DIFFERENCE SCHEME FOR THE "DIRICHLET PARTICLE" METHOD IN CYLINDRICAL COORDINATES, CONSERVING SYMMETRY OF GAS-DYNAMICAL FLOW

A. V. Solov'ev and M. Yu. Shashkov

UDC 519.63

In the numerical investigation of multidimensional gas-dynamical flow, it is often important (see [1, 2]) that the difference scheme used not disrupt a one-dimensional property of a flow, either as a whole or in separate regions. The question of the conservation of various types of flow symmetry in numerical modelling has been investigated in several articles for certain difference-scheme classes (see, for example, [3-7]). In some cases it is convenient to construct schemes with the required property by using an appropriate coordinate system. However this approach to the development of such schemes is not trivial [6].

We note that the conservation of a one-dimensional property in various coordinate systems is practically never considered for methods using irregular meshes [8-15].

We describe a difference scheme of the "Dirichlet-particle" type in cylindrical coordinates which conserves plane, cylindrical, and spherical symmetry for gas-dynamical flow. This scheme is completely conservative.

Conservation of the one-dimensionality of a flow is attained by a special volume formula.

1. We consider first the gas-dynamical equations in rectangular coordinates (x, y) and the corresponding difference scheme described in [14] in the rectangle $E = \{x_{min}<x<x_{min}<y<y_{max}\}$. For definiteness we assume that $(\vec{W}, \vec{n})|_{\partial E}=0$, on the whole boundary, where $\vec{n}$ is the external normal to the boundary ∂E. We also assume that, initially, all quantities depend only on x, i.e., $\rho(x, y)|_{t=0} = \rho(x)$, $p(x, y)|_{t=0} = p(x)$, etc.; consistency requirements of initial and boundary conditions imply that $Wy(x)|_{t=0} = 0$.

The differential problem with these initial and boundary conditions has a solution of the form $p(x, y, t) = p(x, t)$, $\rho(x, y, t) = \rho(x, t)$ etc., $Wy(x, t) = 0$, which we call a one-dimensional solution in x.

Consider the following difference scheme from [14]:

$$\frac{d\rho}{dt}+\frac{1}{\rho}\,\mathrm{DIV}\,\vec{W}=0,\quad \rho\frac{dWx}{dt}+\mathrm{GRAD}_x p=0, \tag{1}$$

$$\rho\frac{dWy}{dt}+\mathrm{GRAD}_y p=0,\quad \rho\frac{d\varepsilon}{dt}+p\,\mathrm{DIV}\,\vec{W}=0,$$

here $Wx = (\vec{W}, \vec{ex})$ and $Wy = (\vec{W}, \vec{ey})$, and the unit basis vectors $\vec{ex}$ and $\vec{ey}$ are in the directions of the x- and y-axes, respectively. The difference operators are defined as follows:

$$(\mathrm{DIV}\,\vec{W})_i=\frac{1}{V_i}\sum_{k\in\omega_i}\left(\frac{\partial V_i}{\partial x_k}Wx_k+\frac{\partial V_i}{\partial y_k}Wy_k\right), \tag{2}$$

$$(\mathrm{GRAD}_x p)_i=-\frac{1}{V_i}\sum_{k\in\omega_i}\frac{\partial V_k}{\partial x_i}p_k,$$

$$(\mathrm{GRAD}_y p)_i=-\frac{1}{V_i}\sum_{k\in\omega_i}\frac{\partial V_k}{\partial y_i}p_k.$$

The volume V_i is chosen to be the area of a Dirichlet cell:

$$V_i=0.5\sum_{k\in\bar{\omega}_i}(x_{k+1/2}+x_{k-1/2})(y_{k+1/2}-y_{k-1/2}). \tag{3}$$

Direct differentiation yields the following formulas:

M. V. Keldysh Applied Mathematics Institute, Academy of Sciences of the USSR. Translated from Differentsial'nye Uravneniya, Vol. 24, No. 7, pp. 1249-1257, July, 1988. Original article submitted February 11, 1988.

