Loop Calculus and Belief Propagation for q-ary Alphabet: Loop Tower

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Outline

- Forney-style graphical model formulation for statistical inference problem
- Traces and graphical traces
- Gauge-invariant formulation of loop calculus
- Loop towers for q-ary alphabet
- Relation to the Bethe free energy approach
- Continuous and supersymmetric cases
- Homotopy approach to loop decomposition

Statistical Inference

σ_{orig}	$\Rightarrow$	$\mathbf{x}$	$\Rightarrow$	$\boldsymbol{\sigma}$
original *data* $\sigma_{\text{orig}} \in \mathcal{C}$ codeword	noisy channel $\mathcal{P}(\mathbf{x}\|\boldsymbol{\sigma})$	corrupted data: log-likelihood magnetic field	statistical inference	possible preimage $\boldsymbol{\sigma} \in \mathcal{C}$

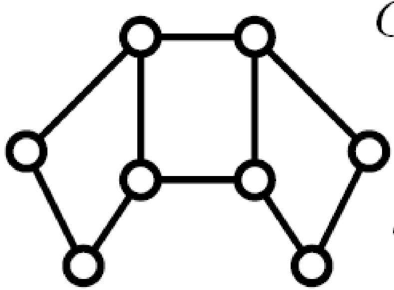
$$\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_N), \quad N \textit{ finite}, \quad \sigma_i = \pm 1 \ (\textit{example})$$

Maximum Likelihood — symbol Maximum-a-Posteriori

$$\mathrm{ML} = \arg\max_{\boldsymbol{\sigma}} \mathcal{P}(\mathbf{x}|\boldsymbol{\sigma}) \qquad \mathrm{MAP}_i = \arg\max_{\sigma_i} \sum_{\boldsymbol{\sigma}\setminus\sigma_i} \mathcal{P}(\mathbf{x}|\boldsymbol{\sigma})$$

Exhaustive search is generally expensive: complexity $\sim 2^N$

Forney-style graphical model formulation



$$C_0 = (\mathcal{V}_0, \mathcal{E}_0)$$
$$\mathcal{V}_0 = \{a\}$$
$$\mathcal{E}_0 = \{(ab)\}$$

q-ary variables reside on edges

$$\sigma_{ab} = \sigma_{ba} = 0, \cdots, (q-1)$$

Forney ’01; Loeliger ‘01

Probability of a configuration

Partition function

$$p(\boldsymbol{\sigma}) = Z_{C_0}^{-1} \prod_a f_a(\boldsymbol{\sigma}_a), \quad Z_{C_0} = \sum_{\boldsymbol{\sigma}} \prod_a f_a(\boldsymbol{\sigma}_a)$$

Marginal probabilities

Reduced variables

$$p_a(\boldsymbol{\sigma}_a) \equiv \sum_{\boldsymbol{\sigma}\setminus\boldsymbol{\sigma}_a} p(\boldsymbol{\sigma}), \quad p_{ab}(\sigma_{ab}) \equiv \sum_{\boldsymbol{\sigma}\setminus\sigma_{ab}} p(\boldsymbol{\sigma}), \qquad \boldsymbol{\sigma}_a \equiv \{\sigma_{ab} | (ab) \in \mathcal{E}_0\}$$

can be expressed in terms of
the derivatives of the free energy
with respect to factor-functions

$$\mathcal{F}_{C_0} = -\ln Z_{C_0}$$

Loop calculus (binary alphabet)

Belief Propagation (BP) is exact on a tree

Loop Series:

Chertkov, Chernyak '06

Exact (!!) expression in terms of BP

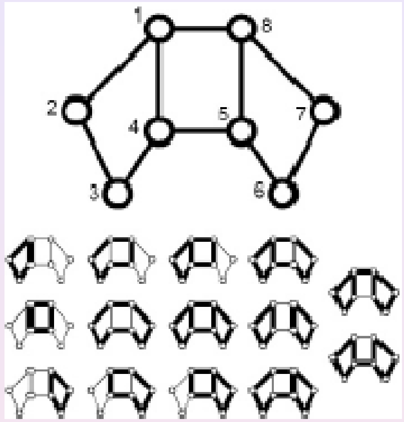
$$Z = \sum_{\boldsymbol{\sigma}_\sigma} \prod_a f_a(\boldsymbol{\sigma}_a) = Z_0 \left(1 + \sum_C r(C)\right)$$

$$r(C) = \frac{\prod_{a \in C} \mu_a}{\prod_{(ab) \in C} (1 - m_{ab}^2)} = \prod_{a \in C} \tilde{\mu}_a$$

$C \in$ Generalized Loops = Loops without loose ends

$$m_{ab} = \int d\boldsymbol{\sigma}_a b_a^{(bp)}(\boldsymbol{\sigma}_a)\sigma_{ab}$$

$$\mu_a = \int d\boldsymbol{\sigma}_a b_a^{(bp)}(\boldsymbol{\sigma}_a) \prod_{b \in a, C} (\sigma_{ab} - m_{ab})$$



- The Loop Series is finite
- All terms in the series are calculated within BP
- BP is exact on a tree
- BP is a Gauge fixing condition. Other choices of Gauges would lead to different representation.

