# Dynamical Structures in Iterative Decoding

**Misha Stepanov**

stepanov@math.arizona.edu

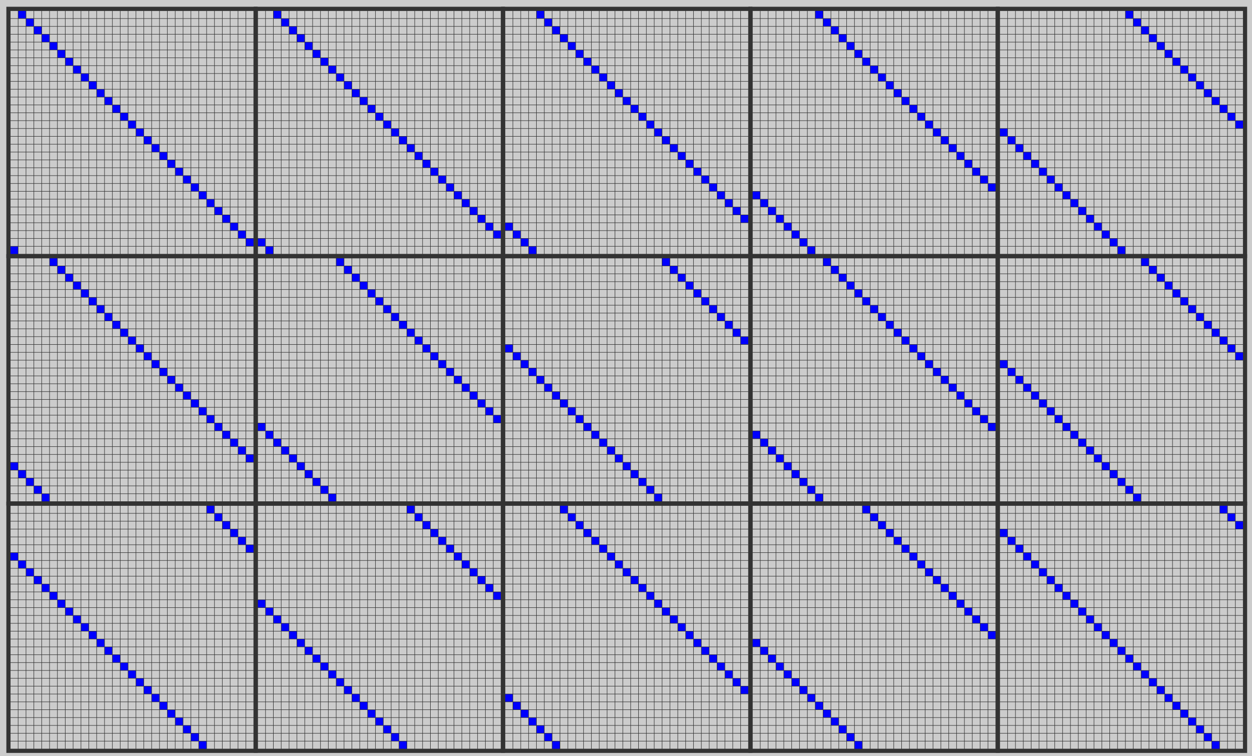
Department of Mathematics,
University of Arizona,
Tucson, AZ 85721, USA

2007-05-04 @ Algorithms, Inference, & Statistical Physics

# Tanner's $[155, 64, 20]$ code

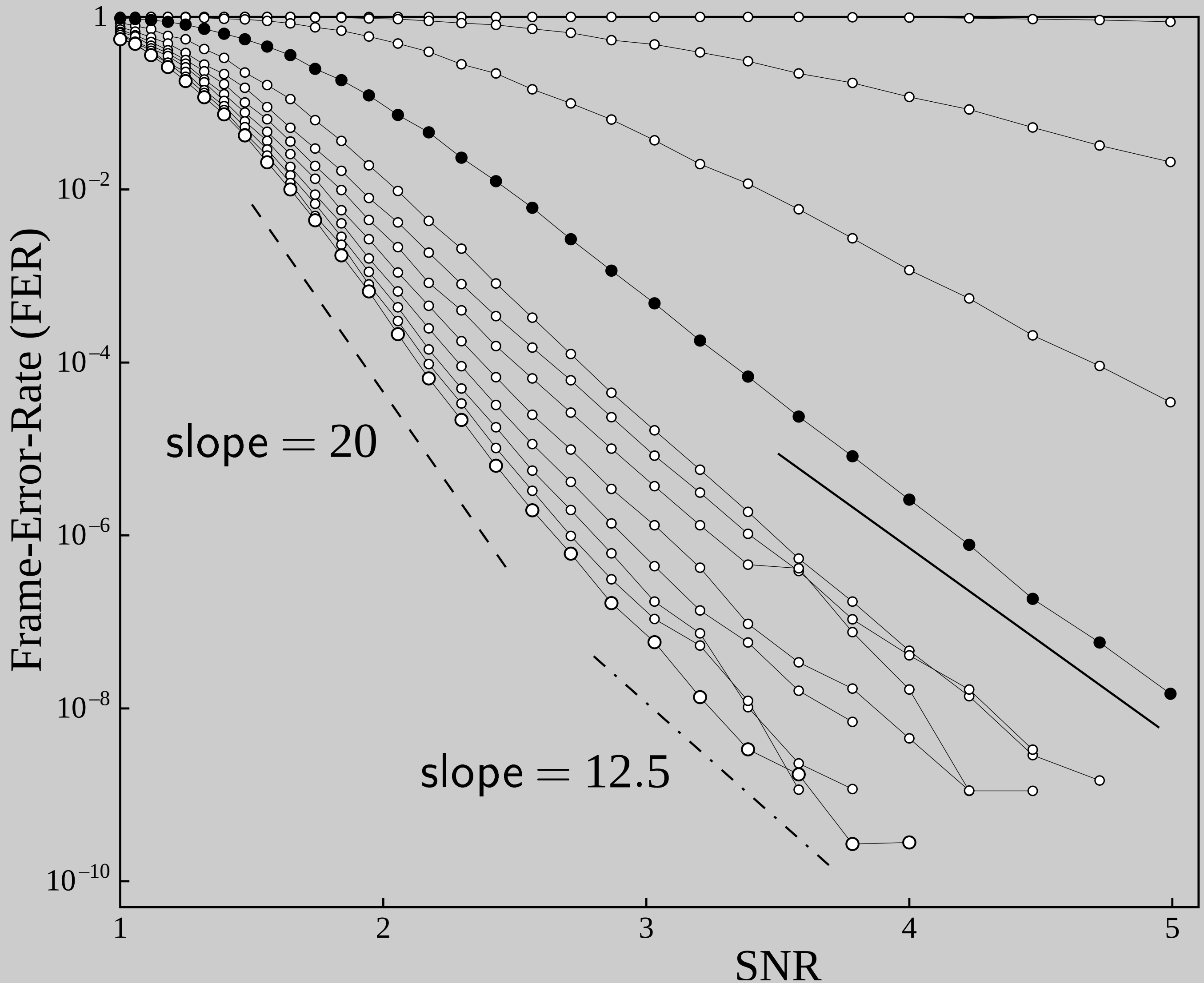
Hamming distance
informational bits
length of encoded message

Parity check matrix:



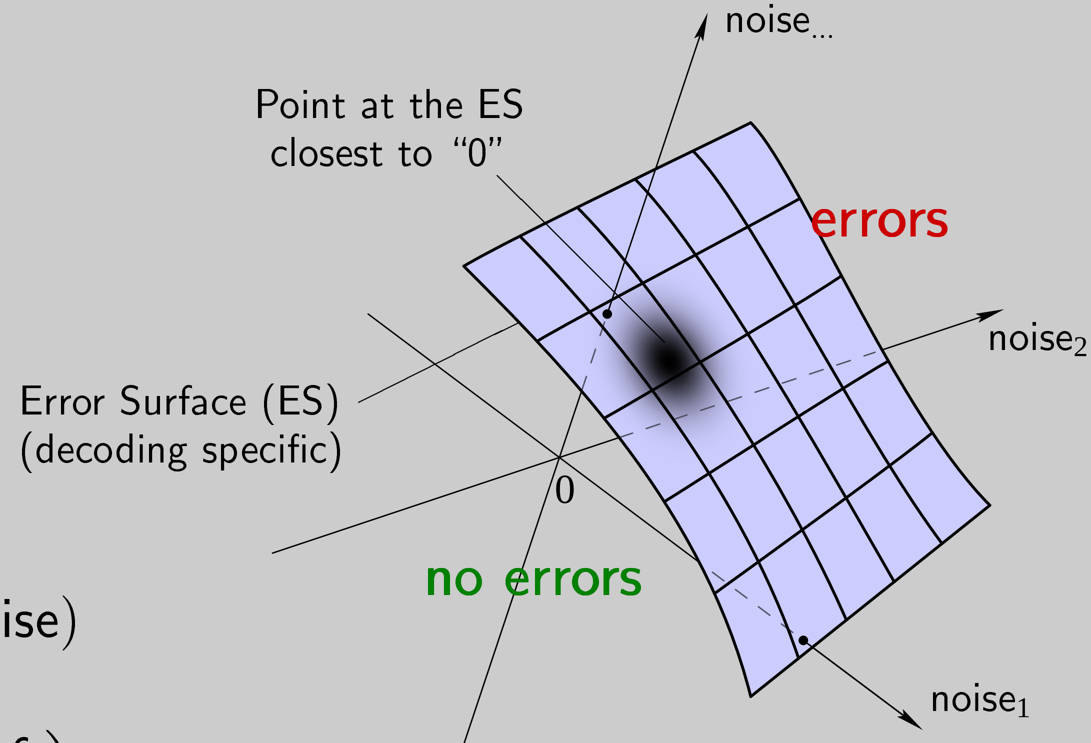
R.M. Tanner, D. Sridhara, T. Fuja, in *Proc. ISCTA 2001* (Ambleside, UK, July 15–20, 2001), p. 365.