0012-2661/88/2407-0817$12.50 © 1989 Plenum Publishing Corporation

2. We next consider a difference scheme in cylindrical coordinates:

$$\frac{d\rho}{dt}+\frac{1}{\rho}\,\mathrm{DIV}\,\vec{W}=0,\quad \rho\frac{dW_r}{dt}+\mathrm{GRAD}_r\,p=0,$$
$$\rho\frac{dW_z}{dt}+\mathrm{GRAD}_z\,p=0,\quad \rho\frac{d\varepsilon}{dt}+p\,\mathrm{DIV}\,\vec{W}=0, \tag{9}$$

where $Wr = (\vec{W}, \vec{er})$ and $Wz = (\vec{W}, \vec{ez})$, and the unit basis vectors $\vec{er}$ and $\vec{ez}$ are in the direction of the r- and x-axes, respectively.

The difference operators are determined as in the rectangular case:

$$(\mathrm{DIV}\,\vec{W})_i=\frac{1}{V_i}\sum_{k\in \mathrm{Pa}_i}\left(\frac{\partial V_i}{\partial r_k}W_{r_k}+\frac{\partial V_i}{\partial z_k}W_{z_k}\right),$$
$$(\mathrm{GRAD}_r\,p)_i=-\frac{1}{V_i}\sum_{k\in \mathrm{Pa}_i}\frac{\partial V_k}{\partial r_i}p_k, \tag{10}$$
$$(\mathrm{GRAD}_z\,p)_i=-\frac{1}{V_i}\sum_{k\in \mathrm{Pa}_i}\frac{\partial V_k}{\partial z_i}p_k.$$

In the present work, we consider a scheme (9), (10) in which the volume $V_i = r_i\bar{V}_i$, is used, where $\bar{V}_i$ is the area of a Dirichlet cell in the (r, z)-plane:

$$\bar{V}_i=\frac{1}{2}\sum_{k\in \overline{\mathrm{Pa}}_i}(r_{k+1/2}+r_{k-1/2})(z_{k+1/2}-z_{k-1/2}).$$

Properties of the volume V_i are as follows. Clearly

$$\frac{\partial V_i}{\partial r_k}=r_i\frac{\partial \bar{V}_i}{\partial r_k}\quad (i\neq k),\quad \frac{\partial V_i}{\partial z_k}=r_i\frac{\partial \bar{V}_i}{\partial z_k}. \tag{11}$$

and

$$\frac{\partial V_i}{\partial r_i}=r_i\frac{\partial \bar{V}_i}{\partial r_i}+\bar{V}_i,\ \text{and}\ \frac{\partial V_i}{\partial z_i}=r_i\frac{\partial \bar{V}_i}{\partial z_i}. \tag{12}$$

We prove that

$$\frac{\partial}{\partial r_i}\sum_{k\mathrm{Pa}_i}V_k=0. \tag{13}$$

Relations (11) and (12) imply that

$$\frac{\partial}{\partial r_i}\sum_{k\in \mathrm{Pa}_i}V_k=r_i\frac{\partial \bar{V}_i}{\partial r_i}+\bar{V}_i+\sum_{k\in \overline{\mathrm{Pa}}_i}r_k\frac{\partial \bar{V}_k}{\partial r_i}. \tag{14}$$

and, since

$$\frac{\partial \bar{V}_i}{\partial r_i}=-\sum_{k\overline{\mathrm{Pa}}_i}\frac{\partial \bar{V}_k}{\partial r_i}$$

by virtue of (5), it follows from (14) that

$$\frac{\partial}{\partial r_i}\sum_{k\mathrm{Pa}_i}V_k=\bar{V}_i+\sum_{k\overline{\mathrm{Pa}}_i}(r_k-r_i)\frac{\partial \bar{V}_k}{\partial r_i}. \tag{15}$$

It follows from (4) that (15) can be written as

$$\frac{\partial}{\partial r_i}\sum_{k\in \mathrm{Pa}_i}V_k=\bar{V}_i+\sum_{k\overline{\mathrm{Pa}}_i}(r_k-r_i)\frac{z_{k+1/2}-z_{k-1/2}}{r_k-r_i}$$
$$\left(r_i-\frac{r_{k+1/2}+r_{k-1/2}}{2}\right)=\bar{V}_i+\sum_{k\overline{\mathrm{Pa}}_i}r_i(z_{k+1/2}-z_{k-1/2})-\sum_{k\overline{\mathrm{Pa}}_i}(z_{k+1/2}-z_{k-1/2})\frac{r_{k+1/2}+r_{k-1/2}}{2}. \tag{16}$$

The second term in the last expression in (16) vanishes because the r_i can be taken out from under the summation sign, and the last term is equal to $-\bar{V}_i$ and cancels the first term.

