Mimetic Discretizations for Maxwell's Equations and the Equations of Magnetic Diffusion *

James M. Hyman † Mikhail Shashkov ‡

Abstract

We construct reliable finite-difference methods for approximating the solutions to Maxwell's equations and equations of magnetic field diffusion using discrete analogs of differential operators that satisfy the identities and theorems of vector and tensor calculus in discrete form. These methods mimic many fundamental properties of the underlying physical problem, including the conservation laws, the symmetries in the solution, the nondivergence of particular vector fields and they do not allow spurious modes. The constructed method can be applied when there are strongly discontinuous properties of the media and nonorthogonal, nonsmooth computational grids.

In this paper we apply discrete vector analysis techniques [1]–[4] to construct mimetic finite-difference methods to Maxwell's first-order curl equations (hyperbolic type) and to the equations of magnetic diffusion (parabolic type).

The system of first-order Maxwell's curl equations can be written as follows:

$$\text{(1-a)} \qquad \partial\vec{B}/\partial t = -\mathbf{curl}\,\vec{E}\,,$$

$$\text{(1-b)} \qquad \partial\vec{E}/\partial t = \epsilon^{-1}\,\mathbf{curl}\,\mu^{-1}\,\vec{B}\,,$$

where $\vec{B}$ is the magnetic flux density, $\vec{E}$ is the electric field intensity, ϵ is the permitivity, and μ is the permeability. These parameters are symmetric positive-definite tensors and can be discontinuous at material interfaces. The normal components of $\vec{B}$ and the tangential components of $\vec{E}$ are continuous at the material interfaces.

Solutions of Eqs. 1 also satisfy the "divergence-free" conditions

$$\text{(2)} \qquad \mathbf{div}\,\vec{D} = 0\,, \quad \mathbf{div}\,\vec{B} = 0\,,$$

for $\vec{B}$ and the electric flux density $\vec{D} = \epsilon\,\vec{E}$ if they are satisfied initially. This property easily follows by taking the divergence of the Eqs. 1 and using the identity $\mathbf{div}\,\mathbf{curl} \equiv 0$.

We require our finite-difference methods to satisfy discrete analogs of the identities of vector and tensor calculus. Consequently, the finite-difference approximation of Eqs. 1 automatically satisfies a discrete analog of Eqs. 2.

We discretize Eqs. 1 as a system of two first-order equations and consider both $\vec{B}$ and $\vec{E}$ as primary variables. The semidiscrete continuous time-discrete space system of equations can be written as

$$\text{(3-a)} \qquad \partial\vec{B}/\partial t = -\mathrm{CURL}\,\vec{E}\,,$$

$$\text{(3-b)} \qquad \partial\vec{E}/\partial t = {}_{\epsilon}\overline{\mathrm{CURL}}_{\mu}\,\vec{B}\,,$$

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†Los Alamos National Laboratory, T-7, MS-B284, Los Alamos, NM, 87545 (jh@lanl.gov).

‡Los Alamos National Laboratory, T-7, MS-B284, Los Alamos, NM, 87545 (misha@t7.lanl.gov).

where $\mathrm{CURL}\,\vec{E}$ is an approximation of $\mathbf{curl}\,\vec{E}$, and ${}_{\epsilon}\overline{\mathrm{CURL}}_{\mu}\,\vec{B}$ approximates the operator $\epsilon^{-1}\,\mathbf{curl}\,\mu^{-1}\,\vec{B}$. We assume that the tensors determining the material properties are given in grid cells and that the interface between different materials coincides with the faces of the cells (edges in 2D).

To define $\mathrm{CURL}\,\vec{E}$, we use a coordinate invariant definition of the $\mathbf{curl}\,\vec{E}$ based on a discrete analog of Faraday's law of electromagnetic induction and Stokes' circulation theorem,

$$(\vec{n}, \mathbf{curl}\,\vec{E}) = \lim_{S \to 0} \left(\oint_L (\vec{E}, \vec{l})\, dl \Big/ S \right). \tag{4}$$

In the discrete version of Stokes' theorem, the faces of the cells are the surfaces S and the "curve" L is formed by the edges of the cell. The tangential components of $\vec{E}$ given on the edges and Eq. 4 define a natural domain for $\mathrm{CURL}\,\vec{E}$. The range of CURL is the normal components of $\mathbf{curl}\,\vec{E}$ given on the faces of the cells. For Eq. 1-a, this approach naturally leads to using the normal components of $\vec{B}$ to describe the magnetic field.