# Frame-Error-Rate



# Instanton method

instanton method
saddle-point method
Laplace method
method of steepest descent
large deviations




$$\text{BER} = \int \text{d(noise)}\,\text{WEIGHT(noise)}$$

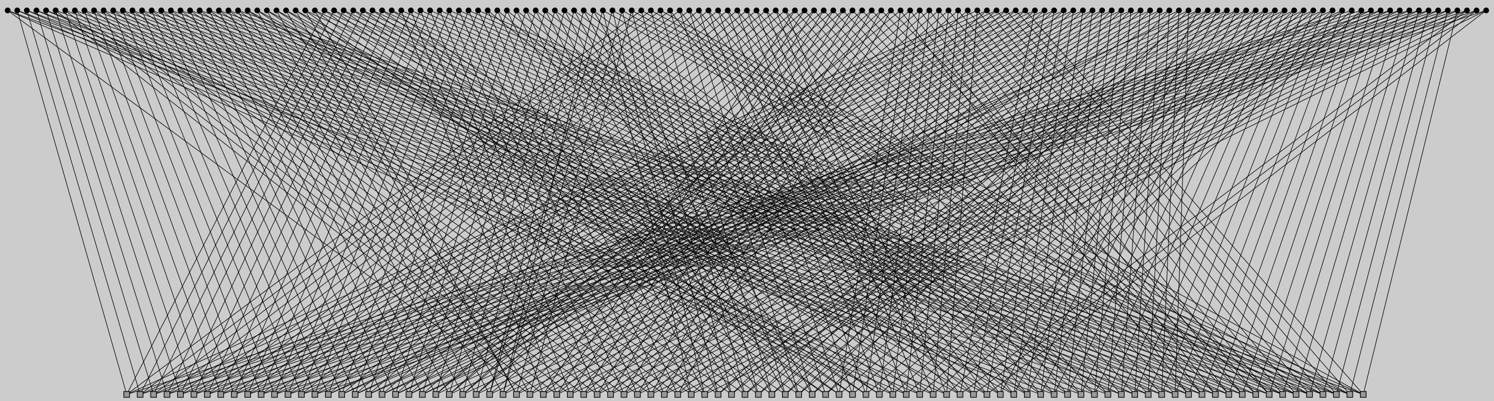
$$\text{BER} \sim \text{WEIGHT}\begin{pmatrix}\text{optimal conf}\\ \text{of the noise}\end{pmatrix}$$

$$\begin{matrix}\text{optimal conf}\\ \text{of the noise}\end{matrix} = \begin{matrix}\text{Point at the ES}\\ \text{closest to "0"}\end{matrix}$$

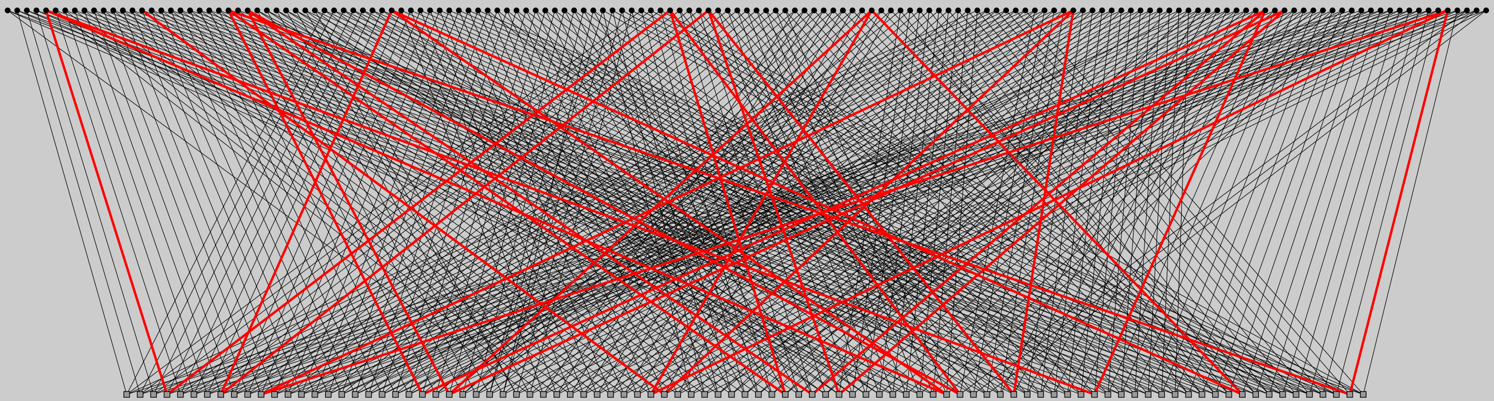

Chernyak, Chertkov, Stepanov, Vasic, Phys. Rev. Lett. **93**, 198702 (2004)

Stepanov, Chertkov, Chernyak, Vasic, Phys. Rev. Lett. **95**, 228701 (2005) [cond-mat/0506037]