Equivalent models: gauge fixing and transformations

Replace the model with an equivalent more convenient model

Invariant approach

(i) Introduce an invariant object that describes partition function Z
(ii) Different equivalent models correspond to different coordinate choices (gauge fixing)
(iii) Gauge transformations are changing the basis sets

Coordinate approach

(i) Introduce a set of gauge transformations that do not change Z
(ii) Gauge transformations build new equivalent models

General strategy (based on linear algebra)

(i) Replace q-ary alphabet with a q-dimensional vector space
(ii) (letters are basis vectors)
(ii) Represent Z by an invariant object graphical trace
(iii) Gauge fixing is a basis set choice
(iv) Gauge transformations are linear transformation of basis sets

Gauge invariance: matrix formulation

Gauge transformations of factor-functions

$$f_a(\boldsymbol{\sigma}_a = (\sigma_{ab}, \cdots)) \to \sum_{\sigma'_{ab}} G_{ab}\left(\sigma_{ab}, \sigma'_{ab}\right) f_a(\sigma'_{ab}, \cdots),$$

with orthogonality conditions

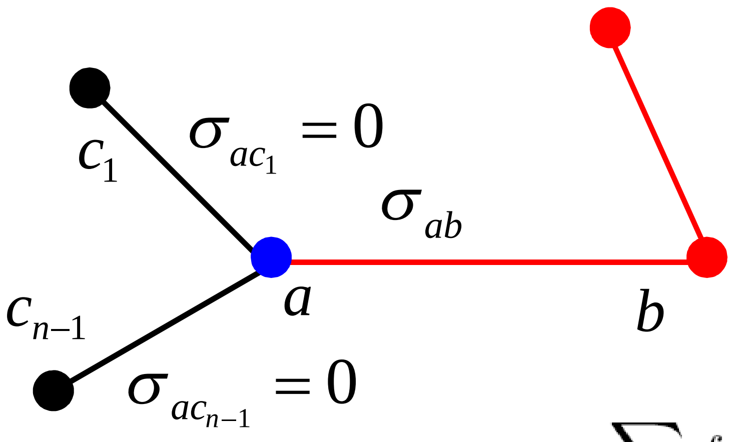
$$\sum_{\sigma_{ab}} G_{ab}(\sigma_{ab}, \sigma')G_{ba}(\sigma_{ab}, \sigma'') = \delta(\sigma', \sigma''),$$

do not change the partition function

$$Z_{C_0} = \sum_{\boldsymbol{\sigma}} \prod_a \left(\sum_{\boldsymbol{\sigma}'_a} f_a(\boldsymbol{\sigma}'_a) \prod_{b \in a} G_{ab}(\sigma_{ab}, \sigma'_{ab}) \right)$$

$$\equiv \sum_{\boldsymbol{\sigma}} \bar{p}\{\hat{G}|\boldsymbol{\sigma}\} \equiv \mathrm{Tr}\left(\bar{p}\{\hat{G}|\boldsymbol{\sigma}\}\right),$$

BP equations: matrix (coordinate) formulation



No-loose-ends condition

$$\sum_{\boldsymbol{\sigma}'_a} f_a(\boldsymbol{\sigma}') G_{ab}^{(bp)}(\sigma_{ab} \neq 0, \sigma'_{ab}) \prod_{c \in a}^{c \neq b} G_{ac}^{(bp)}(0, \sigma'_{ac}) = 0,$$

results in BP equations

$$G_{ba}^{(bp)}(0, \sigma'_{ab}) = \rho_a^{-1} \sum_{\boldsymbol{\sigma}'_a \setminus \sigma'_{ab}} f_a(\boldsymbol{\sigma}') \prod_{c \in a}^{c \neq b} G_{ac}^{(bp)}(0, \sigma'_{ac}).$$

with

$$\rho_a = \sum_{\boldsymbol{\sigma}'_a} f_a(\boldsymbol{\sigma}') \prod_{c \in a} G_{ac}^{(bp)}(0, \sigma'_{ac}).$$

BP equations: standard form

A standard form of BP equations

$$\frac{\exp\left(\eta_{ab}^{(bp)}(\sigma_{ab})\right)}{\sum_{\sigma_{ab}}\exp\left(\eta_{ab}^{(bp)}(\sigma_{ab})+\eta_{ba}^{(bp)}(\sigma_{ab})\right)}$$
$$=\frac{\sum_{\boldsymbol{\sigma}_a\setminus\sigma_{ab}} f_a(\boldsymbol{\sigma}_a)\exp\left(\sum_{b\in a}\eta_{ab}^{(bp)}\sigma_{ab})\right)}{\sum_{\boldsymbol{\sigma}_a} f_a(\boldsymbol{\sigma}_a)\exp\left(\sum_{b\in a}\eta_{ab}^{(bp)}(\sigma_{ab})\right)}$$

is reproduced using the following
representation for the ground state

$$\epsilon_{ab}=G_{ab}(0,\sigma)=\frac{\exp\left(\eta_{ab}(\sigma)\right)}{\sum_{\sigma}\exp\left(\eta_{ab}(\sigma)+\eta_{ba}(\sigma)\right)}$$

Side remark: relation to iterative BP

“Reduced Bethe free energy” (variational approach)

