Wave Structures in Nonlinearly Balanced 1+1 PDEs

D.D. Holm & M.F. Staley

Numerics by: Templated PDE Toolbox

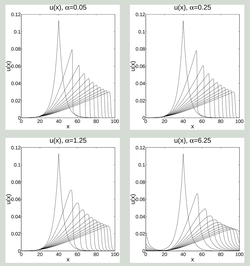
Burgers-$\alpha\beta$, inverse Helmholtz filter

α as given; $\beta = 0.5;\ \nu = 0.005$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left(-\left|\frac{x-40}{5}\right|\right)$

T=200n, n=0…10; 4096 zones

Shows effect of increasing α.



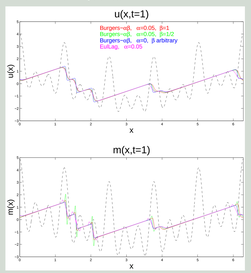
Burgers-$\alpha\beta$, EulLag; inverse Helmholtz filter

α, β as given; $\nu = 0.005$

$u(x,0) = \cos(x) + \cos(5x) + \cos(10x) + \cos(15x)$

T=1; 4096 zones

Shows difference between Burgers-$\alpha\beta$ and EulLag solutions for various parameters.



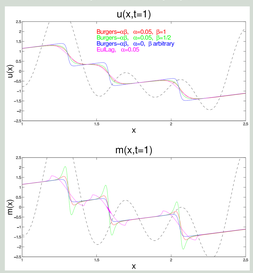
Burgers-$\alpha\beta$, EulLag; inverse Helmholtz filter

α, β as given; $\nu = 0.005$

$u(x,0) = \cos(x) + \cos(5x) + \cos(10x) + \cos(15x)$

T=1; 4096 zones

Shows difference between Burgers-$\alpha\beta$ and EulLag solutions for various parameters. Closeup of plot at left.



Equations

B-Weighted Peakon Equation

$$\mathbf{u_t} + \mathbf{u}\mathbf{u_x} = -\partial_x \tau, \quad (1 - \alpha^2 \partial_x^2)\tau = \frac{\mathbf{b}}{2}\mathbf{u}^2 + \frac{3-\mathbf{b}}{2}\alpha^2 \mathbf{u_x}^2$$

Equivalently,

$$\mathbf{m_t} + \mathbf{u}\mathbf{m_x} + \mathbf{b}\mathbf{u_x}\mathbf{m} = 0, \quad \text{where } \mathbf{m} = (1 - \alpha^2 \partial_x^2)\mathbf{u}$$

Burgers-$\alpha\beta$

$$\mathbf{u_t} + \mathbf{u}\mathbf{u_x} - \nu \mathbf{u_{xx}} = -\alpha^2 \beta \partial_x \tau, \quad (1 - \alpha^2 \partial_x^2)\tau = \mathbf{u_x}^2$$

Note: $\beta = 1/2$ preserves

$$\|\mathbf{u}\|^2_{\mathbf{H}^1_0} = \frac{1}{2}\int_{-\infty}^{\infty} (\mathbf{u}^2 + \alpha^2 \mathbf{u_x}^2)\, \mathbf{dx}$$

EulLag

$$\mathbf{u_t} + \mathbf{u}\mathbf{u_x} - \nu \mathbf{u_{xx}} = -\partial_x \tau, \quad -\partial_x \tau = \alpha^2 (|\mathbf{u_x}| - \mathbf{u_x})\mathbf{u_{xx}}$$

Also, define $\mathbf{m} = (1 - \alpha^2 \partial_x^2)\mathbf{u}$.

Filters

Inversion of the Helmholtz operator $(1 - \alpha^2 \partial_x^2)$ in the equation

$$(1 - \alpha^2 \partial_x^2)\tau = \mathrm{rhs}$$

is easily accomplished in Fourier space:

$$\tau = \mathbf{FFT}^{-1}\left(\frac{1}{1 + \alpha^2 \mathbf{k}^2}\mathbf{FFT}(\mathrm{rhs})\right)$$

where $\mathbf{FFT}$ denotes a Fourier transform and $\mathbf{FFT}^{-1}$ a Fourier synthesis.

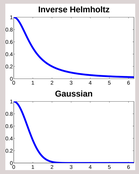
We have experimented with the following filters:

Inverse Helmholtz Filter

$$\frac{1}{1 + \alpha^2 \mathbf{k}^2}$$

Gaussian Filter

$$\exp(-\alpha^2 \mathbf{k}^2)$$



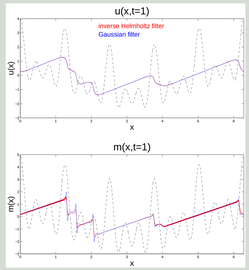
Burgers-$\alpha\beta$, filter as given

$\alpha = 0.05;\ \beta = 1;\ \nu = 0.005$

$u(x,0) = \cos(x) + \cos(5x) + \cos(10x) + \cos(15x)$

T=1; 8192 zones

Shows effect of using the inverse Helmholtz filter versus the Gaussian filter with the Burgers-$\alpha\beta$ equation.