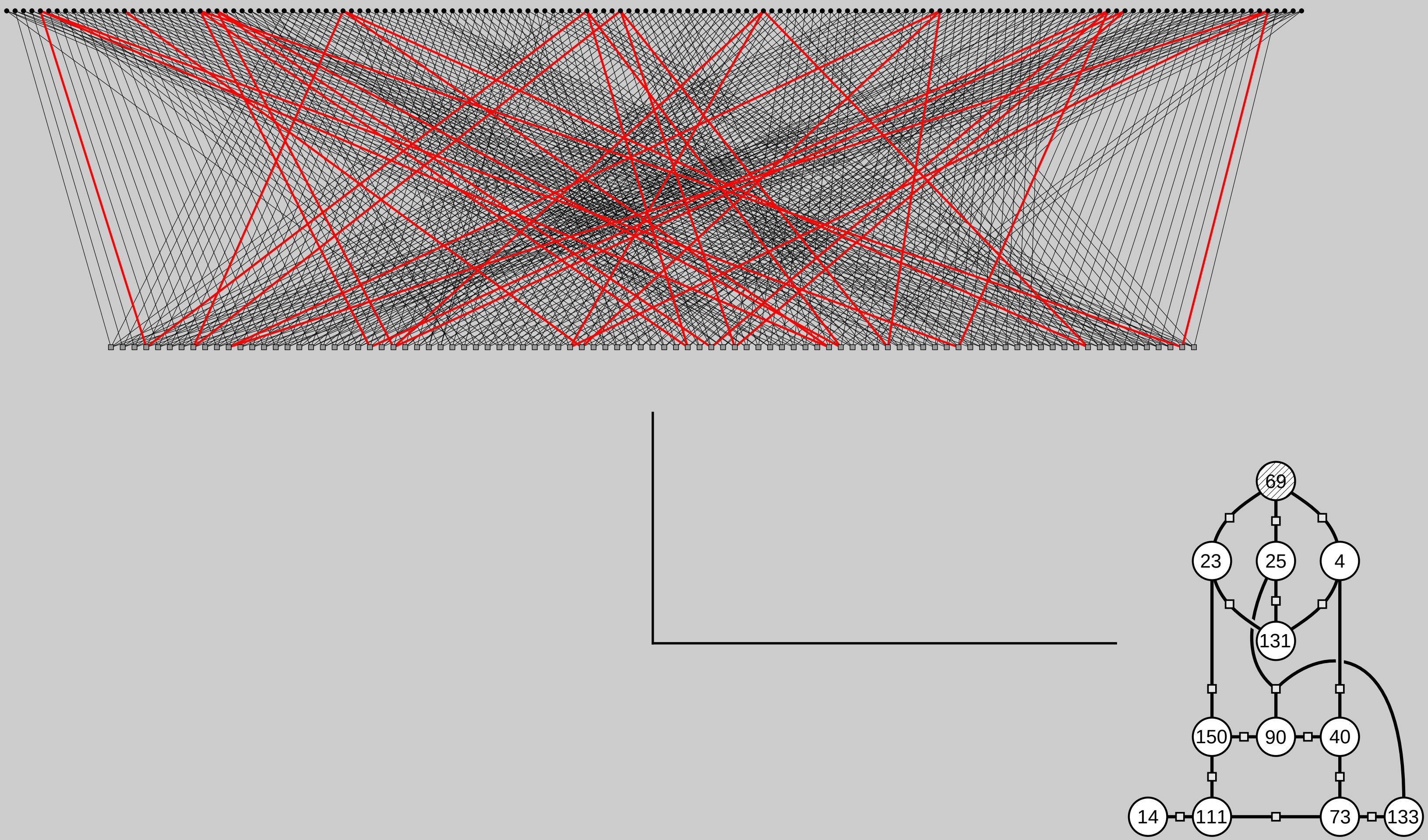
# Tanner graph of $[155, 64, 20]$ code



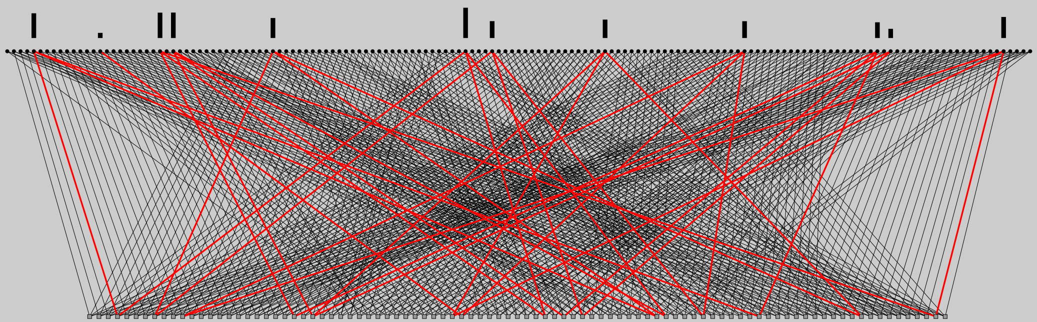
# One special subgraph of Tanner graph

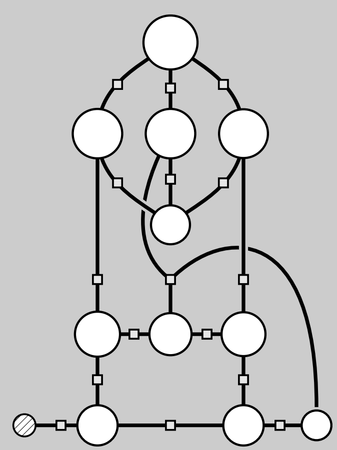


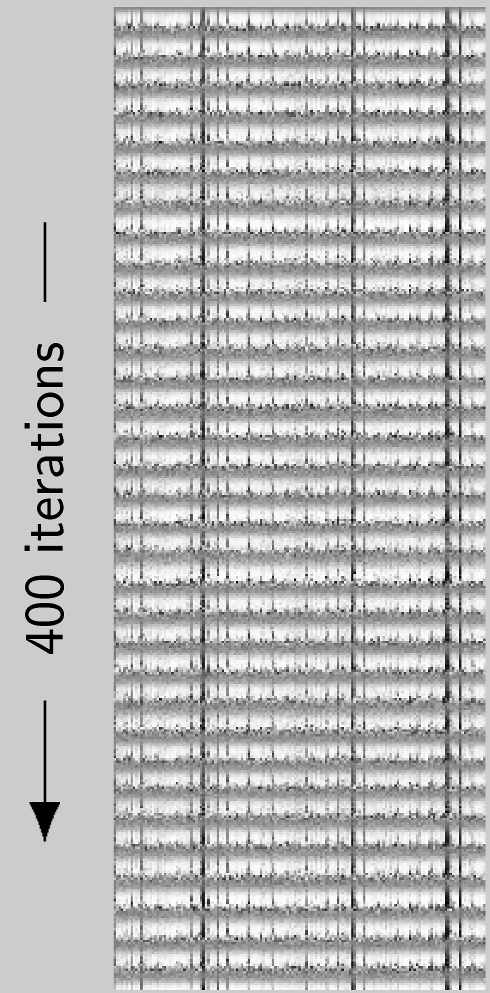
# One special subgraph of Tanner graph



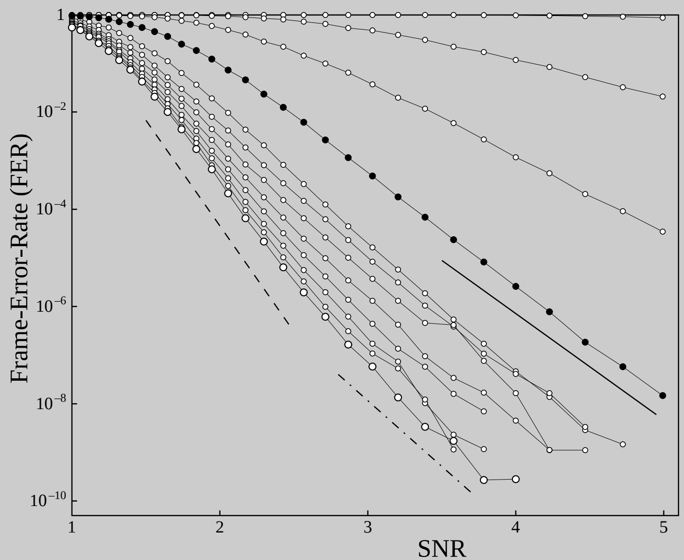
# One special subgraph of Tanner graph

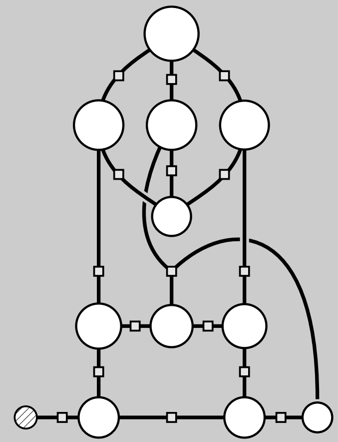




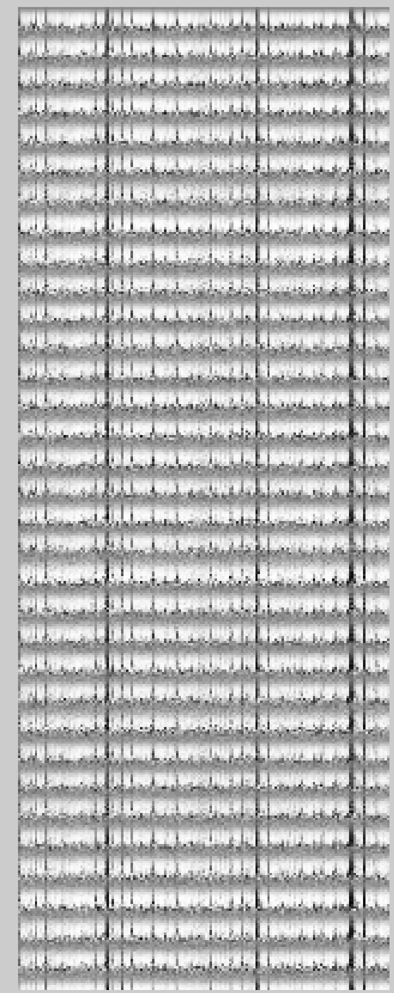


# Instanton for Tanner's $[155, 64, 20]$ code



400 iterations



Effective distances:

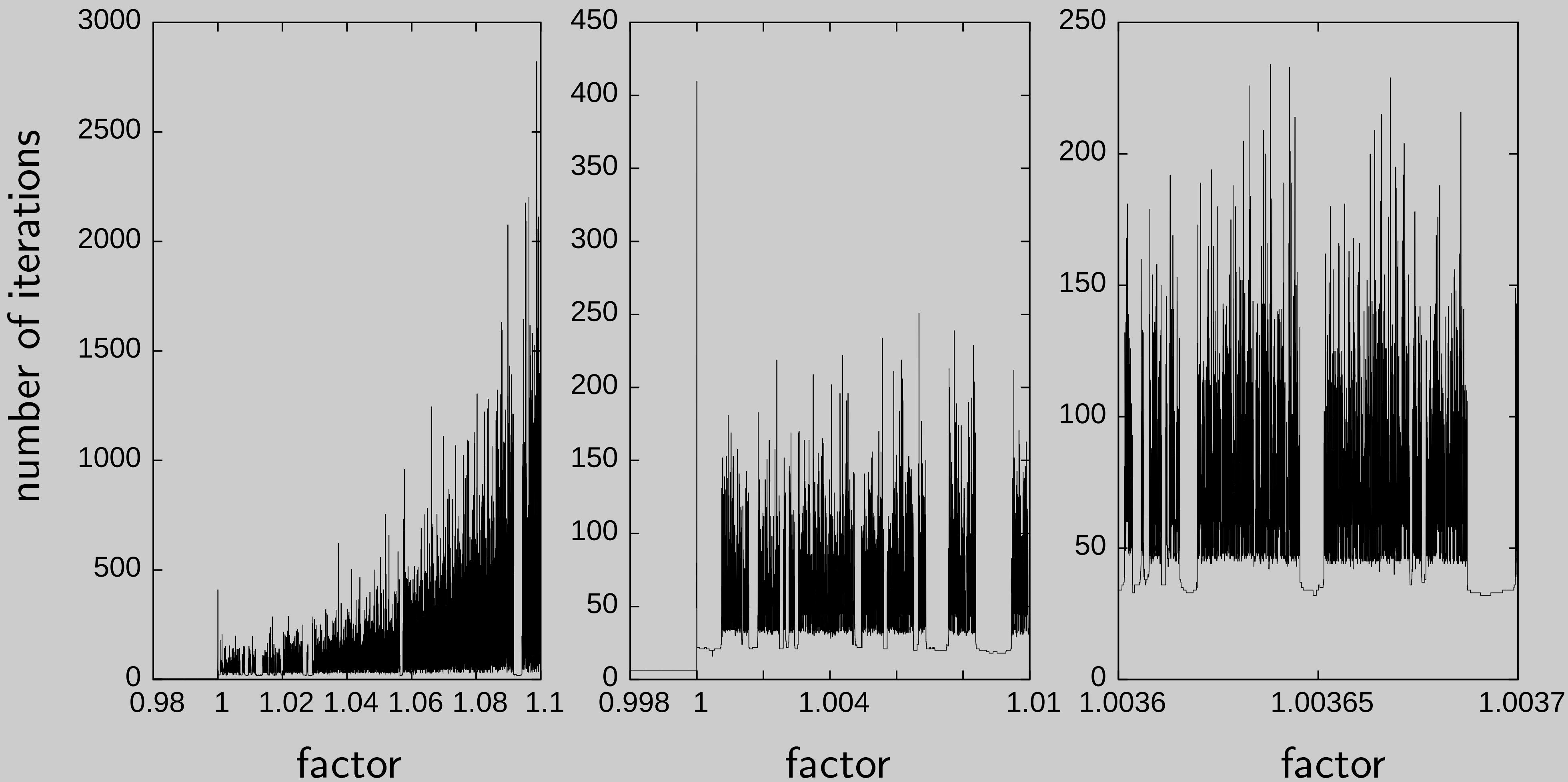
Iterative decoding: 12.5

Linear programming decoding: 16.4

Hamming distance: 20

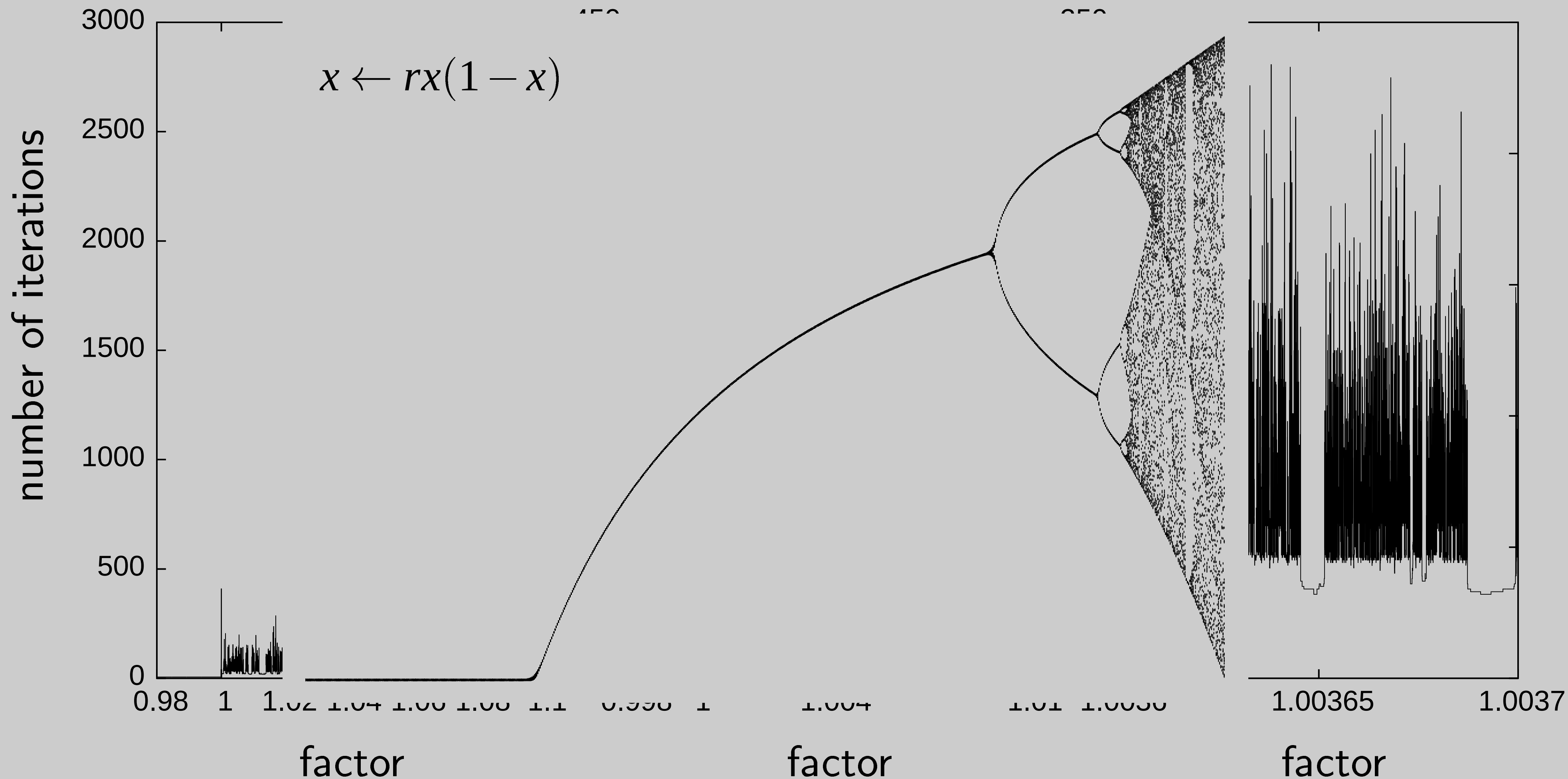
# Instanton “robustness”

number of iterations until a successful decoding



# Instanton “robustness”

number of iterations until a successful decoding



# Smoothed (relaxed, damped) decoding

Iterative scheme (BP): $\eta_{i\alpha}^{(n+1)} = h_i + \sum_{\beta\ni i}^{\beta\neq\alpha} \tanh^{-1}\left(\prod_{j\in\beta} \tanh \eta_{j\beta}^{(n)}\right)$