This proves (13), which implies in particular that

$$\frac{\partial V_i}{\partial r_i} = -\sum_{k\overline{\mathrm{Pa}}_i} \frac{\partial V_k}{\partial r_i}, \tag{17}$$

i.e., there is a formula similar to (6) for V_i. Hence

$$\frac{\partial V_i}{\partial z_i} = -\sum_{k\overline{\mathrm{Pa}}_i} \frac{\partial V_k}{\partial z_i}.$$

Using (11) and (12), we rewrite the difference operators (10) as

$$(\mathrm{DIV}\,\vec{W})_i = \frac{1}{\bar{V}_i}\sum_{k\mathrm{Pa}_i}\left(\frac{\partial \bar{V}_i}{\partial r_k} Wr_k + \frac{\partial \bar{V}_i}{\partial z_k} Wz_k\right) + \frac{Wr_i}{r_i},$$

$$(\mathrm{GRAD}_r p)_i = -\frac{1}{\bar{V}_i}\sum_{k\mathrm{Pa}_i}\frac{\partial \bar{V}_k}{\partial r_i}\left(\frac{r_k}{r_i}p_k\right) - \frac{p_i}{r_i},$$

$$(\mathrm{GRAD}_z p)_i = -\frac{1}{\bar{V}_i}\sum_{k\mathrm{Pa}_i}\frac{\partial \bar{V}_k}{\partial z_i}\left(\frac{r_k}{r_i}p_k\right),$$

and it follows that

$$(\mathrm{DIV}\,\vec{W})_i = (\overline{\mathrm{DIV}}\,\vec{W})_i + Wr_i/r_i,\quad (\mathrm{GRAD}_r p)_i = \tag{18}$$

$$= (1/r_i)\,(\overline{\mathrm{GRAD}}_r \tilde{p})_i - p_i/r_i,\quad (\mathrm{GRAD}_z p)_i = (1/r_i)\,(\overline{\mathrm{GRAD}}_z \tilde{p})_i.$$

Here $\overline{\mathrm{DIV}}$ and $\overline{\mathrm{GRAD}}$ can be obtained from the corresponding operators in (2) by replacing x and y by r and z, respectively.

3. Here we consider the problem in the rectangle $E\{0<r<r_{max},\ z_{min}<z<z_{max}\}$. On the axis of symmetry, we assume that $Wr|_{r=0} = 0$, and we also assume that $Wr|_{r=r_{max}} = 0$ and $Wz|_{z=z_{max}} = 0$. If all functions depend only on r at the initial time, and $W_z|_{t=0} = 0$, then the differential problem has a solution depending only on r.

We prove that the difference scheme has an analogous property. In E choose the rectangular mesh, uniform with respect to z, with the mesh points (r_i, z_j): $z_j = (j - 0.5)hz$, $hz = (z_{max} - z_{min})/Nz$, $0<r_1<\ldots r_{Nr}$. We use the following notation $hr_1 = 2r_1$; $hr_i = r_i - r_{i-1}$, $i = 2, \ldots, Nr$; $hr_{Nr+1} = 2(r_{max} - r_{Nr})$. It follows from (18) and (8) that $(\mathrm{DIV}\,W)_{ij} = (Wr_{i-1})/(hr_i + hr_{i+1})$ and r_i, $(\mathrm{GRAD}_r p)_{ij} = (1/r_i)(p_{i+1}r_{i+1} - p_{i-1}r_{i-1})/(hr_i + hr_{i+1}) - p_i/r_i$, and $(\mathrm{GRAD}_z p)_{ij} = 0$, i.e., there is r-one-dimensionality.

