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# Lagrangian Composite Schemes on Triangular Unstructured Grids

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# LAGRANGIAN COMPOSITE SCHEMES ON TRIANGULAR UNSTRUCTURED GRIDS

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ABSTRACT. *A composite finite difference scheme for a system of conservation laws is given by the global composition of several time steps of a higher order dispersive scheme such as Lax-Wendroff followed by one time step of a diffusive scheme such as Lax-Friedrichs. The diffusive scheme acts as a filter, consistent with the conservation law, that removes dispersive oscillations of the higher order scheme typically appearing around discontinuities of the solution. Composite schemes have been shown to work well in several different settings for Eulerian coordinates. Here we construct a Lax-Wendroff and a Lax-Friedrichs scheme and their composite on an unstructured triangular grid in Lagrangian coordinates. Numerical results for Noh's infinite strength shock problem and the Sedov blast wave problem are presented.*

## 1. INTRODUCTION

Hyperbolic non-linear conservation laws form a very important class of mathematical models describing physical systems. The Euler equations for gas dynamics describe conservation of mass, momenta and energy in the dynamics of gases or more general fluids such as plasmas. A Lagrangian formulation of the Euler equations in the coordinate system moving with the fluid is very valuable in situations in which the total volume or shape of the fluid changes dramaticaly, e.g. by compression or expansion, or when one models a system involving two or more different non-mixing fluids.

There are two very well known finite difference schemes for conservation laws; the first order diffusive Lax-Friedrichs (LF) scheme and the second order dispersive Lax-Wendroff (LW) scheme. The two-step versions of these schemes for a 1D system of conservation laws $u_t + f(u)_x = 0$ have the same predictor

$$u_{i+1/2}^{n+1/2} = \frac{1}{2}[u_i^n + u_{i+1}^n] - \frac{\Delta t}{2\Delta x}[f(u_{i+1}^n) - f(u_i^n)] \tag{1}$$

which advances the numerical solution from the primary grid $x_i$ at time level $n$ to the dual grid $x_{i+1/2}$at time level $n + 1/2$. The LF scheme uses the same formula (1) shifted by one half in both indices in corrector advancing solution from the dual grid at time level $n+1/2$ to the primary grid at time level $n+1$. The LW corrector is

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x}[f(u_{i+1/2}^{n+1/2}) - f(u_{i-1/2}^{n+1/2})]$$

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and the composite scheme LWLFk [4] is defined by global composition of $k-1$ LW time steps followed by one LF time step

$$u^{n+k} = L_F \circ L_W \circ \cdots \circ L_W u^n \tag{2}$$

where the operators $L_F$ and $L_W$ perform one timestep of LF and LW schemes respectively.

Composite schemes have been first used for 1D shallow water equations in [3], extended to 2D and Euler equations in [4], further extended to 3D in [5]. In [6] they have been applied to shallow water equations on trapezoidal meshes, in [7] to shallow water equations on the surface of a sphere, in [9] to gas dynamics in Largangian coordinates on quadrilateral grids and in [2] to Eulerian gas dynamics on triangular grids. Here we develop composite schemes in Lagrangian coordinates on triangular unstructured grids. Such grids can be constructed on an arbitrary domain which is in general not posible for the quadrilateral logicaly rectangular grids used in [9]. Traditional Lagrangian gas dynamics numerical method involving artificial viscosity has been developed on unstructured grids in [1].

The main advantage of composite schemes is their simplicity as they require neither eigenvector decomposition, nor Riemann solver. They are rather fast and well suited to vectorization or parallelization.

## 2. Finite Difference Lagrangian Schemes on Triangular Grids

The Lagrangian formulation of the gas dynamics equations in 2D has the form

$$\rho w_t = f_x + g_y \tag{3}$$

where $w = (\eta, u, v, E)^T, f = (u, -p, 0, -pu), g = (v, 0, -p, -pv)$, $\rho$ is density, $\eta = 1/\rho$, $u, v$ are $x$ and $y$ components of velocity, $E = e + 1/2(u^2+v^2)$ is total energy density with internal energy density $e$ and $p(\rho, e)$ is pressure given by an equation of state. The system of conservation laws (3) stated in coordinates $x, y$ moving with the fluid is amended by two ordinary differential equations (ODEs)

$$x_t = u, \;\; y_t = v \tag{4}$$

for each fluid particle $(a, b)$. At time $t$ the fluid particle labeled by $(a, b)$ appears at point $(x(a, b, t), y(a, b, t))$ and initial position of this particle is $(a, b)$ and defines initial conditions for two ODEs (4) $x(a, b, 0) = a, y(a, b, 0) = b$. For any region $\Omega(t)$ with boundary $\partial\Omega(t)$ evolving with the fluid motion the total mass of this region $\int_{\Omega(t)} \rho(x, y) dx\, dy$ remains constant independent of time and one can from (3) derive the integral formulation of the gas dynamics equations [9]

$$\int\limits_{\Omega(0)} w_t\, ds = \oint\limits_{\partial\Omega(t)} f\, dy - \oint\limits_{\partial\Omega(t)} g\, dx, \tag{5}$$

where $s$ corresponds to mass $ds = \rho\, dx\, dy = \rho\, J\, da\, db$ with $J = x_a y_b - x_b y_a$ being the Jacobian of coordinate transformation from $(a, b)$ to $(x, y)$.