$\downarrow$

$$\eta_{i\alpha}^{(n+1)} + \frac{1}{\Delta}\sum_{\beta\ni i}\eta_{i\beta}^{(n+1)} = h_i + \sum_{\beta\ni i}^{\beta\neq\alpha} \tanh^{-1}\left(\prod_{j\in\beta} \tanh \eta_{j\beta}^{(n)}\right) + \frac{1}{\Delta}\sum_{\beta\ni i}\eta_{i\beta}^{(n)}$$

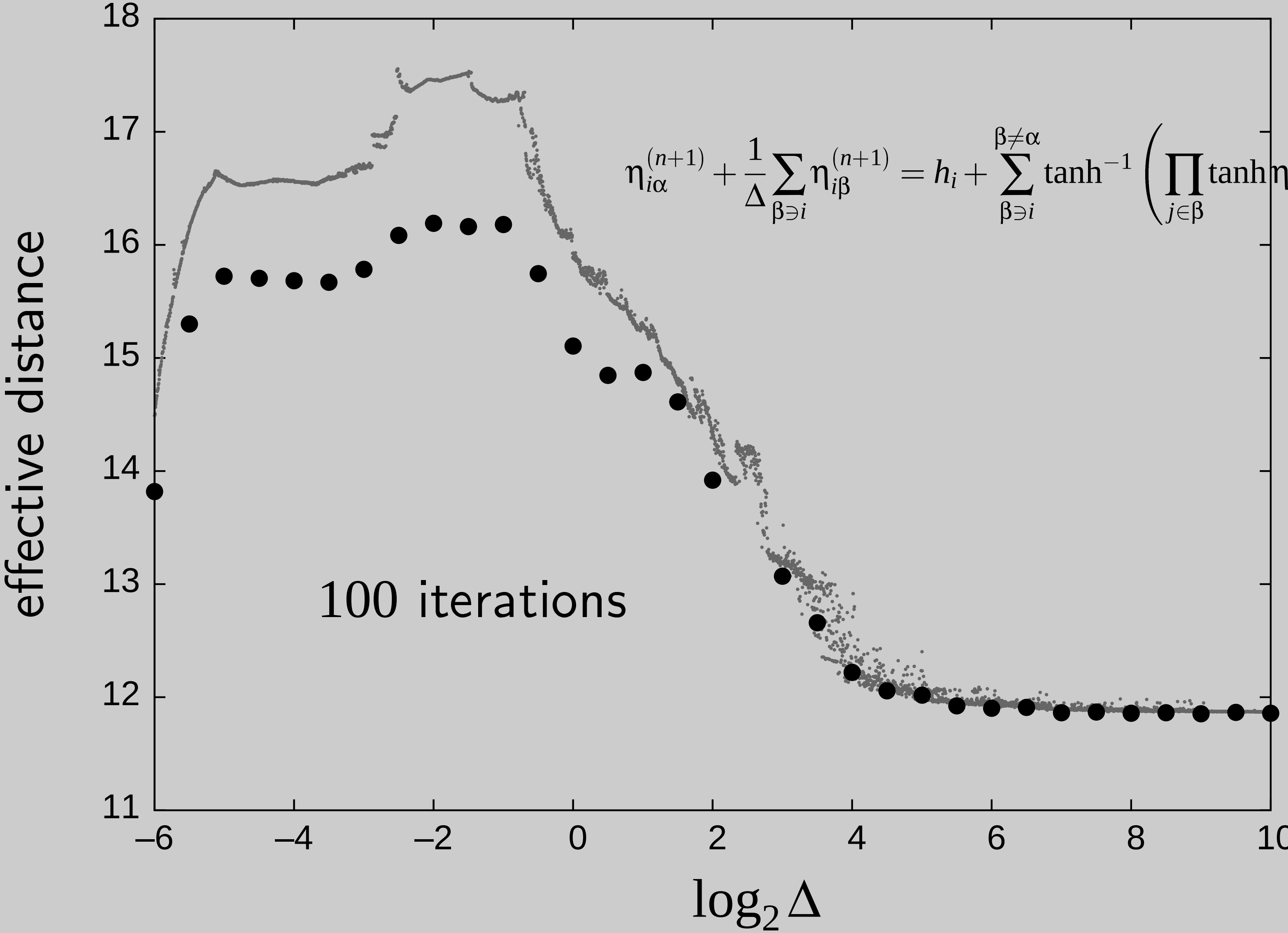
$\Delta \to \infty$ — standard BP

$\Delta \to 0$ — slow dynamics

Stepanov, Chertkov, Allerton 2006 [cs.IT/0607112]

# Instantons effective distance

$$\eta_{i\alpha}^{(n+1)} + \frac{1}{\Delta}\sum_{\beta\ni i}\eta_{i\beta}^{(n+1)} = h_i + \sum_{\beta\ni i}^{\beta\neq\alpha}\tanh^{-1}\left(\prod_{j\in\beta}\tanh\eta_{j\beta}^{(n)}\right) + \frac{1}{\Delta}\sum_{\beta\ni i}\eta_{i\beta}^{(n)}$$

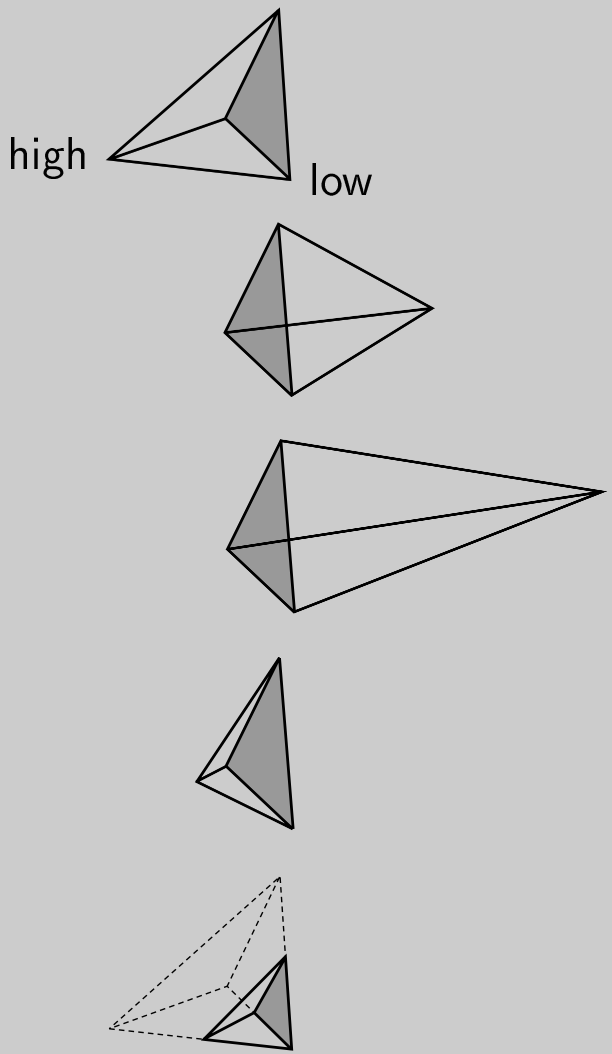


# Summary

- the performance of iterative decoding is determined by most dangerous noise configurations (instantons)
- the fixed point of iterations in decoding is unstable, if the noise configuration is damaging
- the iterative decoding cycles on instantons
- making the iterations smoother helps (shifts the instantons to larger distances)