Reduced Bethe free energy $F_0(\varepsilon) = -\ln(Z_0(\varepsilon))$

$$\mathcal{Z}_0(\hat{\epsilon}) \equiv \bar{p}\{G|\mathbf{0}\} = \prod_a \rho_a(\boldsymbol{\epsilon}_a),$$ with $$\epsilon_{ab}(\sigma_{ab}) \equiv G_{ab}(0, \sigma_{ab})$$

$$\varepsilon_{ab} \cdot \varepsilon_{ba} = 1$$

is an attempt to approximate the partition function Z in terms of the ground-state contribution in a proper gauge

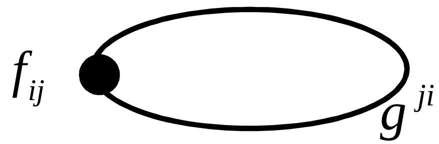
BP equations are recovered by the stationary point conditions $$\frac{\partial F_0(\varepsilon)}{\partial \varepsilon_{ab}} = 0$$

Not a standard variational scheme: corrections can be of either sign

What is the relation of the introduced functional to the Bethe free energy (Yedidia, Freeman, Weiss ‘01)?

Graphical representation of trace and cyclic trace

Trace

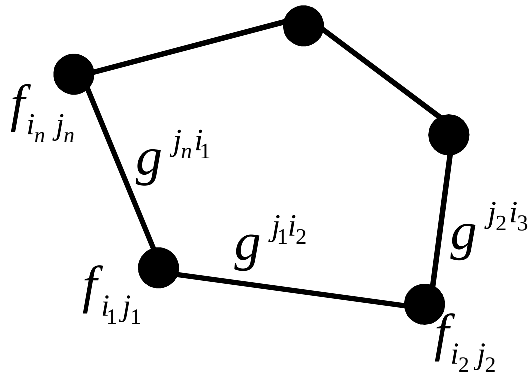


scalar product

$$Tr(f) = \sum f_j^{\,j} = \sum f_{ij} g^{ji}$$

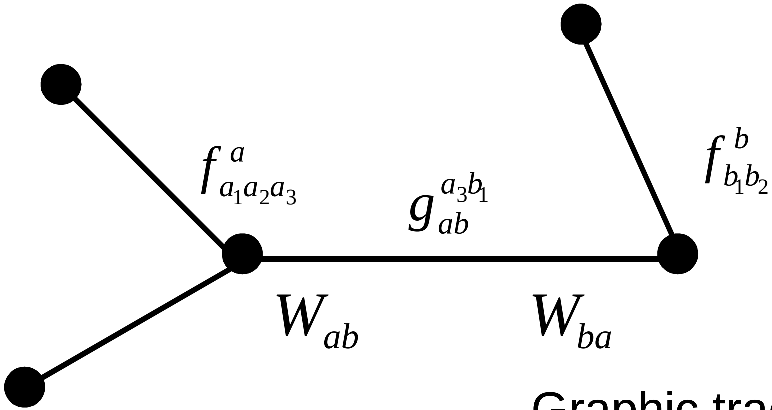
Summation over repeating subscripts/superscripts

Cyclic trace



$$Tr(f^n) = \sum f_{j_1}^{\,j_2} f_{j_2}^{\,j_3} \ldots f_{j_n}^{\,j_1} = \sum f_{i_1 j_1} g^{j_1 i_2} f_{i_2 j_2} g^{j_2 i_3} \ldots f_{i_n j_n} g^{j_n i_1}$$

Graphic trace and partition function



Collection of tensors (poly-vectors)

$$f = \{ f^{a} \}_{a=1,\ldots,N} = \{ f^{a}_{a_1 \ldots a_{n_a}} \}_{a=1,\ldots N}$$

$$g = \{ g_{ab} \}_{a \in b} = \{ g^{i_a i_b}_{ab} \}_{a \in b}$$

Graphic trace

$$g^{a_k b_j}_{ab}$$

$$Tr(f) = Tr_g\left(\prod_a f^{a}\right) = \sum \left(\ldots f^{a}_{a_1 \ldots a_k \ldots a_{n_a}} \ldots f^{b}_{b_1 \ldots b_j \ldots b_{n_b}} \ldots\right)$$

Scalar products

$$u \cdot w = g_{ab}(u \otimes w)$$

$$u \in W_{ab} \quad w \in W_{ba}$$

Orthogonality condition

$$g^{ij}_{ab} = g_{ab}(e^{i}_{ab} \otimes e^{j}_{ba}) = e^{i}_{ab} \cdot e^{j}_{ba} = \delta^{ij}$$

Tensors and factor-functions

$$f^{a} = \sum_{\sigma_1 \ldots \sigma_n} f_a(\sigma_1, \ldots, \sigma_n) e^{\sigma_1}_{ab_1} \otimes \ldots \otimes e^{\sigma_n}_{ab_n}$$

$$Z = Tr(f)$$

Partition function and graphic trace: gauge invariance

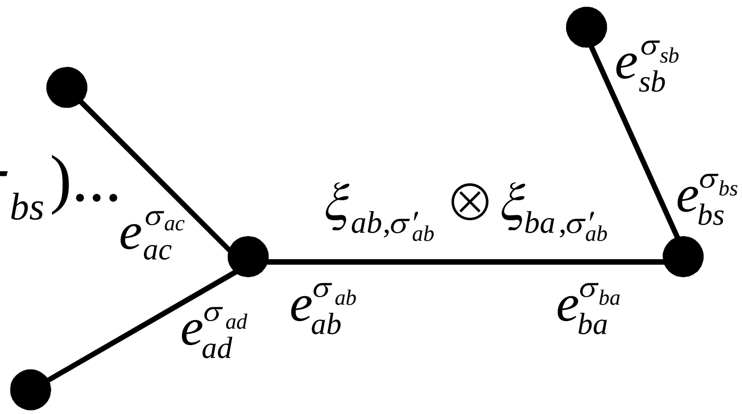
Dual basis set of co-vectors
(elements of the dual space)