To prove that the scheme is one-dimensional with respect to z, we use a rectangular mesh, uniform with respect to r, and impose the appropriate boundary conditions. Since $Wr_{ij} = 0$, the expression for DIV in this case coincides with (8) for the same operator with y replaced by z and the assumption that $Wx_{ij} = 0$. Hence it follows from Sec. 1 that $(\mathrm{DIV}\,\vec{W})_{ij} = S_j$. Since $\tilde{p}_{ij} = r_{ij}p_{ij}$ and by virtue of the assumption that $r_{ij} = r_j$ and $p_{ij} = p_j$, we conclude that $\tilde{p}_{ij} = \tilde{p}_j$. It therefore follows from Sec. 1 and the last formula in (18) that $(\mathrm{GRAD}_z p)_{ij} = G_j$. It remains to prove that $(\mathrm{GRAD}_r p)_{ij} = 0$.

Using (17) in (10), we obtain

$$(\mathrm{GRAD}_r p)_{ij} = -\left[\frac{\partial V_{ij+1}}{\partial r_{ij}}(p_{j+1} - p_i) + \frac{\partial V_{ij-1}}{\partial r_{ij}}(p_{j-1} - p_i)\right]/V_{ij}.$$

The required relation now follows from relations (11), $\partial \bar{V}_{ij+1}/\partial r_{ij} = 0$, and $\partial \bar{V}_{ij-1}/\partial r_{ij} = 0$.

This proves that our difference scheme conserves the one-dimensionality of gas-dynamical flow with respect to z and r.

4. We now consider the scheme (9), (10) for spherically symmetric flow. In the (r, z)-plane, we introduce a polar mesh, uniform with respect to the angle. The coordinates of the mesh points are as follows:

$$r_{ij} = R_j \sin\Theta_i,\ \Theta_i = (i - 0.5)\Delta\Theta; \tag{19}$$

$$z_{ij} = R_j \cos\Theta_i,\ \Delta\Theta = 0.5\pi/N\Theta,$$

here R_j is the spherical radius of a point; Θ_i is the angle measured from the z-axis. We assume that, at the initial time, $p_{ij} = p_j$, $\varepsilon_{ij} = \varepsilon_j$, $\rho_{ij} = \rho_j$, $W\Theta_{ij} = -\cos\Theta_i Wr_{ij} + \sin\Theta_i Wz_{ij}$.

, and $WR_{ij} = \sin\theta_i Wr_{ij} + \cos\theta_i Wz_{ij} = U_j$ and that appropriate boundary conditions are imposed. We wish to prove that these conditions are satisfied at each instant of time, i.e., that

$$-\cos\Theta_i(\mathrm{GRAD}_r p)_{ij}+\sin\Theta_i(\mathrm{GRAD}_z p)_{ij}=0,\ \sin\Theta_i(\mathrm{GRAD}_r p)_{ij}+\cos\Theta_i(\mathrm{GRAD}_z p)_{ij}=G_j,\ (\mathrm{DIV}\,\vec{W})_{ij}=D_j. \quad (20)$$

y virtue of our assumptions, the operator GRAD is defined as follows:

$$(\mathrm{GRAD}_r p)_{ij}=-\left[\frac{\partial V_{ij+1}}{\partial r_{ij}}(p_{j+1}-p_j)+\frac{\partial V_{ij-1}}{\partial r_{ij}}(p_{j-1}-p_j)\right]/V_{ij},$$
$$(\mathrm{GRAD}_z p)_{ij}=-\left[\frac{\partial V_{ij+1}}{\partial z_{ij}}(p_{j+1}-p_j)+\frac{\partial V_{ij-1}}{\partial z_{ij}}(p_{j-1}-p_j)\right]/V_{ij}. \quad (21)$$

rom (11) and (19) we conclude that

$$\frac{\partial V_{ij+1}}{\partial r_{ij}}=r_{ij+1}\frac{\partial \nabla_{ij+1}}{\partial r_{ij}}=R_{j+1}\sin\Theta_i\frac{\partial \nabla_{ij+1}}{\partial r_{ij}},$$
$$\frac{\partial V_{ij-1}}{\partial r_{ij}}=R_{j-1}\sin\Theta_i\frac{\partial \nabla_{ij-1}}{\partial r_{ij}},\quad \frac{\partial V_{ij+1}}{\partial z_{ij}}=R_{j+1}\sin\Theta_i\frac{\partial \nabla_{ij+1}}{\partial z_{ij}},$$
$$\frac{\partial V_{ij-1}}{\partial z_{ij}}=R_{j-1}\sin\Theta_i\frac{\partial \nabla_{ij-1}}{\partial z_{ij}},$$