this page was intentionally left blank

# Amoeba (downhill simplex method)




simplex at the beginning of the step

reflection

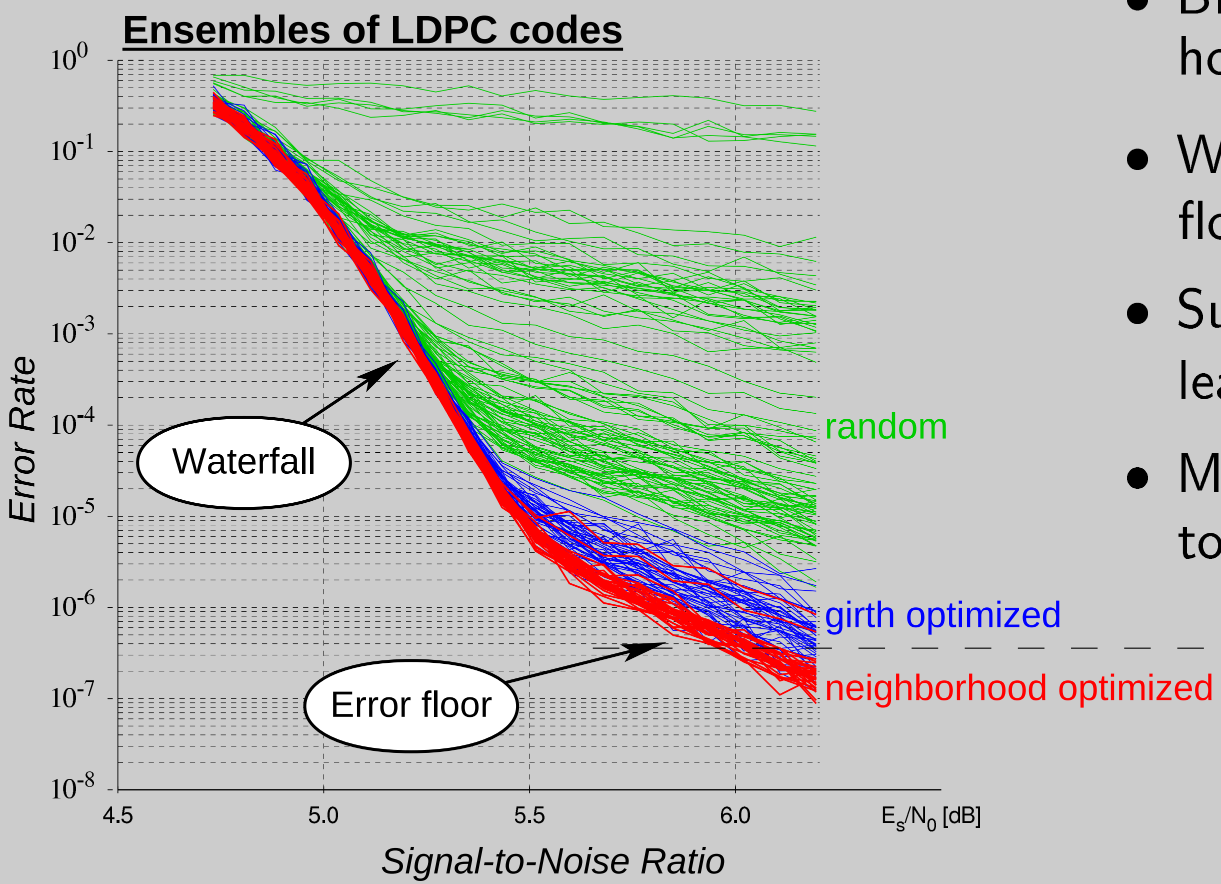
reflection and expansion

contraction

multiple contraction

Numerical Recipes,
ch. 10, part 4

# Error floors of LDPC codes




Tom Richardson, Error floors of LDPC codes

- BER *vs* SNR — how good is the code?
- Waterfall $\leftrightarrow$ error floor transition
- Suboptimal decoding leads to error floor
- Monte Carlo fails to reach lowest BER

no go zone

too few errors for Monte Carlo

# Binary, linear error-correcting codes

$$\underbrace{\begin{pmatrix} \begin{smallmatrix}0\\1\\1\\0\\1\end{smallmatrix} \\ N \\ \times \\ 1 \end{pmatrix}}_{\text{codeword } \vec{y}} = \underbrace{\begin{pmatrix} \hat{\mathbb{G}} & \begin{smallmatrix}1&0&1\\1&1&0\\0&1&1\\1&0&0\\0&0&1\end{smallmatrix} \\ N \times L & \end{pmatrix}}_{\text{generator matrix}} \underbrace{\begin{pmatrix} L & \begin{smallmatrix}1\\0\\1\end{smallmatrix} \\ \times & \\ 1 & \end{pmatrix}}_{\text{information } \vec{x}}$$

$$\vec{y} \to \vec{y} + \vec{\xi}$$

distortion of signal by channel

$$\underbrace{\begin{pmatrix} \hat{\mathbb{H}} & \begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix} \\ M \times N & \end{pmatrix}}_{\text{parity check matrix}} \underbrace{\begin{pmatrix} N & \begin{smallmatrix}0\\1\\1\\0\\1\end{smallmatrix} \\ \times & \\ 1 & \end{pmatrix}}_{\text{codeword } \vec{y}} = \underbrace{\begin{pmatrix} M & \begin{smallmatrix}0\\0\\0\end{smallmatrix} \\ \times & \\ 1 & \end{pmatrix}}_{\text{syndrome vector}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}$$

# Decoding

$$\begin{array}{c}\text{decoded}\\ \text{codeword}\end{array} = \underset{\text{all codewords}}{\operatorname{argmax}} \ \mathcal{P}\left( \text{codeword} \;\middle|\; \begin{array}{c}\text{channel}\\ \text{output}\end{array} \right) \qquad 2^{\sharp\text{bits}} \text{ operations}$$

Iterative decoding $\qquad (\sharp\text{edges}) \cdot (\sharp\text{iterations})$ operations

Checks vote for the bits value (unsatisfied check votes to flip the bit)

Proceed voting iteratively until convergence

$$\eta_{i\alpha}^{(n+1)} = h_i + \sum_{\beta\neq\alpha}^{\beta\ni i} \tanh^{-1} \prod_{j\neq i}^{j\in\beta} \tanh \eta_{j\beta}^{(n)}, \quad h_i = \frac{1}{2}\log\frac{p(y_i|+1)}{p(y_i|-1)}, \quad \eta_{i\alpha}^{(0)} \equiv 0$$

$$\eta_{i\alpha}^{(n+1)} = h_i + \sum_{\beta\neq\alpha}^{\beta\ni i} \left( \prod_{j\neq i}^{j\in\beta} \operatorname{sign} \eta_{j\beta}^{(n)} \right) \ \min_{j\neq i}^{j\in\beta} \left| \eta_{j\beta}^{(n)} \right|$$

# Decoding

$$\begin{matrix}\text{decoded}\\ \text{codeword}\end{matrix} = \underset{\text{all codewords}}{\mathrm{argmax}}\ \mathcal{P}\left(\text{codeword}\ \middle|\ \begin{matrix}\text{channel}\\ \text{output}\end{matrix}\right) \qquad 2^{\sharp\text{bits}}\ \text{operations}$$

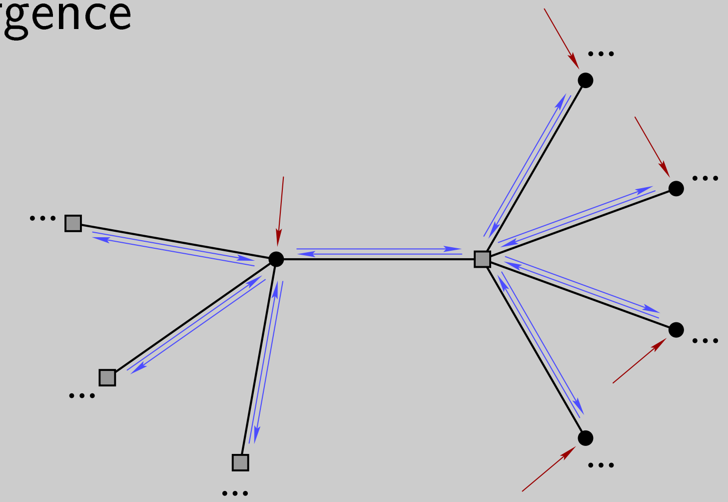
Iterative decoding $\qquad (\sharp\text{edges}) \cdot (\sharp\text{iterations})$ operations

Checks vote for the bits value (unsatisfied check votes to flip the bit)

Proceed voting iteratively until convergence

Message passing
Belief propagation
on a graph

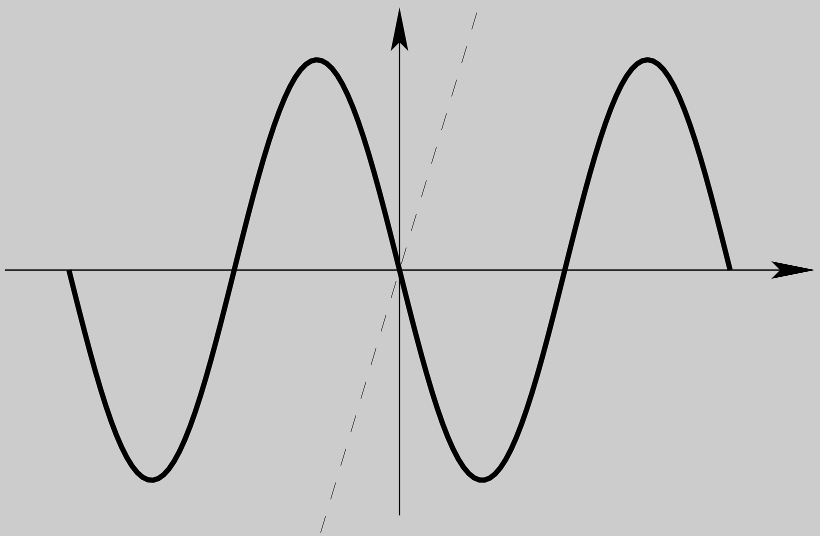
Linear programming




Iterative solution, works for trees

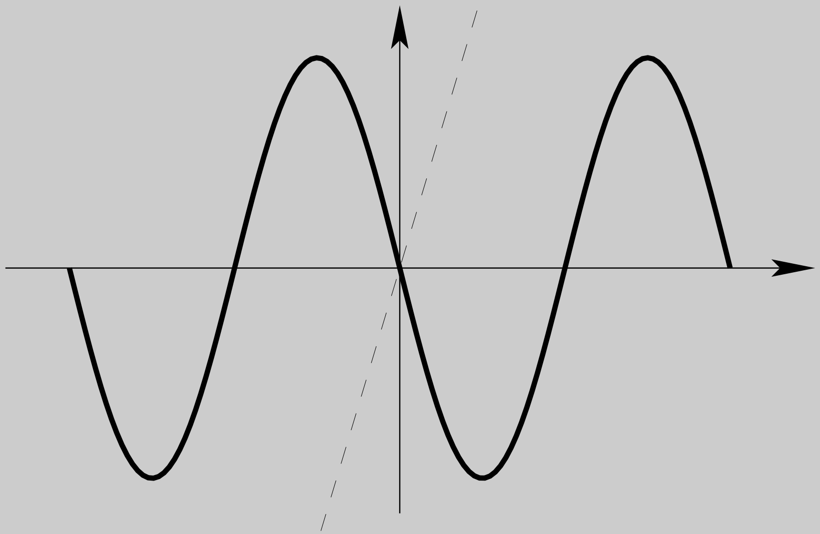
Bethe (1935), Pieirls (1936), Gallager (1962), Pearl (1986), MacKay (1995)