Orthogonality condition (two equivalent forms)

$$\xi_{ab} \in W_{ab}^{*} \qquad \xi_{ab,i}(e_{ab}^{j}) = \delta_i^j \qquad \xi_{ab,i} \cdot \xi_{ba,j} = \delta_{ij} \qquad g_{ab} = \sum_j \xi_{ab,j} \otimes \xi_{ba,j}$$

Graphic trace: Evaluate scalar products (reside on edges) on tensors (reside vertices)

$$Tr(f) = \sum_{\sigma\sigma'} f_a(\sigma_{ab}, \sigma_{ac}, \sigma_{ad})\, f_b(\sigma_{ba}, \sigma_{bs}) \ldots$$



Gauge invariance: graphic trace is an invariant object,
factor-functions are basis-set dependent

$$f_a(\sigma_1, \ldots, \sigma_n) = \xi_{ab_1,\sigma_1} \otimes \ldots \otimes \xi_{ab_n,\sigma_n}(f^a)$$

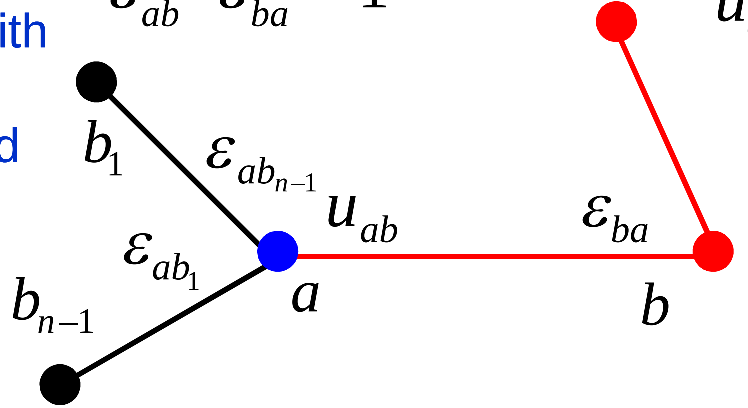
“Gauge fixing” is a choice of an orthogonal basis set

Belief propagation gauge and BP equations

Introduce local ground $\varepsilon_{ab} \in W_{ab}^{*}$ and excited (painted) states $u_{ab} \in W_{ab}^{*}$

$$\varepsilon_{ab} \cdot \varepsilon_{ba} = 1 \qquad u_{ab} \cdot \varepsilon_{ba} = 0$$

BP gauge: painted structures with loose ends should be forbidden (in particular, no allowed painted structures in a tree case)



$$u_{ab} \otimes \varepsilon_{ab_1} \otimes \ldots \otimes \varepsilon_{ab_{n-1}}(f^{a}) = u_{ab}(\varepsilon_{ab_1} \otimes \ldots \otimes \varepsilon_{ab_{n-1}}(f^{a})) = 0$$

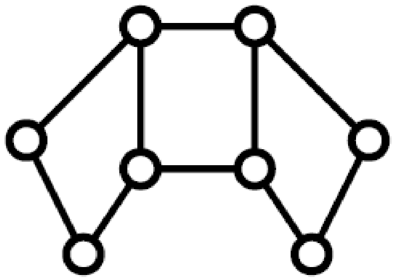
$$\varepsilon_{ab_1} \otimes \ldots \otimes \varepsilon_{ab_{n-1}}(f^{a}) \in W_{ab}$$

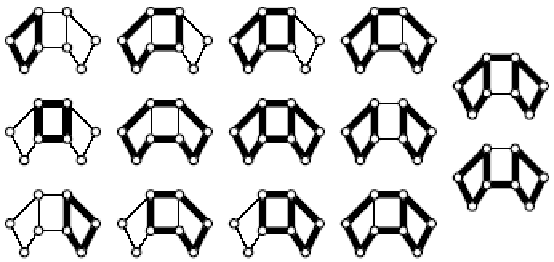
or, stated differently, results in BP equations in invariant form:

$$g_{ab}(\varepsilon_{ab_1} \otimes \ldots \otimes \varepsilon_{ab_{n-1}}(f^{a})) = \rho_{ab}\varepsilon_{ba}$$

$$g_{ab}(\varepsilon_{ab_1} \otimes \ldots \otimes \varepsilon_{ab_{n-1}}(f^{a})) \in W_{ba}^{*}$$