here $V_{ij} = R_j\sin\theta_i\cdot V_{ij} = R_j\sin\theta_i\cdot V_j$. Hence relations (21) can be written as

$$(\mathrm{GRAD}_r p)_{ij}=-\frac{1}{R_j\nabla_j}\left[\frac{\partial \nabla_{ij+1}}{\partial r_{ij}}(R_{j+1}(p_{j+1}-p_j))+\frac{\partial \nabla_{ij-1}}{\partial r_{ij}}(R_{j-1}(p_{j-1}-p_j))\right],$$
$$(\mathrm{GRAD}_z p)_{ij}=-\frac{1}{R_j\nabla_j}\left[\frac{\partial \nabla_{ij+1}}{\partial z_{ij}}(R_{j+1}(p_{j+1}-p_j))+\frac{\partial \nabla_{ij-1}}{\partial z_{ij}}(R_{j-1}(p_{j-1}-p_j))\right]. \quad (22)$$

ence, since p_j and R_j are arbitrary, the first relation in (20) is satisfied only if

$$-\cos\Theta_i\frac{\partial \nabla_{ij+1}}{\partial r_{ij}}+\sin\Theta_i\frac{\partial \nabla_{ij+1}}{\partial z_{ij}}=0,\ -\cos\Theta_i\frac{\partial \nabla_{ij-1}}{\partial r_{ij}}+\sin\Theta_i\frac{\partial \nabla_{ij-1}}{\partial z_{ij}}=0. \quad (23)$$

y using the formulas

$$r_{k+1/2}=2^{-1}[(z_k-z_i)(r^2_{k+1}-r^2_i+z^2_{k+1}-z^2_i)-(z_{k+1}-z_i)(r^2_k-r^2_i+z^2_k-z^2_i)]/F_{k+1/2},$$

and

$$z_{k+1/2}=2^{-1}[(r_i-r_k)(r^2_{k+1}-r^2_i+z^2_{k+1}-z^2_i)-(r_i-r_{k+1})(r^2_k-r^2_i+z^2_k-z^2_i)]/F_{k+1/2},$$
$$F_{r+1/2}=(z_k-z_i)(r_{k+1}-r_i)-(z_{k+1}-z_i)(r_k-r_i) \quad (24)$$

rom [14], we can prove that

$$\frac{\partial \nabla_{ij+1}}{\partial r_{ij}}=\alpha\frac{R_{j+1/2}}{R_{j+1}-R_j}\cdot\frac{\sin\Theta_{i+1/2}-\sin\Theta_{i-1/2}}{\cos\Theta_i}\left(R_j\sin\Theta_i-\alpha R_{j+1/2}\frac{\sin\Theta_{i+1/2}+\sin\Theta_{i-1/2}}{2}\right), \quad (25)$$
$$\frac{\partial \nabla_{ij+1}}{\partial z_{ij}}=\alpha\frac{R_{j+1/2}}{R_{j+1}-R_j}\cdot\frac{\cos\Theta_{i-1/2}-\cos\Theta_{i+1/2}}{\sin\Theta_i}\left(R_j\cos\Theta_i-\alpha R_{j+1/2}\frac{\cos\Theta_{i+1/2}+\cos\Theta_{i-1/2}}{2}\right),$$

here $R_{j+1/2} = (R_j + R_{j+1})/2$, $\Theta_{i+1/2} = (\theta_i + \theta_{i+1})/2$, and $\alpha = \cos^{-1}(\Delta\theta/2)$. Applying known rigonometric relations in (25), we conclude that the first identity in (23) is satisfied.