# Unstable iterations, period doubling

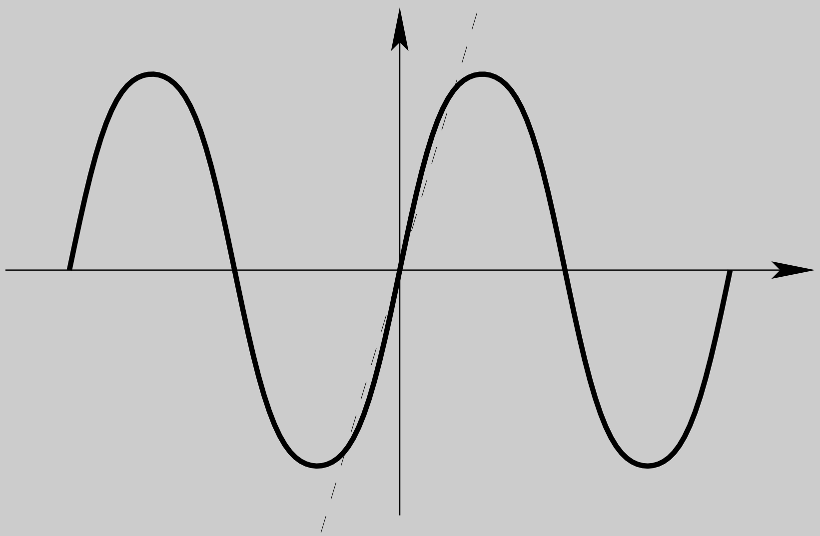


$$f(x) = -1.2 \sin(x)$$

# Unstable iterations, period doubling

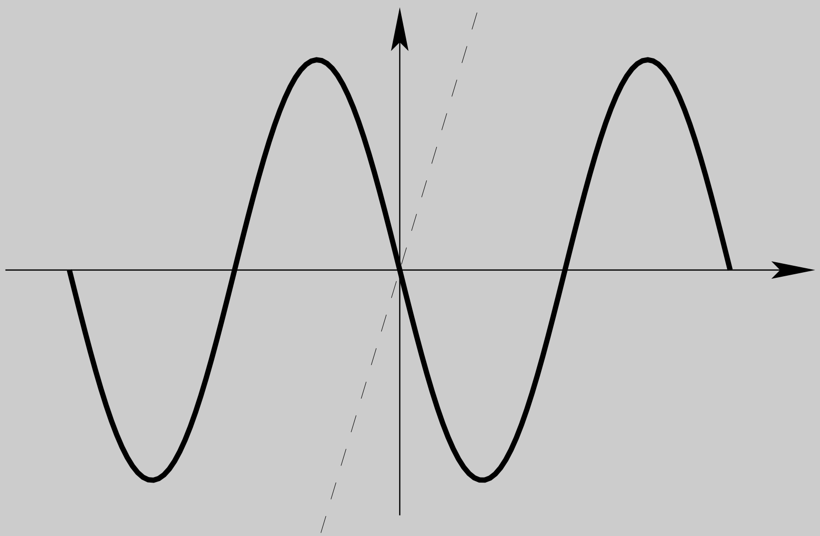


$$f(x) = -1.2\sin(x)$$

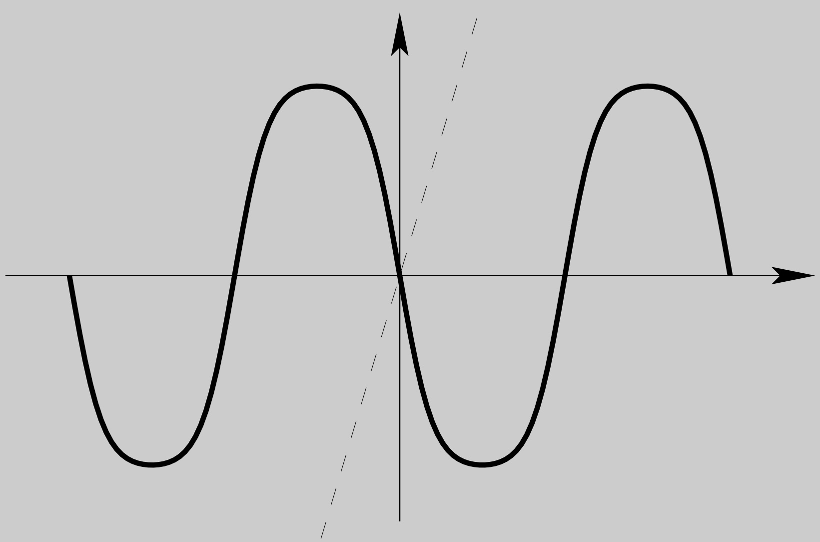


$$f(f(x))$$

# Unstable iterations, period doubling

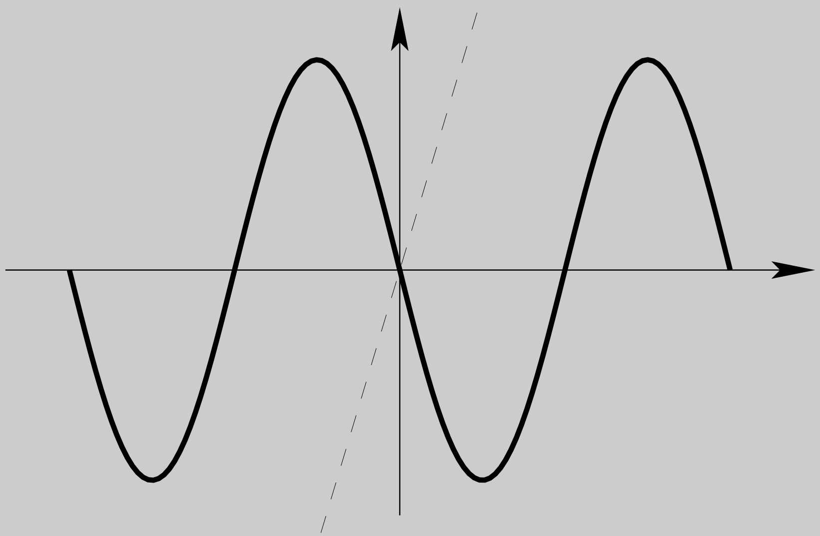


$$f(x) = -1.2\sin(x)$$

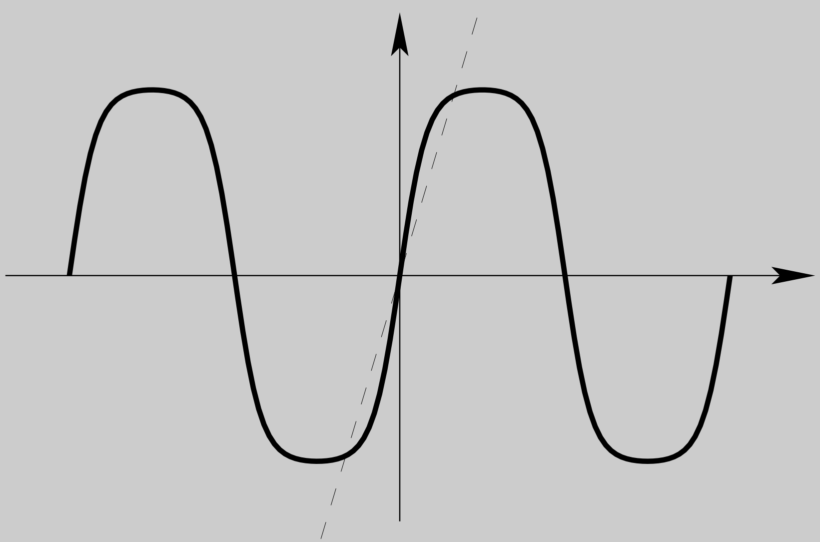


$$f(f(f(x)))$$

# Unstable iterations, period doubling

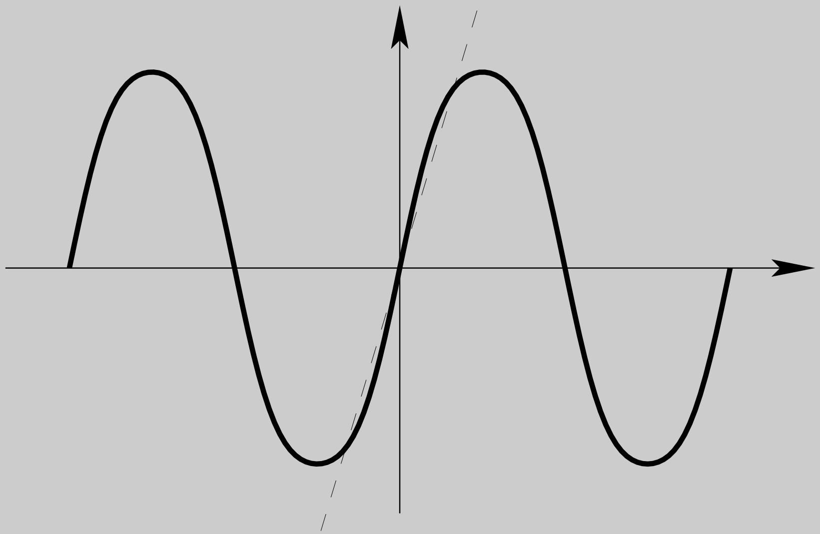


$$f(x) = -1.2\sin(x)$$

