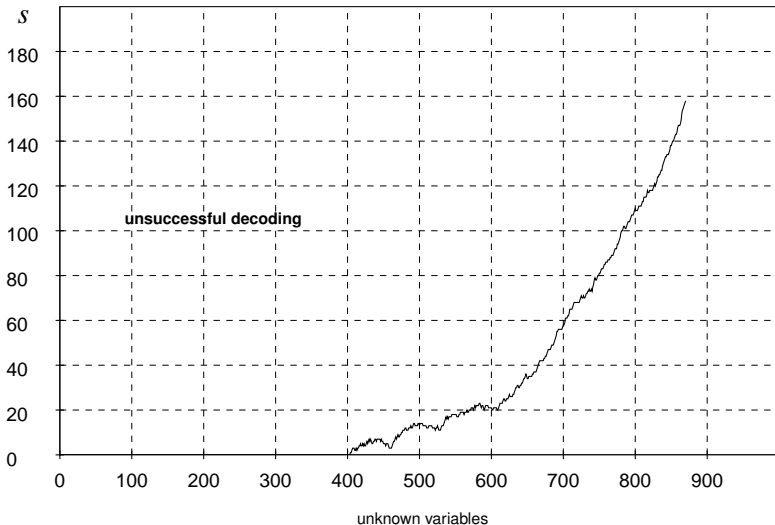
Admissible



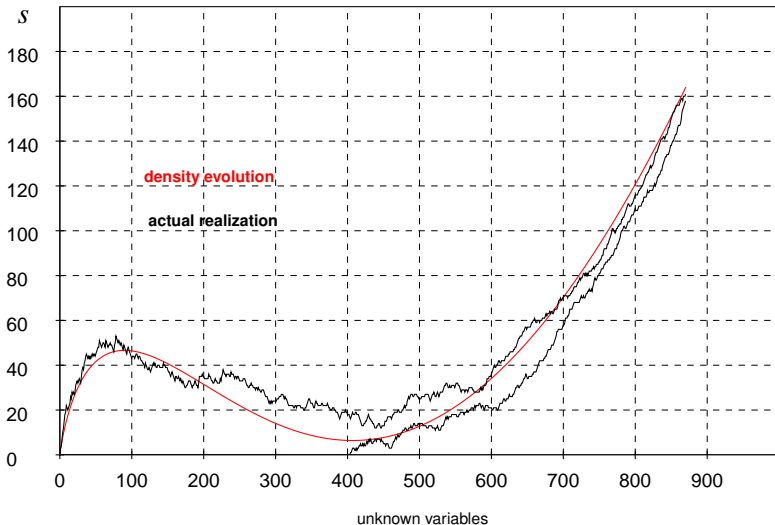
Not Admissible



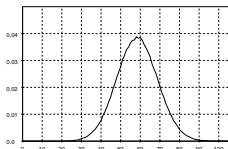
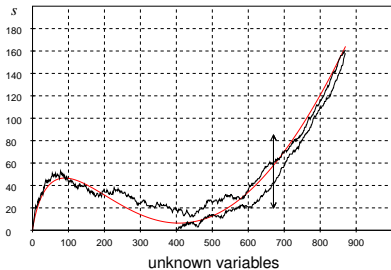
(3,6) code, length 2048, fixed erasure $\epsilon = 0.425$. $P_B = 0.47528$.



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empirical dist. of degree one check nodes

(3,6) code, length 2048, fixed erasure $\epsilon = 0.425$. $P_B = 0.47528$.

$$P_B \sim Q \left(\frac{\sqrt{n}(\epsilon - \beta n^{-\frac{2}{3}} - \epsilon^*)}{\frac{\partial^2 \epsilon(x)}{\partial x^2} \Big|_{x^*} \lim_{\epsilon \rightarrow \epsilon^*} (x - x^*) \sqrt{\frac{\mathcal{V}}{\Lambda'(1)}}} \right)$$

$$\mathcal{V} = \frac{\mathbb{E}(X_\epsilon - \bar{X}_\epsilon)^2}{n\Lambda'(1)}$$

