JOURNAL OF COMPUTATIONAL PHYSICS **139,** 406–409 (1998)
ARTICLE NO. CP975877

Local Reconstruction of a Vector Field from Its Normal Components on the Faces of Grid Cells

Mikhail Shashkov, Blair Swartz, and Burton Wendroff

Los Alamos National Laboratory, T-7, MS-B284, Los Alamos, New Mexico 87545
E-mail: misha@t7.lanl.gov

Received May 27, 1997; revised October 1, 1997

Key Words: discrete vector fields; reconstruction; non-smooth grids; discrete operators; Lagrangian gas dynamics.

We compare two local least squares approximations for the reconstruction of the Cartesian components of a vector field at the nodes of a logically rectangular grid, when this vector field is given by its components normal to the faces (edges in 2D) of the cells of the grid. Such a problem appears naturally, for example, for Lagrangian gas dynamics codes based on Godunov's method, where the normal component of a vector on an edge between two cells is computed from the solution of a 1D Riemann problem [1], but the Cartesian components are needed at nodes in order to compute the nodal motion. This kind of reconstruction is also interesting in the framework of discrete vector analysis [2] as a map between different spaces of vector functions. Reconstructed vector fields are often used to compute discrete analogs of differential operators such as divergence and curl. Therefore, to estimate the quality of the reconstruction we need to evaluate not only the accuracy of the vector itself but also the accuracy of the approximate discrete div and **curl**. More specifically we are given the exact values at the edge centroids of the projection of a vector function on the directions normal to the sides of the cells (see Fig. 1a for notation). Our goal is to reconstruct the Cartesian components $A^x_{i,j}$, $A^y_{i,j}$ of the vector $\mathbf{A}$ at the node (i, j).

The zero-order reconstruction algorithm described in [1] is based on choosing $\mathbf{A}_{i,j}$ to minimize the local quadratic functional

$$\sum_{l=j-\frac{1}{2},j+\frac{1}{2}} \left(\mathbf{A}_{i,j}\cdot\mathbf{n}^{\xi}_{i,l} - A^{\xi}_{i,l}\right)^2 + \sum_{k=i-\frac{1}{2},i+\frac{1}{2}} \left(\mathbf{A}_{i,j}\cdot\mathbf{n}^{\eta}_{k,j} - A^{\eta}_{k,j}\right)^2,$$

where $\mathbf{n}^{\xi}$ and $\mathbf{n}^{\eta}$ are the unit normals to the grid lines $i = \text{const}$ and $j = \text{const}$, respectively, and A^{ξ}, A^{η} are the given projections of $\mathbf{A}$ onto these normals. This procedure produces a 2×2 system of equations that can easily be solved for the unknowns $A^x_{i,j}$, $A^y_{i,j}$. This algorithm is exact only for constant vector functions.

0021-9991/98 $25.00

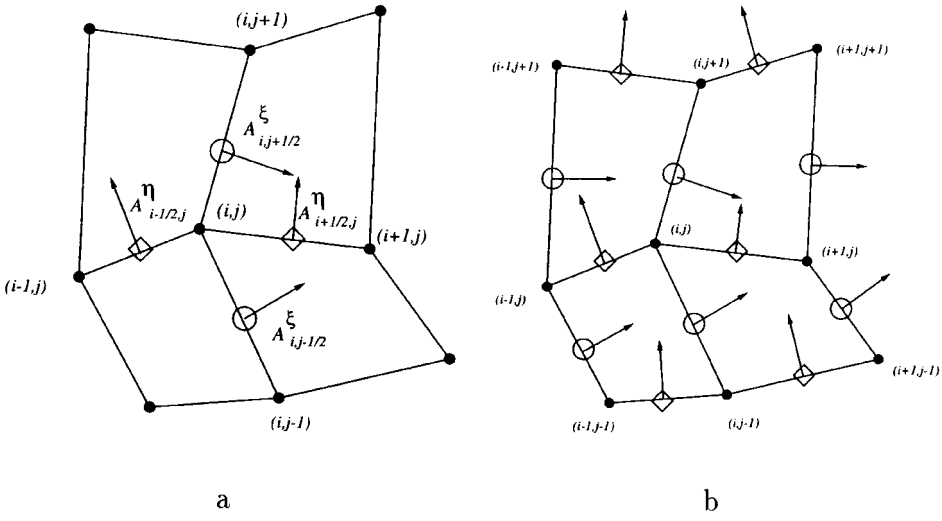

FIG. 1. Normal components A^{ξ} and A^{η} involved in reconstructing the Cartesian components $A^{x}_{i,j}$ and $A^{y}_{i,j}$ at node (i, j): (a) zero-order reconstruction; (b) first-order reconstruction.