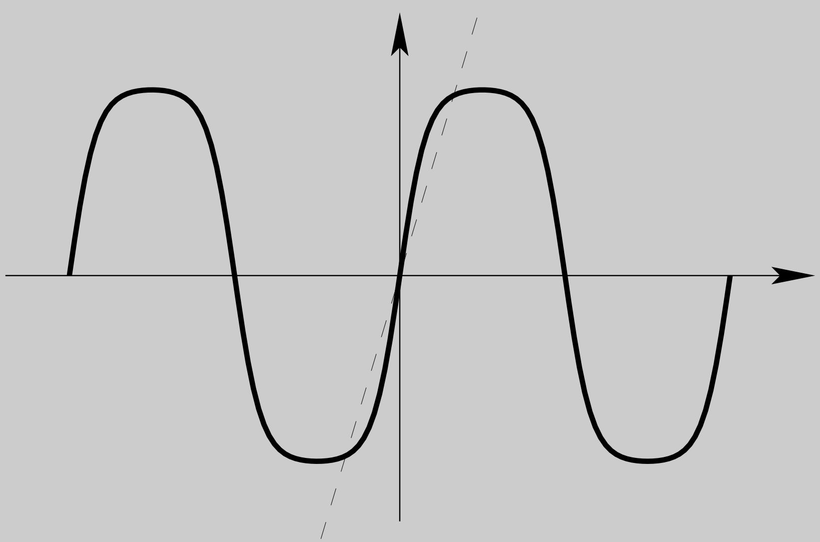


$$f(f(f(f(x))))$$

# Unstable iterations, period doubling

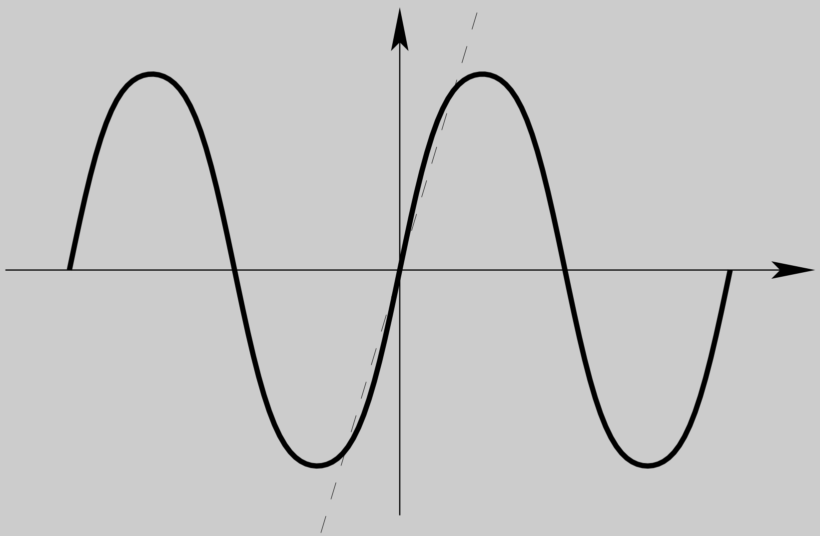


$$f(f(x))$$

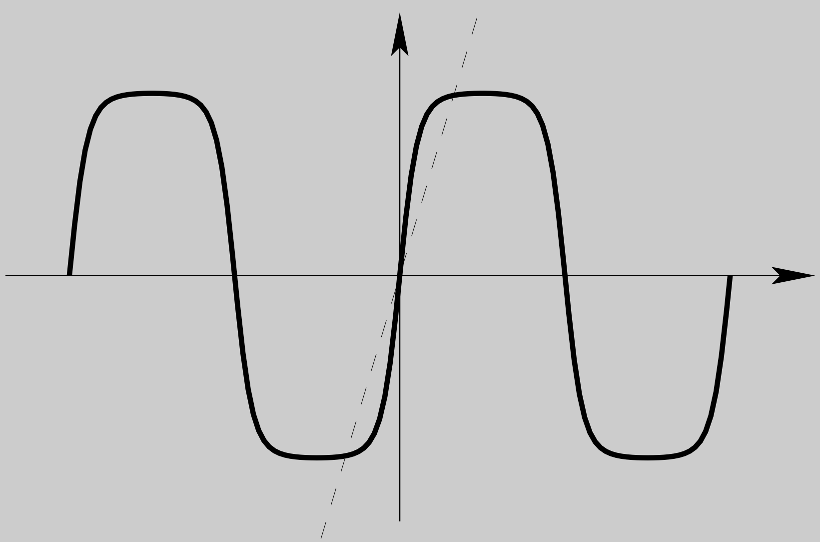


$$f(f(f(f(x))))$$

# Unstable iterations, period doubling

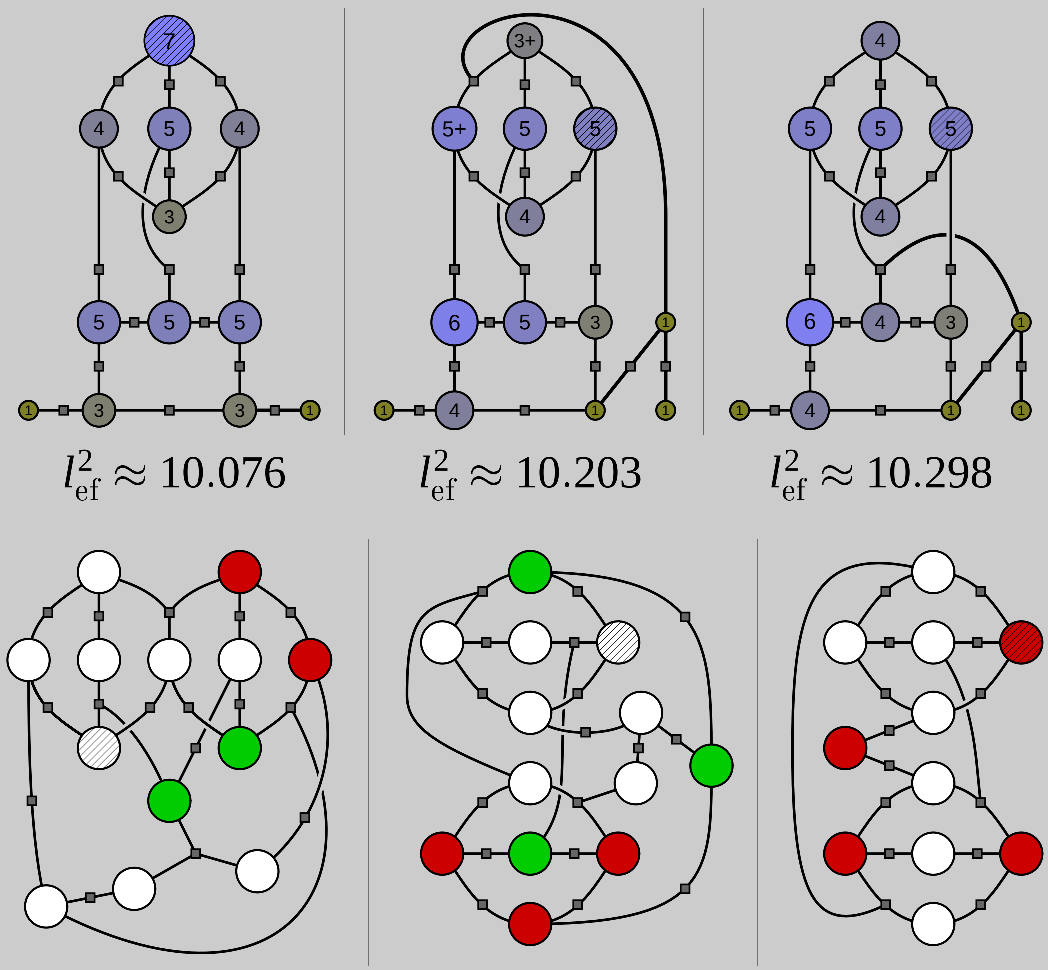


$$f(f(x))$$



$$f(f(f(f(f(f(x))))))$$

# Instantons, Laplacian channel




$l_{\rm ef}^2 \approx 10.076$ $l_{\rm ef}^2 \approx 10.203$ $l_{\rm ef}^2 \approx 10.298$

Gaussian

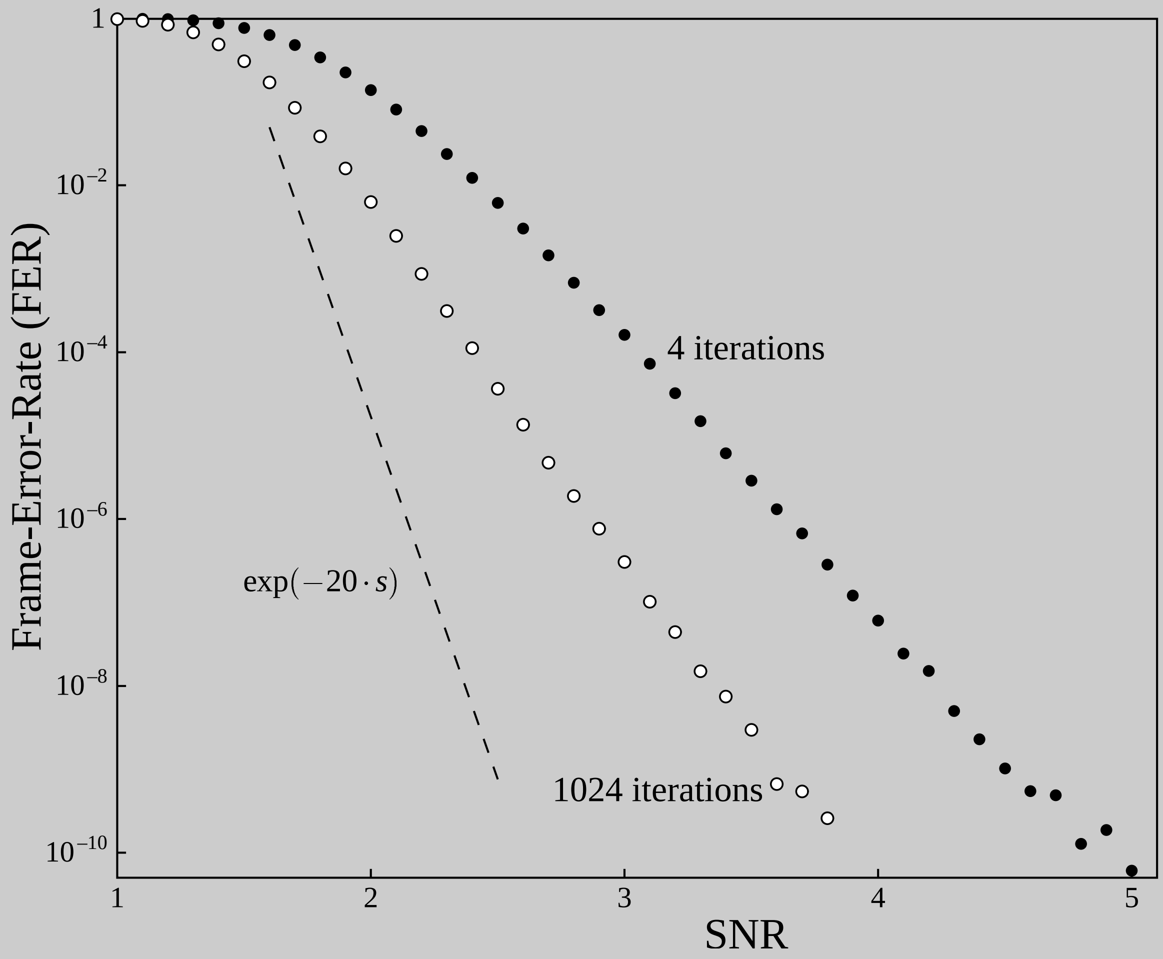
$l_{\rm ef} = 7.6$ $l_{\rm ef} = 8.$ $l_{\rm ef} = 8.$