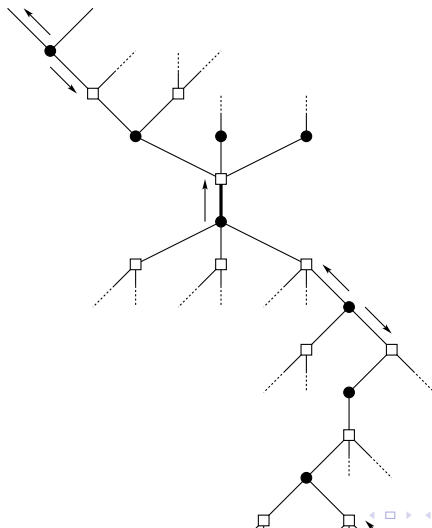
binary erasure channel

$$P_B \sim Q \left(\frac{\sqrt{n}(h - \beta n^{-\frac{2}{3}} - h^*)}{\frac{\partial^2 h(x)}{\partial x^2} \Big|_{x^*} \lim_{\epsilon \rightarrow \epsilon^*} (x - x^*) \sqrt{\frac{\mathcal{V}}{\Lambda'(1)}}} \right)$$

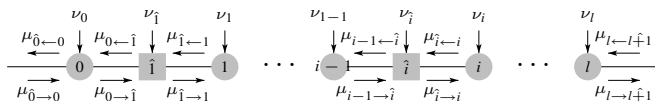
$$\mathcal{V} = \frac{\mathbb{E}(X_\epsilon - \bar{X}_\epsilon)^2}{n\Lambda'(1)}$$

general case

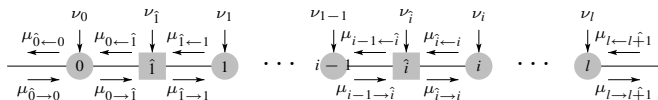
How to Compute Correlation



How to Compute Correlation

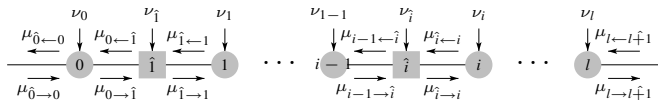


How to Compute Correlation



$$cM^{l/2}K(M^T)^{l/2}c^T$$

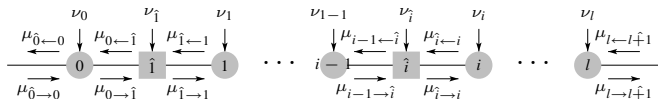
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$M : \lambda_1 = 1; \lambda_2 \text{ degenerated}$

How to Compute Correlation

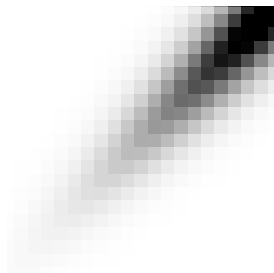


$$\text{correlation for depth } l = \frac{2(1 - 1)c_3^2}{\lambda_2} e_2 K e_3^T l (\gamma \lambda_2)^l (1 + O(x - x^*))$$

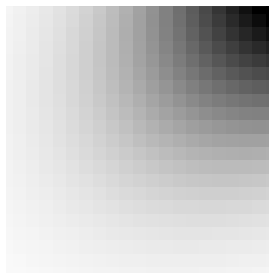
$M : \lambda_1 = 1; \lambda_2 \text{ degenerated}$

$$\mathcal{V}(1 - \gamma^2 \lambda_2^2)^2 \triangleq \frac{(1 - 1)c_3^2}{2\lambda_2} e_2 K e_3^T$$

Flipping Probabilities



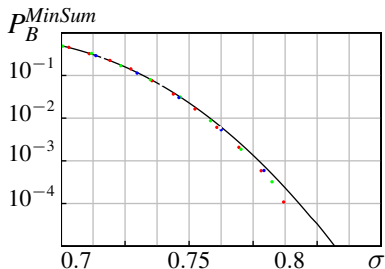
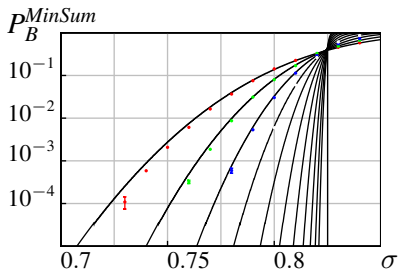
Quantized BP