When constructing ${}_{\epsilon}\overline{\mathrm{CURL}}_{\mu}\,\vec{B}$ in Eq. 3-b, it is important to note that when ϵ and μ are discontinuous and the computational grid is not smooth, then it is not possible to separate the discretization of the **curl** operator from the rest of the expression. We will construct the discrete analog of the operator $\epsilon^{-1}\,\mathbf{curl}\,\mu^{-1}\,\vec{B}$ based on the integral identity

$$\int_V (\vec{H}, \mathbf{curl}\,\vec{E})\, dV - \int_V (\vec{E}, \mathbf{curl}\,\vec{H})\, dV = \oint_{\partial V} ([\vec{E} \times \vec{H}], \vec{n})\, dS\,, \tag{5}$$

where $\vec{H} = \mu^{-1}\vec{B}$ is magnetic field density. The conservation of electromagnetic energy is a consequence of this identity and is preserved in the discrete model.

The discrete operator ${}_{\epsilon}\overline{\mathrm{CURL}}_{\mu}\,\vec{B}$ is defined using the support-operator method (SOM) [5, 6] by enforcing a discrete analog of the identity 5,

$$\int_V \mu^{-1}(\vec{B}, \mathbf{curl}\,\vec{E})\, dV - \int_V \epsilon\,(\vec{E}, \epsilon^{-1}\,\mathbf{curl}\,\mu^{-1}\vec{B})\, dV = \oint_{\partial V} \mu^{-1}([\vec{E} \times \vec{B}], \vec{n})\, dS\,. \tag{6}$$

and using the previously defined $\mathrm{CURL}\,\vec{E}$. This approach leads to a ${}_{\epsilon}\overline{\mathrm{CURL}}_{\mu}\,\vec{B}$ operator with compatible domain and range.

We discretize $\mathbf{div}\,\vec{B} = 0$ in Eqs. 2 using the natural coordinate invariant definition of the divergence based on Gauss' divergence theorem,

$$\mathbf{div}\,\vec{B} = \lim_{V \to 0} \left(\oint_{\partial V} (\vec{B}, \vec{n})\, dS \Big/ V \right). \tag{7}$$

The domain of this discrete divergence is the normal components of the vectors defined on the faces of the cells, and the range of values is the scalars defined in the cells. This discrete divergence satisfies a discrete analog of $\mathbf{div}\,\mathbf{curl}\,\vec{E} = 0$ in each cell [1]. Therefore, if initially the discrete analog of $\mathbf{div}\,\vec{B} = 0$ is satisfied, then it will hold at later times.

To discretize the condition $\mathbf{div}\,\vec{D} = \mathbf{div}\,\epsilon\,\vec{E} = 0$, we define a discrete analog of the operator $\mathbf{div}\,\epsilon\,\cdot$ with the domain consisting of the tangential components of the vectors. We use a discrete analog of the identity

$$\int_V \epsilon(\vec{W}, \mathbf{grad}\ u)\, dV = -\int_V u\,\mathbf{div}\,\epsilon\,\vec{W}\, dV \tag{8}$$

and the natural discretization of the **grad** operator based on its connection with the directional derivative $\partial u / \partial l = (\mathbf{grad}\,u, \vec{l})$. This definition, applied to the edges of the cell, defines a discrete gradient with a domain containing the scalar functions given by its values in nodes and a range of values given by the tangential components of the vector function on the edges. Therefore, the discrete analog of the condition $\mathbf{div}\,\vec{D}$ will hold at the nodes.

It is easily shown by an analysis, similar to the proofs in [2], that the discrete operators satisfy the discrete analog of identity $\mathbf{div}\,\mathbf{curl}\,\vec{H} = \mathbf{div}\,\epsilon \cdot (\epsilon^{-1}\,\mathbf{curl}\,\mu^{-1}\,\vec{B}) = 0$. Therefore, if the discrete analog of $\mathbf{div}\,\epsilon\,\vec{E} = 0$ holds initially, it will hold at later times.

These discrete operators can also approximate the equations of magnetic diffusion

$$\partial \vec{B}/\partial t = -\mathbf{curl}\,\vec{E}\,, \quad \vec{E} = \sigma^{-1}\,\mathbf{curl}\,\mu^{-1}\,\vec{B}\,, \tag{9}$$

where the conductivity σ is a symmetric positive-definite discontinuous tensor. The discrete approximation of Eqs. 9 is guaranteed to be symmetric and positive definite on arbitrary, nonorthogonal logically rectangular grids. These properties insure the spatial approximation is stable, and powerful conjugate-gradient-based iteration methods can be used when integrating the system with an implicit method.

Using numerical examples illustrating the reflection and refraction of waves at the boundary of two nonconducting media, and the scattering of a plane wave on a circular perfect conductor, we have demonstrated that on nonsmooth, nonorthogonal grids the methods converge between first and second order, depending on the smoothness of the grid.

References

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