We suggest here a new first-order approximation which leads to a 6×6 system of equations that is exact for linear vector functions. The unknowns in this method will be the values of the Cartesian components of the vector function and their first derivatives $dA^x/dx, dA^x/dy, dA^y/dx, dA^y/dy$ at the nodes. Because now we have six unknowns we need to use more information, and we will try to match the 12 normal components that are naturally related to node (i, j) (see Fig. 1b). If we define $\mathbf{r} = (x, y)$, then the functional to minimize is

$$\sum_{k=i-1,i,i+1} \sum_{l=j-\frac{1}{2},j+\frac{1}{2}} \left[(\mathbf{A}_{i,j} + (\mathbf{r}_{k,l} - \mathbf{r}_{i,j}) \cdot (\nabla \mathbf{A})_{i,j}) \cdot \mathbf{n}^{\xi}_{k,l} - A^{\xi}_{k,l}\right]^2$$
$$+ \sum_{k=i-\frac{1}{2},i+\frac{1}{2}} \sum_{l=j-1,j,j+1} \left[(\mathbf{A}_{i,j} + (\mathbf{r}_{k,l} - \mathbf{r}_{i,j}) \cdot (\nabla \mathbf{A})_{i,j}) \cdot \mathbf{n}^{\eta}_{k,l} - A^{\eta}_{k,l}\right]^2.$$

(Note. While a *quadratic* field would involve exactly 12 degrees of freedom, the resulting (12×12) system for their determination from the given data is singular for a uniform square grid.)

The accuracy of these reconstructions strongly depends on the smoothness of the grid. We will consider two grids. The first grid is the smooth grid (shown in Fig. 2a) obtained by mapping a uniform grid on the square $[-0.5, 0.5] \times [-0.5, 0.5]$ in the space (ξ, η) into the same square in computational space (x, y) via $x(\xi, \eta) = \xi + 0.1 \sin(2\pi\xi) \sin(2\pi\eta)$, $y(\xi, \eta) = \eta + 0.1 \sin(2\pi\xi) \sin(2\pi\eta)$. The second grid is a nonsmooth (random) grid that is obtained from a uniform grid with mesh size $h = 1/(M - 1)$ by moving each node to a random position in a square with side $0.25h$ centered at the original position of the node (shown in Fig. 2b). The error of the approximation to $\mathbf{A}$ was computed in the max norm over interior nodes.

To compute a discrete divergence, DIV, and discrete curl, **CURL**, for a cell we use the approximate Cartesian components of $\mathbf{A}$ at the nodes and a standard difference approximation

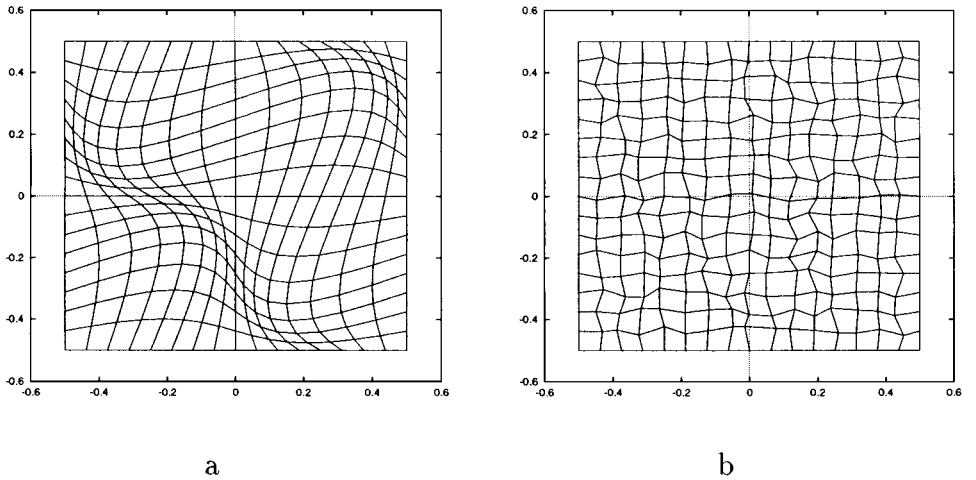

FIG. 2. (a) Smooth grid; (b) non-smooth (random) grid.

for derivatives. For this, the discrete analog of $\partial u/\partial x$ is

$$\left(\frac{\Delta u}{\Delta x}\right)_{i+1/2,\,j+1/2} = \frac{(u_{i+1,\,j+1}-u_{i,\,j})(x_{i,\,j+1}-x_{i+1,\,j}) - (u_{i,\,j+1}-u_{i+1,\,j})(x_{i+1,\,j+1}-x_{i,\,j})}{2V_{i+1/2,\,j+1/2}},$$

where $V_{i+1/2,\,j+1/2}$ is the volume of the cell $(i+1/2,\, j+1/2)$, with a similar expression for $\partial u/\partial y$ (see, for example, [3]). The error of the resulting approximation for div and **curl** (evaluated at the average of the cell's vertices) is also measured in the max norm over (strictly) interior cells.