As in 1D we have two grids, primary and dual. The primary grid is an unstructured triangular grid with triangular cells and vertices $(x, y)_i$ and the dual grid has $m$-lateral cells whose vertices are the medians $(\tilde{x}, \tilde{y})_j$ of the triangular primary cells, as shown in Fig. 1. At time level $n$ we assume the values of $w$ being constant inside

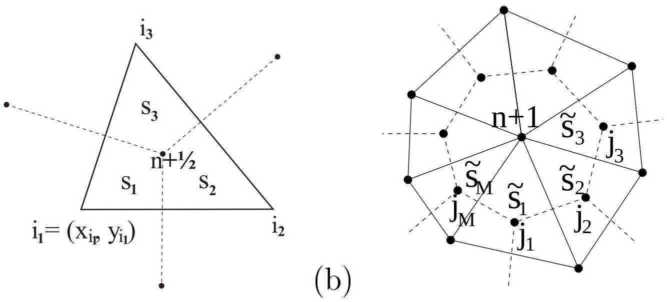


FIGURE 1. Cell $j$ of primary grid (a) and cell $i$ of dual grid (b). Edges of primary grid are shown by solid lines and edges of dual grid by dashed lines.

each dual cell $i$. Now setting region $\Omega(t)$ to the primary triangular cell $j$ shown in Fig. 1(a), performing integration of (5) and rearranging we get the finite volume predictor

$$w_j^{n+1/2} = \sum_{k=1}^{3} \left( \frac{s_k}{s} w_{i_k}^n + \left[ f_{i_k}^n \cdot \frac{\Delta y_k^n}{4s} - g_{i_k}^n \cdot \frac{\Delta x_k^n}{4s} \right] \cdot \Delta t \right), \tag{6}$$

where $s_k, k = 1, 2, 3$ are submasses of triangular cell $j$ (masses of the intersection of the primary cell $j$ with dual cells $i_k$), $s = \sum_{k=1}^{3} s_k$ is the total mass of cell $j$ and differences $\Delta$ are given by $\Delta x_k^n = x_{k+1}^n - x_{k-1}^n, \Delta y_k^n = y_{k+1}^n - y_{k-1}^n$ with cyclic extension for $k = 0, -1$. The predictor gives us the values of $w$ inside the primary triangular cells $j$ on time level $n+1/2$. Now we advance all vertices of primary grid $(x, y)_i$ by the simplest approximation to (4)

$$x_i^{n+1/2} = x_i^n + u_i^n \Delta t/2, \;\; y_i^{n+1/2} = y_i^n + v_i^n \Delta t/2 \tag{7}$$

and compute the medians $(\tilde{x}, \tilde{y})_j^{n+1/2}$ of each triangle $j$ which are vertices of the dual grid on time level $n + 1/2$.

Now we choose the dual $m$-lateral cell in Fig. 1(b) as the region $\Omega(t)$ in (5) from which with LF type forward time difference we get the LF corrector

$$w_i^{n+1} = \sum_{k=1}^{m} \left( \frac{\tilde{s}_k}{\tilde{s}} w_{j_k}^{n+1/2} - \left[ f_{j_k}^{n+1/2} \cdot \frac{\Delta \tilde{y}_k^{n+1/2}}{4\tilde{s}} - g_{j_k}^{n+1/2} \cdot \frac{\Delta \tilde{x}_k^{n+1/2}}{4\tilde{s}} \right] \cdot \Delta t \right)$$

and with central time difference we get the LW corrector

$$w_i^{n+1} = w_i^n + \Delta t \sum_{k=1}^{m} \left[ f_{j_k}^{n+1/2} \cdot \frac{\Delta \tilde{y}_k}{4\tilde{x}} - g_{j_k}^{n+1/2} \cdot \frac{\Delta \tilde{x}_k}{4\tilde{s}} \right],$$

where $\tilde{s}_k, k = 1, \cdots, m$ and $\tilde{s} = \sum_{k=1}^{m} \tilde{s}_k$ are submasses and total mass of dual $m$-lateral cell $i$ and differences $\Delta$ are given by $\Delta \tilde{x}_k^{n+1/2} = \tilde{x}_{k+1}^{n+1/2} - \tilde{x}_{k-1}^{n+1/2}, \Delta \tilde{y}_k^{n+1/2} = \tilde{y}_{k+1}^{n+1/2} - \tilde{y}_{k-1}^{n+1/2}$ again with cyclic extension. The corrector gives us the values of $w$ at time level $n+1$ inside the dual $m$-lateral cells. To close the loop we advance the primary points $i$ (vertices of primary triangular grid) from time level $n+1/2$ to $n+1$ by the analog of (7) with velocities at points $i$ obtained by linear interpolation of velocities at dual points $j$ (triangle medians, $m$-lateral vertices) at time level $n+1/2$.