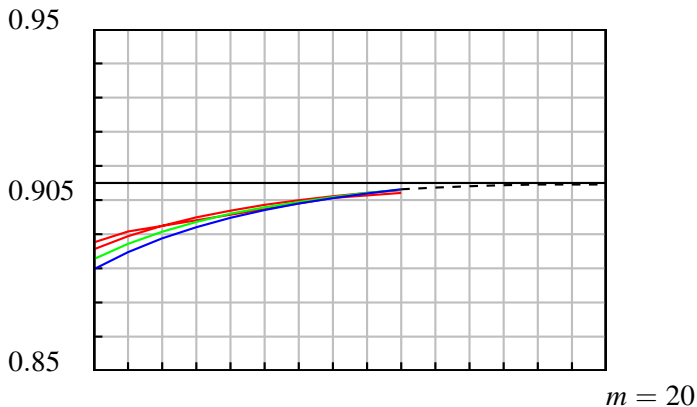
Quantized MinSum

Results - (3, 6), BAWGNC, MinSum, $MAXL = 5$, $m = 10$

$$\sigma^* = 0.825, \alpha \approx 0.842,$$

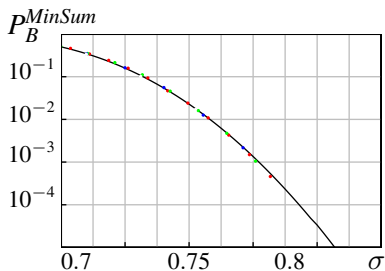
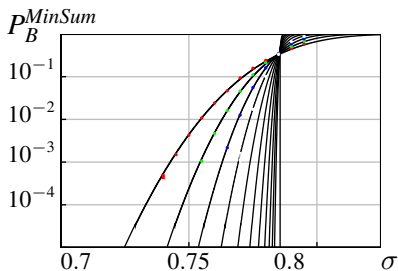


(3, 6), Sequence of Quantized MinSum Decoders



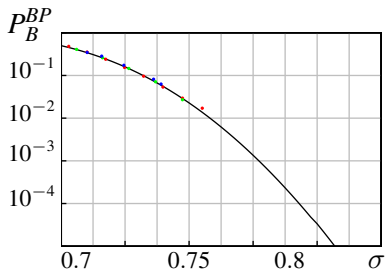
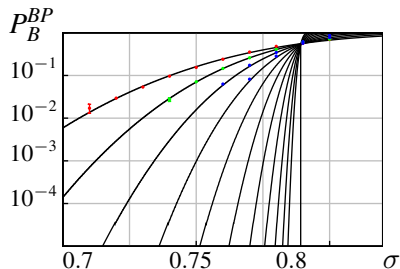
Results - (3, 6), BAWGNC, MinSum, $MAXL = 20$, $m = \infty$

$$\sigma^* = 0.82125, \alpha \approx 0.905,$$



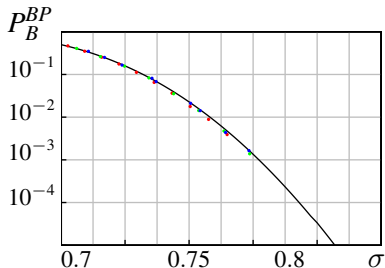
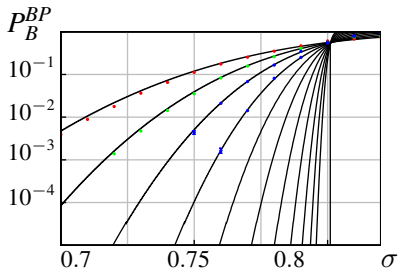
Results - (3, 6), BAWGNC, BP, $MAXL = 5.13625$, $m = 7$

$$\sigma^* = 0.86915, \alpha \approx 0.900005,$$



Results - (3, 6), BAWGNC, BP, $MAXL = 20$, $m = \infty$

$$\sigma^* = 0.881, \alpha \approx 0.97,$$



- scaling in principle allows joint optimization of code and decoder
- computational complexity (m^6)
- irregular
- general ensembles
- error floor
- optimization
- proof :-)