Laplacian

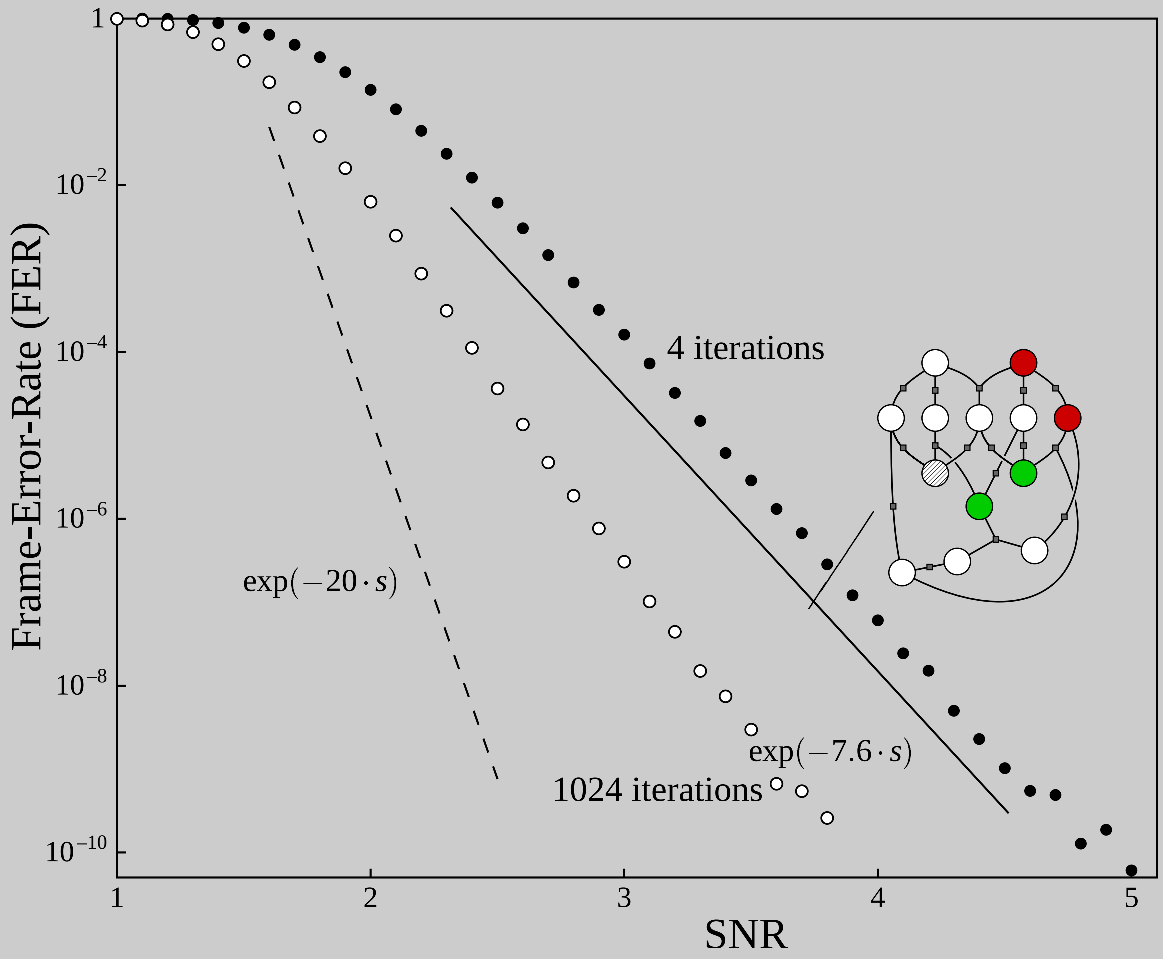
exponential tails

Stepanov, Chertkov,
2005 Allerton Conference [arxiv.org: cs.IT/0507031]

# Frame-Error-Rate, Laplacian channel



# Frame-Error-Rate, Laplacian channel



# BP as minimization of Bethe free energy

Yedidia, Freeman, Weiss

Variables: beleifs $0 \le b_i(\sigma_i), b_\alpha(\sigma_\alpha) \le 1$

Conditions: normalization $\sum_{\sigma_i} b_i(\sigma_i) = \sum_{\sigma_\alpha} b_\alpha(\sigma_\alpha) = 1$

consistency $\sum_{\sigma_\alpha \setminus \sigma_i} b_\alpha(\sigma_\alpha) = b_i(\sigma_i)$

Function: Bethe free energy

$$\mathcal{F}_{\rm Bethe} = -\sum_\alpha \sum_{\sigma_\alpha} b_\alpha(\sigma_\alpha) \ln f_\alpha(\sigma_\alpha) + \sum_\alpha \sum_{\sigma_\alpha} b_\alpha(\sigma_\alpha) \ln b_\alpha(\sigma_\alpha) - \sum_i (q_i - 1) \sum_{\sigma_i} b_i(\sigma_i) \ln b_i(\sigma_i)$$

$$f_\alpha(\sigma_\alpha) \equiv \exp\Big(\sum_{i \in \alpha} h_i \sigma_i / q_i\Big) \delta\Big(\prod_{i \in \alpha} \sigma_i, 1\Big)$$

Minimization $\Longrightarrow$ BP equation: $\eta_{i\alpha} = h_i + \sum_{\beta \ni i}^{\beta \neq \alpha} \tanh^{-1}\left(\prod_{j \in \beta} \tanh \eta_{j\beta}\right)$

# Relaxed (smoothed) decoding

$$\mathcal{L} = \mathcal{F}_{\text{Bethe}} + (\text{Lagrangian multipliers}) \cdot (\text{conditions})$$

Minimization: $$\frac{\delta \mathcal{L}}{\delta(\text{beliefs})} = 0, \quad \frac{\delta \mathcal{L}}{\delta(\text{Lagrangian multipliers})} = 0$$

Iterative scheme (BP): $$\eta_{i\alpha}^{(n+1)} = h_i + \sum_{\beta \ni i}^{\beta \neq \alpha} \tanh^{-1}\left(\prod_{j \in \beta} \tanh \eta_{j\beta}^{(n)}\right)$$

# Relaxed (smoothed) decoding

$$\mathcal{L} = \mathcal{F}_{\mathrm{Bethe}} + (\text{Lagrangian multipliers}) \cdot (\text{conditions})$$

Minimization: $$\frac{\delta\mathcal{L}}{\delta(\text{beliefs})} = 0, \quad \frac{\delta\mathcal{L}}{\delta(\text{Lagrangian multipliers})} = 0$$

Iterative scheme (BP): $$\eta_{i\alpha}^{(n+1)} = h_i + \sum_{\beta \ni i}^{\beta \neq \alpha} \tanh^{-1}\left(\prod_{j \in \beta} \tanh \eta_{j\beta}^{(n)}\right)$$

tree-based re-parametrization

Wainwright, Jaakola, Willsky

concave-convex procedure

Yuille

Heskes, Albers, Kappen

# Relaxed (smoothed) decoding

$$\mathcal{L} = \mathcal{F}_{\mathrm{Bethe}} + (\text{Lagrangian multipliers}) \cdot (\text{conditions})$$

Minimization: $$\frac{\delta \mathcal{L}}{\delta(\text{beliefs})} = 0, \quad \frac{\delta \mathcal{L}}{\delta(\text{Lagrangian multipliers})} = 0$$

Iterative scheme (BP): $$\eta_{i\alpha}^{(n+1)} = h_i + \sum_{\beta \ni i}^{\beta \neq \alpha} \tanh^{-1} \left( \prod_{j \in \beta} \tanh \eta_{j\beta}^{(n)} \right)$$

# Relaxed (smoothed) decoding

$$\mathcal{L} = \mathcal{F}_{\text{Bethe}} + (\text{Lagrangian multipliers}) \cdot (\text{conditions})$$

Minimization: $$\frac{\delta \mathcal{L}}{\delta(\text{beliefs})} = 0, \quad \frac{\delta \mathcal{L}}{\delta(\text{Lagrangian multipliers})} = 0$$

Iterative scheme (BP): $$\eta_{i\alpha}^{(n+1)} = h_i + \sum_{\beta \ni i}^{\beta \neq \alpha} \tanh^{-1}\left(\prod_{j \in \beta} \tanh \eta_{j\beta}^{(n)}\right)$$

Dynamics: $$\frac{d(\text{beliefs})}{d\tau} = -\frac{\delta \mathcal{L}}{\delta(\text{beliefs})}$$

$$\frac{d(\text{Lagrangian multipliers})}{d\tau} = -\frac{\delta \mathcal{L}}{\delta(\text{Lagrangian multipliers})}$$

# Relaxed (smoothed) decoding

$$\mathcal{L} = \mathcal{F}_{\text{Bethe}} + (\text{Lagrangian multipliers}) \cdot (\text{conditions})$$

Minimization: $$\frac{\delta \mathcal{L}}{\delta(\text{beliefs})} = 0, \quad \frac{\delta \mathcal{L}}{\delta(\text{Lagrangian multipliers})} = 0$$

Iterative scheme (BP): $$\eta_{i\alpha}^{(n+1)} = h_i + \sum_{\beta \ni i}^{\beta \neq \alpha} \tanh^{-1}\left( \prod_{j \in \beta} \tanh \eta_{j\beta}^{(n)} \right)$$

Dynamics: $$\frac{d(\text{beliefs})}{d\tau} = -\frac{\delta \mathcal{L}}{\delta(\text{beliefs})}$$

$$\frac{d(\text{Lagrangian multipliers})}{d\tau} = -\frac{\delta \mathcal{L}}{\delta(\text{Lagrangian multipliers})}$$

$$\eta_{i\alpha}^{(n+1)} + \frac{1}{\Delta} \sum_{\beta \ni i} \eta_{i\beta}^{(n+1)} = h_i + \sum_{\beta \ni i}^{\beta \neq \alpha} \tanh^{-1}\left( \prod_{j \in \beta} \tanh \eta_{j\beta}^{(n)} \right) + \frac{1}{\Delta} \sum_{\beta \ni i} \eta_{i\beta}^{(n)}$$

# Relaxed (smoothed) decoding

Iterative scheme (BP): $\eta_{i\alpha}^{(n+1)} = h_i + \sum_{\beta \ni i}^{\beta \neq \alpha} \tanh^{-1}\left(\prod_{j \in \beta} \tanh \eta_{j\beta}^{(n)}\right)$

$$\eta_{i\alpha}^{(n+1)} + \frac{1}{\Delta}\sum_{\beta \ni i} \eta_{i\beta}^{(n+1)} = h_i + \sum_{\beta \ni i}^{\beta \neq \alpha} \tanh^{-1}\left(\prod_{j \in \beta} \tanh \eta_{j\beta}^{(n)}\right) + \frac{1}{\Delta}\sum_{\beta \ni i} \eta_{i\beta}^{(n)}$$

# Smoothed (relaxed, damped) decoding

Iterative scheme (BP): $$\eta_{i\alpha}^{(n+1)} = h_i + \sum_{\beta \ni i}^{\beta \neq \alpha} \tanh^{-1}\left(\prod_{j \in \beta} \tanh \eta_{j\beta}^{(n)}\right)$$

$$\eta_{i\alpha}^{(n+1)} + \frac{1}{\Delta}\sum_{\beta \ni i} \eta_{i\beta}^{(n+1)} = h_i + \sum_{\beta \ni i}^{\beta \neq \alpha} \tanh^{-1}\left(\prod_{j \in \beta} \tanh \eta_{j\beta}^{(n)}\right) + \frac{1}{\Delta}\sum_{\beta \ni i} \eta_{i\beta}^{(n)}$$