The composite scheme LWLFk is defined by the same way as in 1D (2) by global composition of $k-1$ LW time steps with one LF step. Note that both LF and LW use the same predictor (6).

## 3. Numerical Results

First we present results for Noh's infinite strength shock problem [8]. Initial conditions for this problem consist of constant density and pressure $\rho_0 = 1, p_0 = 0$ everywhere and velocity of magnitude one, $u_0^2 + v_0^2 = 1$, directed towards the origin. This problem for an ideal gas with$\gamma = 5/3$ has the exact solution consisting of a circular infinite strength shock moving with velocity $1/3$ out from the origin. Inside the circular shock the density is 16, pressure is $16/3$ and velocity is zero. Outside the shock, i.e. for $\sqrt{x^2+y^2} > t/3$, the density is $(1+t/\sqrt{x^2+y^2})$, pressure remains zero and velocity remains the same as in the initial conditions. Fig. 2 (a),(b) shows numerical results by the LWLF4 composite scheme at time $t = 1$ on a hexagonal grid in the initial hexagonal region inscribed in the circle with radius 2. Our results have overshoot on the shock, however the typical (for numerical treatment of Noh problem) dip in origin is much less pronounced than that of standard Lagrangian methods, see [9, 1].

For the Sedov problem the initial density, velocity and pressure are constants $\rho_0 = 1, u_0 = 0, v_0 = 0, p_0 = 0$ except in a very small region around the origin where we set a very high pressure as a "point" explosion. Numerical results by LWLF3 scheme are presented in Fig. 2(c),(d).

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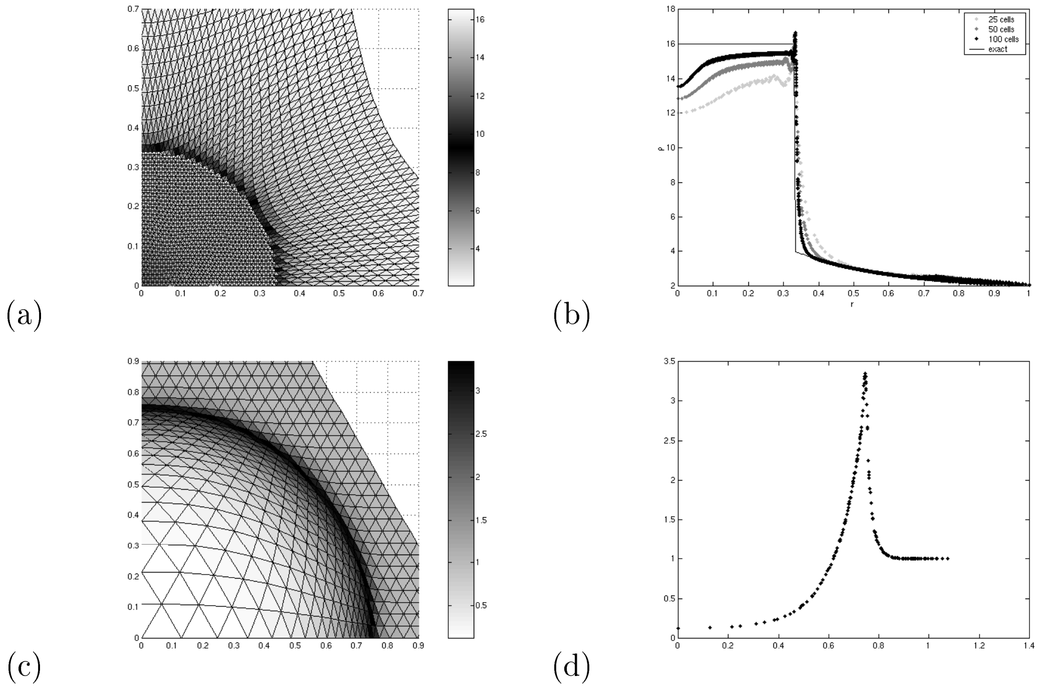


FIGURE 2. Noh problem: (a) colormap of density with the grid with 50 cells on radius; (b) scatter plot of density depending on the radius for 25, 50 and 100 cells on radius with exact solution. Sedov problem with 25 cells on radius: (c) colormap of density with the grid; (d) scatter plot of density depending on the radius

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