Loop decomposition: binary case





$$Z_{C_0} = Z_{0;C_0}\Big(1 + \sum_{C_1} r(C_1)\Big), \quad r(C_1) \equiv Z_{0;C_0}^{-1}\bar{p}(G|\boldsymbol{\sigma}_{C_1}),$$

$$Z_{0;C_0} \equiv \bar{p}(G|\boldsymbol{\sigma}_0), \quad \boldsymbol{\sigma}_0 \equiv \{\sigma_{ab}=0| \quad (ab)\in C_0\},$$

$$\boldsymbol{\sigma}_{C_1} \equiv \left\{ \begin{array}{ll} \sigma_{ab}=1| & (ab)\in C_1 \\ \sigma_{ab}=0| & (ab)\in C_0 \setminus C_1. \end{array} \right\}.$$

Beliefs (marginal probabilities)

$$b_{ab}^{(bp)}(\sigma_{ab}) = G_{ab}^{(bp)}(0,\sigma_{ab}),$$

$$b_a^{(bp)}(\boldsymbol{\sigma}_a) = \frac{f_a(\boldsymbol{\sigma}_a)\prod_{b\in a} G_{ab}^{(bp)}(0,\sigma_{ab})}{\sum_{\boldsymbol{\sigma}_a} f_a(\boldsymbol{\sigma}_a)\prod_{b\in a} G_{ab}^{(bp)}(0,\sigma_{ab})}.$$

$$r(C_1) = \frac{\prod\limits_{a\in C_1}\mu_a}{\prod\limits_{(ab)\in C_1}(1-m_{ab}^2)}, \quad m_{ab} \equiv \sum_{\sigma_{ab}} \sigma_{ab} b_{ab}^{(bp)}(\sigma_{ab}),$$

$$\mu_a \equiv \sum_{\boldsymbol{\sigma}_a}\left(\prod_{b\in a, C_1}(\sigma_{ab}-m_{ab})\right) b_a^{(bp)}(\boldsymbol{\sigma}_a).$$

A generalized loop visualizes a single-configuration contribution to the partition function in BP gauge

Loop towers for q-ary alphabet: first step

A generalized loop defines a vertex model on the corresponding subgraph with (q-1)-ary alphabet (first store above the ground store)

$$Z_{C_0} = Z_{0;C_0} + \sum_{C_1 \in \Omega(C_0)} Z_{C_1}, \quad Z_{C_1} = \sum_{\boldsymbol{\sigma}_{C_1}} \bar{p}(G^{(bp)}|\boldsymbol{\sigma}_{C_1})$$

q>2 (non-binary case): more than one local excited state

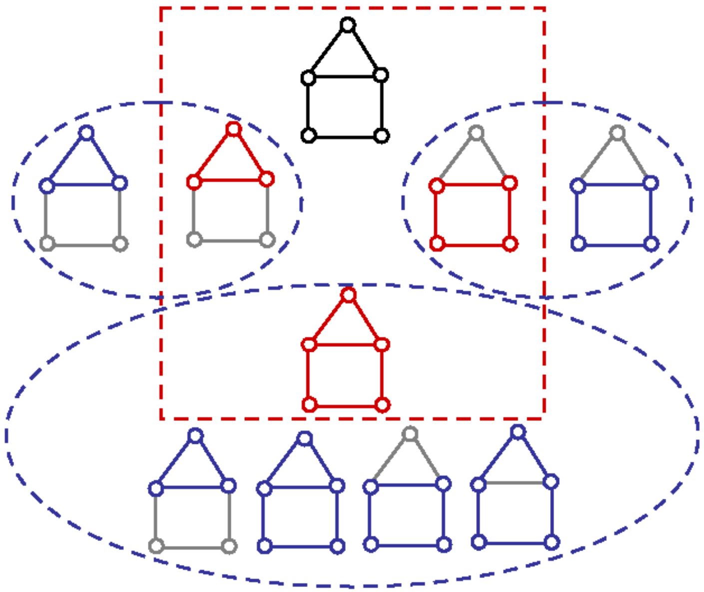
Partition function for the subgraph model

$$Z_{C_1} = \sum_{\boldsymbol{\sigma}_{C_1}} \prod_{a \in C_1} f_{1;a}(\boldsymbol{\sigma}_{a;C_1}), \quad f_{1;a}(\boldsymbol{\sigma}_{a;C_1}) =$$

$$= \sum_{\boldsymbol{\sigma}'_a} f_a(\boldsymbol{\sigma}'_a) \prod_{b \in a, C_0} G^{(bp)}_{ab;C_0}(\sigma_{ab}, \sigma'_{ab}) \prod_{b \in a, C_0}^{b \notin C_1} \delta(\sigma_{ab}, 0)$$

Loop-tower expansion for q-ary alphabet

$$j = 1, \cdots, q-2: \quad Z_{C_j} = Z_{0;C_j} + \sum_{C_{j+1} \in \Omega(C_j)} Z_{C_{j+1}}.$$



Building the next level (store)

$$Z_{C_j} = \sum_{\boldsymbol{\sigma}_{C_j}} \prod_{a \in C_j} f_{j;a}(\boldsymbol{\sigma}_{a;C_j}),$$