We test the reconstruction procedures on two vector fields. The first is a "smooth" vector field, $A^x(x, y) = x - y + x^2 - y^2$, $A^y(x, y) = x + y + x^2 + y^2$. In the second field we try to model the structure of the velocity field in a 1D shock, and we take **A** to have the form

$$A^x(x, y) = e^{20x}/(1 + e^{20x}), \quad A^y(x, y) = 0.$$

The errors in the max norm for the "smooth" vector function itself and for its div and curl, for both smooth and random grids are given in Table 1. This table suggests that both methods

TABLE 1
Maximum Errors for the "Smooth" Vector Field; Smooth and Random Grids

		Smooth			Nonsmooth		
Grid type	M	**A**	DIV	**CURL**	**A**	DIV	**CURL**
Zeroth	33	9.28E-3	1.12E-1	3.49E-1	2.75E-2	1.08	1.30
	65	2.43E-3	3.37E-2	1.11E-1	1.36E-2	1.13	1.48
	129	6.13E-4	8.72E-3	2.94E-2	6.49E-3	1.38	1.68
First	33	2.22E-3	1.01E-2	9.74E-3	1.26E-3	3.46E-2	2.32E-2
	65	5.55E-4	2.56E-3	2.57E-3	3.17E-4	1.73E-2	1.34E-2
	129	1.39E-4	6.41E-4	6.53E-4	8.06E-5	9.02E-3	7.07E-3

TABLE 2
Maximum Errors for the "Shock" Vector Field, Random Grid

Type	M	**A**	A^y	DIV	**CURL**
Zeroth	33	2.90E-2	2.90E-2	6.71E-1	2.21
	65	1.91E-2	1.55E-2	1.05	1.97
	129	9.10E-3	9.10E-3	1.40	3.25
First	33	1.45E-2	1.96E-3	3.65E-1	2.26E-1
	65	4.06E-3	5.68E-4	1.22E-1	1.29E-1
	129	1.03E-3	1.51E-4	6.83E-2	7.20E-2

yield second-order convergence on the smooth grid, not only for the vector function itself, but also for divergence and curl. It is interesting to note that the zeroth order method, which is exact only for constant functions, still converges with second-order in h even for div and **curl**. This phenomenon is due to the smoothness of both the grid and the function to be interpolated. Readers can easily understand this phenomenon by considering the 1D analog of our procedure. For the random grid the convergence of the zeroth order approximate vector is first order, and there is no convergence for its DIV and **CURL**. However, as expected, the first-order method has second-order convergence for a vector function and first-order convergence for its DIV and **CURL**.

Real Lagrangian grids can be very nonsmooth; therefore, we compare the two methods for the "shock" vector field only on a random grid. The results are presented in Table 2. Here we also present the max norm of the error only for the y component of the vector **A** because the original vector field is "one-dimensional." For the "shock" case on the nonsmooth grid the zero-order reconstruction gives first-order convergence for the vector itself and no convergence for DIV and **CURL**. First-order reconstruction gives second-order convergence for the vector and first-order convergence for divergence and curl. Also notice that for the zero-order reconstruction the main error is in the y-component, while for the first-order reconstruction it is not only smaller but is in the x-component. This is important for the one-dimensional nature of this flow.

In conclusion we can say it seems that the zero-order method should be used only on smooth grids or for visualization purposes. If the method of solution of the gas dynamics equations involves computation of derivatives of the reconstructed field on a nonsmooth grid, then the zero-order method should not be used because it generates artificial divergence and vorticity that cannot be eliminated by refinement of the grid. The extension to 3D is clear.

REFERENCES

1. J. K. Dukowicz and B. J. A. Meltz, Vorticity errors in multidimensional Lagrangian codes, *J. Comput. Phys.* **99**, 115 (1992).
2. J. M. Hyman and M. Shashkov, Natural discretizations for the divergence, gradient, and curl on logically rectangular grids, *Comput. Math. Appl.* **33**, 81 (1997).
3. M. Shashkov, Conservative finite-difference schemes on general grids (CRC Press, Boca Raton, FL, 1995), p. 60.