Relations (22) imply that, to establish the second identity in (20), we need only prove hat

$$\sin\Theta_i\frac{\partial \nabla_{ij+1}}{\partial r_{ij}}+\cos\Theta_i\frac{\partial \nabla_{ij+1}}{\partial z_{ij}}=G^1_j,$$
$$\sin\Theta_i\frac{\partial \nabla_{ij-1}}{\partial r_{ij}}+\cos\Theta_i\frac{\partial \nabla_{ij-1}}{\partial z_{ij}}=G^2_j. \quad (26)$$

e now use the following formulas, analogous to formulas (25):

$$\frac{\partial \nabla_{ij+1}}{\partial r_{ij}}=\alpha\frac{R_{j+1/2}}{R_{j+1}-R_j}\cdot\frac{\cos\Theta_{i-1/2}-\cos\Theta_{i+1/2}}{\sin\Theta_i}\left(R_j\sin\Theta_i-\alpha R_{j+1/2}\frac{\sin\Theta_{i+1/2}+\sin\Theta_{i-1/2}}{2}\right),$$
$$\frac{\partial \nabla_{ij+1}}{\partial z_{ij}}=\alpha\frac{R_{j+1/2}}{R_{j+1}-R_j}\cdot\frac{\sin\Theta_{i+1/2}-\sin\Theta_{i-1/2}}{\cos\Theta_i}\left(R_j\cos\Theta_i-\alpha R_{j+1/2}\frac{\cos\Theta_{i+1/2}+\cos\Theta_{i-1/2}}{2}\right),$$

This yields $G_j^1 = \alpha(R_{j+1/2}/(R_{j+1} - R_j))(R_j^2 - 2^{-1}\alpha R_{j+1/2}\sin\Delta\theta)$. The second formula in (26) can be proved similarly.

We next prove that $(\mathrm{DIV}\,\vec{W})_{ij} = D_j$ when $Wr_{ij} = \sin\Theta_i \cdot U_j$ and $Wz_{ij} = \cos\Theta_i \cdot U_j$, i.e., $\vec{W}$ is in the direction of the radius vector. From the first formula in (18), we obtain

$$(\mathrm{DIV}\,\vec{W})_{ij} = \frac{1}{\bar{V}_{ij}}\left[\frac{\partial\bar{V}_{ij}}{\partial r_{ij+1}}(U_{i+1}-U_i)\sin\Theta_i + \frac{\partial\bar{V}_{ij}}{\partial z_{ij+1}}(U_{j+1}-U_j)\cos\Theta_i + \frac{\partial\bar{V}_{ij}}{\partial r_{ij-1}}(U_{i-1}-U_i)\sin\Theta_i + \right.$$
$$+\frac{\partial\bar{V}_{ij}}{\partial z_{ij-1}}(U_{j-1}-U_j)\cos\Theta_i + \frac{\partial\bar{V}_{ij}}{\partial r_{i-1j}}(\sin\Theta_{i-1}-\sin\Theta_i)U_j +$$
$$+\frac{\partial\bar{V}_{ij}}{\partial z_{i-1j}}(\cos\Theta_{i-1}-\cos\Theta_i)U_i + \frac{\partial\bar{V}_{ij}}{\partial r_{i+1j}}(\sin\Theta_{i+1}-\sin\Theta_i)U_j +$$
$$\left.+\frac{\partial\bar{V}_{ij}}{\partial z_{i+1j}}(\cos\Theta_{i+1}-\cos\Theta_i)U_i\right] + \frac{U_j}{R_i}. \qquad (27)$$

The last term in (27) depends only on j; hence, to prove that $(\mathrm{DIV}\,\vec{W})_{ij} = D_j$, it is sufficient to establish that the expression in square brackets depends only on j. Since DIV is a linear operator, it is sufficient to consider separately the cases $U_j = 1$, $U_k = 0$, $k \neq j$ etc., and this leads to the following conditions:

$$D_{ij}^1 = \frac{\partial\bar{V}_{ij}}{\partial r_{ij+1}}\sin\Theta_i + \frac{\partial\bar{V}_{ij}}{\partial z_{ij+1}}\cos\Theta_i = D_j^1,$$

$$D_{ij}^2 = \frac{\partial\bar{V}_{ij}}{\partial r_{ij-1}}\sin\Theta_i + \frac{\partial\bar{V}_{ij}}{\partial z_{ij-1}}\cos\Theta_i = D_j^2,$$