Loop tower

$$C_0 \supset C_1 \supset \cdots \supset C_{q-2}$$

$$f_{j;a}(\boldsymbol{\sigma}_{a;C_j}) = \sum_{\boldsymbol{\sigma}'_{a;C_{j-1}}} f_{j-1;a}(\boldsymbol{\sigma}'_{a;C_{j-1}})$$

$$\times \prod_{b \in a, C_{j-1}} G^{(bp)}_{ab;C_{j-1}}(\sigma_{ab}, \sigma'_{ab}) \prod_{b \in a, C_{j-1}}^{b \notin C_j} \delta(\sigma_{ab}, j-1),$$

Bethe free energy for q-ary alphabet

BP equations can be obtained as stationary points of the Bethe free energy functional of beliefs

$$\Phi_{Bethe} = \sum_{a} \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln \left(\frac{b_a(\boldsymbol{\sigma}_a)}{f_a(\boldsymbol{\sigma}_a)} \right)$$

$$- \sum_{(ab)} \sum_{\sigma_{ab}} b_{ab}(\sigma_{ab}) \ln b_{ab}(\sigma_{ab}).$$

with natural constraints

$$0 \leq b_a(\boldsymbol{\sigma}_a), b_{ac}(\sigma_{ac}) \leq 1,$$

$$\sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) = 1, \quad \sum_{\sigma_{ab}} b_{ab}(\sigma_{ab}) = 1,$$

$$b_{ac}(\sigma_{ac}) = \sum_{\boldsymbol{\sigma}_a \backslash \sigma_{ac}} b_a(\boldsymbol{\sigma}_a), \quad b_{ac}(\sigma_{ca}) = \sum_{\sigma_c \backslash \sigma_{ca}} b_c(\boldsymbol{\sigma}_c).$$

Bethe effective Lagrangian

$$\mathcal{L}_{Bethe} = \sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln\left(\frac{b_a(\boldsymbol{\sigma}_a)}{f_a(\boldsymbol{\sigma}_a)}\right) - \sum_{(ab)} \sum_{\sigma_{ab}} b_{ab}(\sigma_{ab}) \ln b_{ab}(\sigma_{ab})$$
$$+ \sum_{(ab)} \left(\sum_{\sigma_{ab}} \ln(\varepsilon_{ab}(\sigma_{ab})) \left(b_{ab}(\sigma_{ab}) - \sum_{\boldsymbol{\sigma}_a \setminus \sigma_{ab}} b_a(\boldsymbol{\sigma}_a) \right) \right.$$
$$\left. + \sum_{\sigma_{ba}} \ln(\varepsilon_{ba}(\sigma_{ba})) \left(b_{ab}(\sigma_{ba}) - \sum_{\boldsymbol{\sigma}_b \setminus \sigma_{ba}} b_b(\boldsymbol{\sigma}_b) \right) \right)$$

Values of beliefs

$$b_a^{(*)}(\boldsymbol{\sigma}_a) = (\varrho_a(\boldsymbol{\varepsilon}_a))^{-1} f_a(\boldsymbol{\sigma}_a) \prod_{b \in a} \varepsilon_{ab}(\sigma_{ab})$$

$$b_{ab}^{(*)}(\sigma_{ab}) = \varrho_{ab}^{-1}(\varepsilon_{ab}, \varepsilon_{ba}) \varepsilon_{ab}(\sigma_{ab}) \varepsilon_{ba}(\sigma_{ab}),$$
$$\varrho_a(\boldsymbol{\varepsilon}_a) \equiv \sum_{\boldsymbol{\sigma}_a} f_a(\boldsymbol{\varepsilon}_a) \prod_{c \in a} \varepsilon_{ac}(\sigma_{ac}),$$
$$\varrho_{ab}(\varepsilon_{ab}, \varepsilon_{ba}) \equiv \sum_{\boldsymbol{\sigma}_{ab}} \varepsilon_{ab}(\sigma_{ab}) \varepsilon_{ba}(\sigma_{ab}),$$

Variation of beliefs

$$\mathcal{F}_B(\hat{\varepsilon}) = -\sum_a \ln \varrho_a(\boldsymbol{\varepsilon}_a) + \sum_{(ab)} \ln\left(\varrho_{ab}(\varepsilon_{ab}, \varepsilon_{ba})\right))$$

Relation to Bethe free energy

$$\mathcal{L}_{Bethe} = \sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln\left(\frac{b_a(\boldsymbol{\sigma}_a)}{f_a(\boldsymbol{\sigma}_a)}\right) - \sum_{(ab)} \sum_{\sigma_{ab}} b_{ab}(\sigma_{ab}) \ln b_{ab}(\sigma_{ab})$$
$$+ \sum_{(ab)} \left(\sum_{\sigma_{ab}} \ln(\varepsilon_{ab}(\sigma_{ab})) \left(b_{ab}(\sigma_{ab}) - \sum_{\boldsymbol{\sigma}_a \backslash \sigma_{ab}} b_a(\boldsymbol{\sigma}_a) \right) \right.$$
$$\left. + \sum_{\sigma_{ba}} \ln(\varepsilon_{ba}(\sigma_{ba})) \left(b_{ab}(\sigma_{ba}) - \sum_{\boldsymbol{\sigma}_b \backslash \sigma_{ba}} b_b(\boldsymbol{\sigma}_b) \right) \right)$$

Variation of the ground state

Variation of beliefs

$$\mathcal{F}_B(\hat{\varepsilon}) = -\sum_a \ln \varrho_a(\boldsymbol{\varepsilon}_a) + \sum_{(ab)} \ln\left(\varrho_{ab}(\varepsilon_{ab}, \varepsilon_{ba})\right)$$

$$\Phi_{Bethe} = \sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln\left(\frac{b_a(\boldsymbol{\sigma}_a)}{f_a(\boldsymbol{\sigma}_a)}\right)$$
$$- \sum_{(ab)} \sum_{\sigma_{ab}} b_{ab}(\sigma_{ab}) \ln b_{ab}(\sigma_{ab}).$$

Gauge fixing

$$\sum_{\sigma_{ab}} \varepsilon_{ab}(\sigma_{ab}) \varepsilon_{ba}(\sigma_{ab}) = 1.$$

$\mathcal{F}_0(\hat{\epsilon})$ Reduced Bethe free energy

Summary

- We have extended the loop expansion for general statistical inference problem to the case of general q-ary alphabet
- In the general case the loop decomposition goes over the loop towers
- We have formulated the statistical inference problem in terms of a graphical trace, which leads to the invariance of the partition function under a set of gauge transformations
- BP equations have been interpreted as a special choice of gauge
- The introduced Bethe effective Lagrangian establishes a connection between the gauge-invariant and Bethe free-energy approaches
- Generalization to the continuous and supersymmetric cases

Path forward: interplay of topological and geometrical equivalence

Topological structure: the graph

Geometrical structure: factor-functions

Use topologically equivalent models

Use geometrically equivalent models

e.g. Weitz '06

Combine

+

- improving BP
- quantum version
- etc

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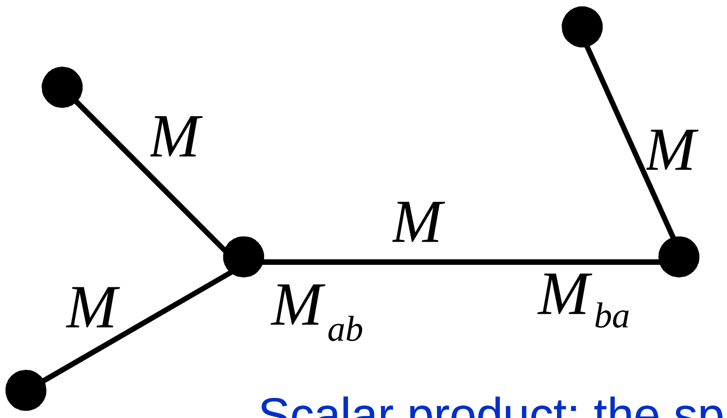
Continuous and supersymmetric case: graphical sigma-models



$$(\psi,\psi') = \int_M d\sigma \psi'(\sigma)\psi(\sigma)$$

$$M_a = \prod_{b\in a} M_{ab}$$

Scalar product: the space of states and its dual are equivalent

$$\hat{\varphi}(\psi) = \int_M d\sigma \varphi(\sigma)\psi(\sigma)$$

No-loose-end requirement

$$\psi_{ba}^{(0)} = \lambda_{ab} \int \prod_{c\in a}^{c\neq b} d\sigma_{ac} f_a(\boldsymbol{\sigma}_a) \prod_{c\in a}^{c\neq b} \psi_{ac}^{(0)}(\sigma_{ac})$$

Continuous version of BP equations

$$\psi_{ba}^{(j-1)} = \lambda_{ab} P_{C,ab} \int \prod_{c\in a}^{c\neq b} d\sigma_{ac} f_a(\boldsymbol{\sigma}_a) \prod_{c\in a}^{c\neq b} \psi_{ac}^{(j-1)}(\sigma_{ac}); \quad P_{C,ab}\psi_{ba}^{(j-1)} = 0$$

Supersymmetric sigma-models: supermanifolds

$\mathbb{Z}_2$-graded manifolds (supermanifolds)

dimension

$$m = (m_+, m_-)$$

substrate (usual) manifold $\bar{M} \subset M$

additional Grassman (anticommuting variables) $\theta_i \theta_j = -\theta_j \theta_i$

Functions on a supermanifold

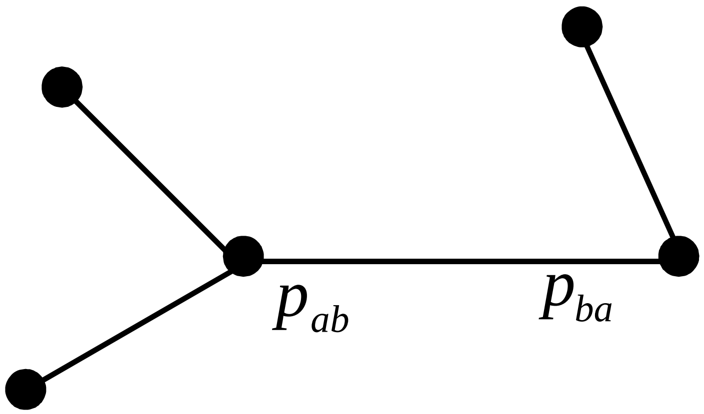
$$\psi(\sigma) = \psi(x,\theta) = \psi^{(0)}(x) + \sum_{i_1} \psi^{(1)}_{i_1}(x)\theta_{i_1} + \sum_{i_1 i_2}^{i_1<i_2} \psi^{(2)}_{i_1 i_2}(x)\theta_{i_1}\theta_{i_2} + \ldots$$