$$D_{ij}^1 + D_{ij}^2 + D_{ij}^3 = D_{ij}^1 + D_{ij}^2 + \frac{\partial\bar{V}_{ij}}{\partial r_{i-1j}}(\sin\Theta_{i-1}-\sin\Theta_i) +$$

$$\frac{\partial\bar{V}_{ij}}{\partial z_{i-1j}}(\cos\Theta_{i-1}-\cos\Theta_i) + \frac{\partial\bar{V}_{ij}}{\partial r_{i+1j}}(\sin\Theta_{i+1}-\sin\Theta_i) + \frac{\partial\bar{V}_{ij}}{\partial z_{i+1j}}(\cos\Theta_{i+1}-\cos\Theta_i) = D_j^4$$

The first two identities can be obtained from (26) by a translation of the indices with respect to j; to prove the last relation, it is sufficient to show that D_{ij}^3 depends only on j. From (22) we obtain

$$\frac{\partial\bar{V}_{ij}}{\partial r_{i+1j}} = \frac{\alpha}{R_j}\frac{(R_{j+1/2}-R_{j-1/2})\cos\Theta_{i+1/2}}{\sin\Theta_{i+1}-\sin\Theta_i}\left(R_j\sin\Theta_{i+1} - \alpha\sin\Theta_{i+1/2}\frac{R_{j-1/2}+R_{j+1/2}}{2}\right)$$

$$\frac{\partial\bar{V}_{ij}}{\partial z_{i+1j}} = \frac{\alpha}{R_j}\frac{(R_{j+1/2}-R_{j-1/2})\sin\Theta_{i+1/2}}{\cos\Theta_{i+1}-\cos\Theta_i}\left(R_j\cos\Theta_{i+1} - \alpha\cos\Theta_{i+1/2}\frac{R_{j-1/2}+R_{j+1/2}}{2}\right)$$

and similar formulas hold for $\partial\bar{V}_{ij}/\partial r_{i-1j}$ and $\partial\bar{V}_{ij}/\partial z_{i-1j}$. It can now be directly verified that $D_{ij}^3 = 2\alpha(R_{j+1/2} - R_{j-1/2})\sin(\Delta\Theta/2) = D_j^3$, which is the required result.

We have proved that all conditions in (20) are satisfied, and so our difference scheme conserves the spherical symmetry of the flow.

Examples of test problems verifying our theoretical results can be found in [15].

The authors take this opportunity to thank A. A. Samarskii for his interest in this work, and A. P. Favorskii and V. F. Tishkin for their valuable discussions of our results.

LITERATURE CITED

1. A. A. Samarskii (ed.), An Investigation of Hydrodynamical Instability of Numerical Methods [in Russian], Moscow (1980).
2. G. P. Voskresenskii and A. V. Zabrodin, "Problems in numerical modelling of supersonic spatial flow and questions concerning the representation of algorithms for computers," Moscow (1986) (Preprint IMP Akad. Nauk SSSR: 83).
3. V. F. Tishkin, N. N. Tyurina, and A. P. Favorskii, "Difference schemes for calculating hydrodynamical flows in cylindrical coordinates," Moscow (1978) (Preprint IPM Akad. Nauk SSSR: 23).
4. S. M. Bakhrakh, G. V. Zharova, and V. F. Spiridonov, Questions in Atomic Science and Technology. Methods and Programs for the Numerical Solution of Problems of Mathematical Physics [in Russian], (1982), No. 3, pp. 15-21.

5. V. V. Drozdov and M. Yu. Shashkov, Development of Difference Schemes for Laplace's Equation Conserving Symmetry: A Collection of Theses, Lectures and Reports. All-Union School for Young Scientists. Numerical Methods and Mathematical Modelling [in Russian], Minsk (1984), pp. 69-70.
6. V. M. Goloviznin, V. F. Tishkin, and A. P. Favorskii, "A variational approach to the construction of difference schemes for hydrodynamics equations in spherical coordinates," Moscow (1977). (Preprint IPM Akad. Nauk SSSR: 16).
7. L. V. Shurshalov, Zh. Vychisl. Mat. Mat. Fiz., 20, No. 3, pp. 793-800 (1980).
8. V. F. D'yachenko, Zh. Vychisl. Mat. Mat. Fiz., 5, No. 4, pp. 680-688 (1965).
9. F. Kh. Kharlou, Numerical Methods in Hydrodynamics, B. Older, S. Fernbach, and M. Rotenberg (eds.) [in Russian], Moscow (1967), pp. 316-342.
10. N. N. Anuchina, Numerical Methods in the Mechanics of Continuous Media [in Russian], Novosibirsk (1970), Vol. 1, No. 4, pp. 3-84.
11. Yu. P. Glagoleva, B. M. Zhogov, Yu. F. Kir'yanov, et al., Numerical Methods in the Mechanics of Continuous Media [in Russian], Novosibirsk (1972), Vol. 3, No. 2, pp. 18-55.
12. K. V. Kosmachevskii, Mathematical Modelling of Two-Dimensional MHD Flow in a Magnetoplasma Compressor on an Adjustable Lagrange Mesh, Dissertation, Candidate in Physicomathematical Sciences [in Russian], Moscow (1985).
13. R. A. Volkova, N. V. Mikhailova, V. F. Tishkin, and A. P. Favorskii, "Application of a variational approach to the calculation of gas-dynamical flow on meshes of variable structure," Moscow (1982) (Preprint IPM Akad. Nauk SSSR: 17).
14. N. V. Mikhailova, V. F. Tishkin, N. N. Tyurina, et al., Zh. Vychisl. Mat. Mat. Fiz., 26, No. 9, 1392-1406 (1986).
15. A. V. Solov'ev, E. V. Solov'eva, V. F. Tishkin, et al., "Artificial viscosity in the 'Dirichlet particle' method," Moscow (1986) (Preprint IPM Akad. Nauk SSSR: 133).

SECOND-ORDER DIFFERENCE SCHEME FOR GAS-DYNAMICS EQUATIONS WITH A CONSISTENT APPROXIMATION OF CONVECTIVE FLOWS

S. I. Tkachenko

UDC 519.63

1. Introduction. We investigate a method of solving gas-dynamics problems introduced in [1, 2], using difference schemes with consistent approximation of convective flow. The flow is described in Euler variables. Consistency of approximations of flows is, in particular, an element in the construction of completely conservative difference schemes (c.c.d.s.) and imposes rather stringent restrictions on the form of the difference equations. This decreases the range of possible improvements in the quality of the corresponding algorithms. One way of improving c.c.d.s. is to raise the order of approximation of convective flows while conserving the condition for their consistency. We use this method.

2. Original Equations. In a domain $D(x, y)\in R^2$, with x, y rectangular coordinates, we consider the following equation system describing nonstationary flow of a compressible heat-conducting viscous medium, closed by the equation of state $P = (\gamma - 1)\rho\varepsilon$ of a perfect gas:

$$\begin{aligned} &\partial\rho/\partial t+\operatorname{div}(\rho\vec{V})=0,\\ &\partial\rho\vec{V}/\partial t+\operatorname{div}(\rho\vec{V}\otimes\vec{V})=-\operatorname{grad}P+\operatorname{div}d,\\ &\partial\rho\varepsilon/\partial t+\operatorname{div}(\rho\varepsilon\cdot\vec{V})=-P\operatorname{div}\vec{V}+\operatorname{div}(\varkappa\operatorname{grad}\varepsilon)+R. \end{aligned} \tag{1}$$

Here $\vec{V}(u, v)$ is the velocity; ρ is the density; P is the pressure; ε is the specific internal energy; d is the viscous-stress tensor; R is the energy release caused by the action of viscosity; γ is the adiabatic exponent; $\varkappa$ is the thermal conductance; $\vec{V}\otimes\vec{V}$ is a direct product of vectors.

We consider an initial boundary-value problem for (1) in D.

M. V. Keldysh Applied Mathematics Institute, Academy of Sciences of the USSR. Translated from Differentsial'nye Uravneniya, Vol. 24, No. 7, pp. 1258-1262, July, 1988. Original article submitted February 11, 1988.

 © 1989 Plenum Publishing Corporation