Berezin integral (measure in a supermanifold)

$$\int d\theta_i = 0; \quad \int \theta_i d\theta_i = 1; \quad d\theta_i d\theta_j = -d\theta_j d\theta_i; \quad d\theta_i \theta_j = -\theta_j d\theta_i$$

Any function on a supermanifold can be represented as a sum of its even and odd components

Supersymmetric sigma models: graphic supertrace I



Natural assumption: factor-functions are even functions on

$$M_a = \prod_{b \in a} M_{ab}$$

Introduce parities of the beliefs

$$p_{ab} = 0,1$$

BP equations for parities $(\psi^{(0)}_{ab}, \psi^{(0)}_{ba}) \neq 0$ Follows from the first two

$$p_{ba} = \sum_{c \in a}^{c \neq b} p_{ac}; \qquad p_{ba} = p_{ab}; \qquad \sum_{c \in a} p_{ac} = 0$$

Edge parity is well-defined $p \in H_1(X; \mathbb{Z}_2)$

2^{B_1} elements

$H_0(X; \mathbb{Z}_2) = \mathbb{Z}_2$

$B_0 = 1$

(number of connected components)

Euler characteristic

$$B_1 - B_0 = card(E) - card(V)$$

Supersymmetric sigma models: graphic supertrace II

Decompose the vector spaces $\mathcal{W}_{ab} = \mathcal{W}_{ab}^{(+)} \oplus \mathcal{W}_{ab}^{(-)}$

into reduced vector spaces $\mathcal{W}_{ab}^{(\alpha_{ab})}$ with $\alpha_{ab} = (-1)^{p_{ab}}$

Graphic supertrace decomposition (generalizes the supertrace)

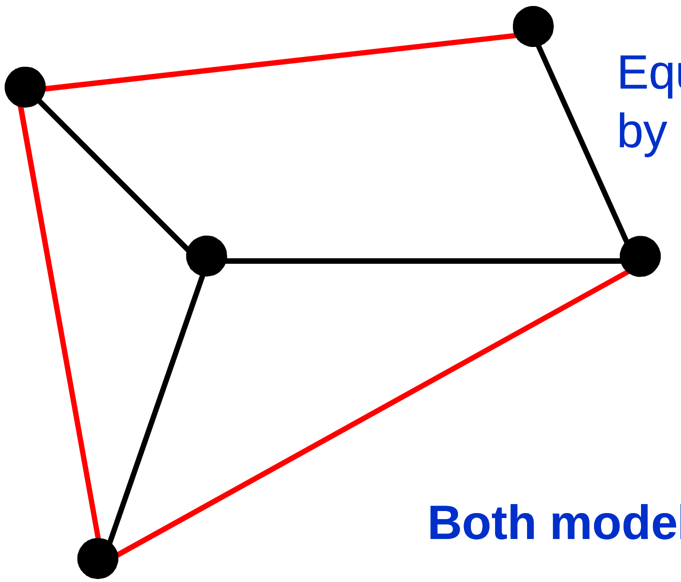
$$Z = \sum_{p \in H_1(X;\mathbb{Z}_2)} Z_p; \qquad \mathrm{Tr}\mathcal{F} = \sum_{p \in H_1(X;\mathbb{Z}_2)} \mathrm{Tr}\mathcal{F}_p$$

results in a multi-reference loop expansion

$\mathcal{F}_p$ is the graphic trace (partition function) of a reduced model

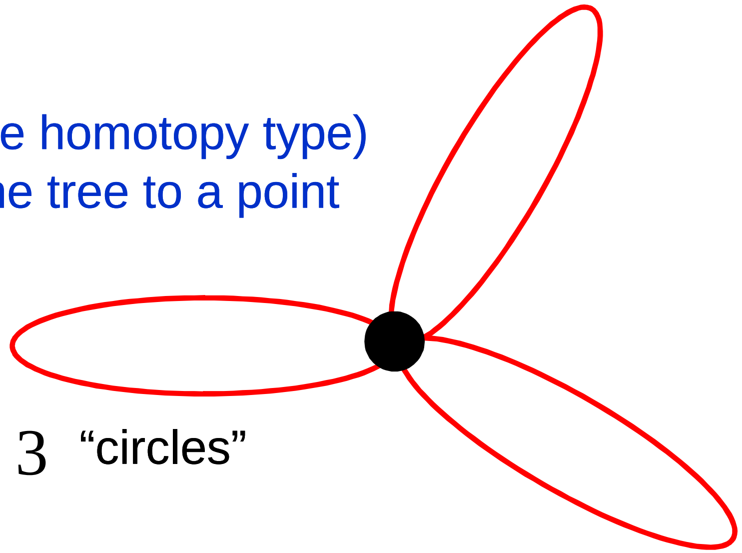
Homotopy approach to loop decomposition

Graph (arbitrary)



Bouquet of circles

Equivalent (same homotopy type)
by contracting the tree to a point



$B_1 = 3$ “circles”

Both models are equivalent

Loop calculus for the bouquet model (independent loops) constitutes a resummation for the original model (generalized loops)