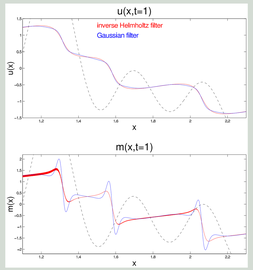
Burgers-$\alpha\beta$, filter as given

$\alpha = 0.05;\ \beta = 1;\ \nu = 0.005$

$u(x,0) = \cos(x) + \cos(5x) + \cos(10x) + \cos(15x)$

T=1; 8192 zones

Shows effect of using the inverse Helmholtz filter versus the Gaussian filter with the Burgers-$\alpha\beta$ equation.



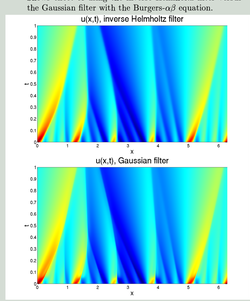
Burgers-$\alpha\beta$, filter as given

$\alpha = 0.05;\ \beta = 1;\ \nu = 0.005$

$u(x,0) = \cos(x) + \cos(5x) + \cos(10x) + \cos(15x)$

T=1; 8192 zones

Shows effect of using the inverse Helmholtz filter versus the Gaussian filter with the Burgers-$\alpha\beta$ equation.



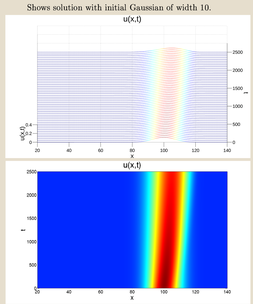
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = -1$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-100}{10}\right)^2\right]$

T=2500; 8192 zones

Shows solution with initial Gaussian of width 10.



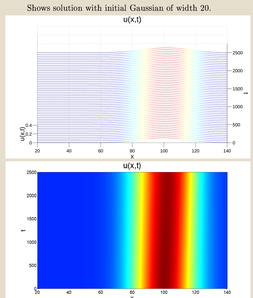
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = -1$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-100}{20}\right)^2\right]$

T=2500; 8192 zones

Shows solution with initial Gaussian of width 20.



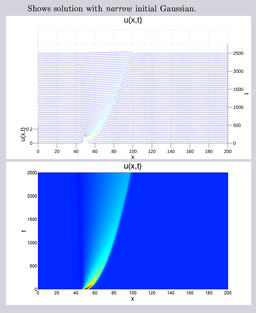
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = 0$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-30}{2.5}\right)^2\right]$

T=2500; 16384 zones

Shows solution with *narrow* initial Gaussian.



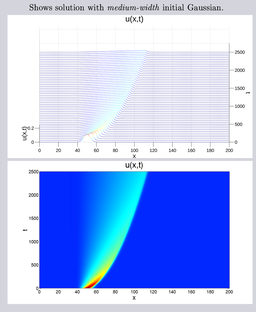
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = 0$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-30}{5}\right)^2\right]$

T=2500; 16384 zones

Shows solution with *medium-width* initial Gaussian.



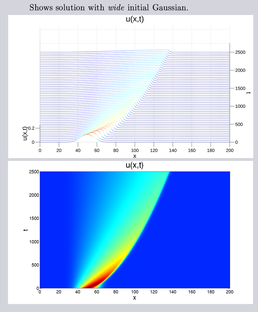
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = 0$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-30}{10}\right)^2\right]$

T=2500; 16384 zones

Shows solution with *wide* initial Gaussian.



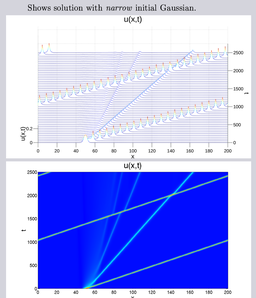
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = 2$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-30}{2.5}\right)^2\right]$

T=2500; 16384 zones

Shows solution with *narrow* initial Gaussian.



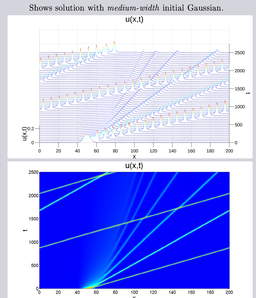
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = 2$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-30}{5}\right)^2\right]$

T=2500; 16384 zones

Shows solution with *medium-width* initial Gaussian.



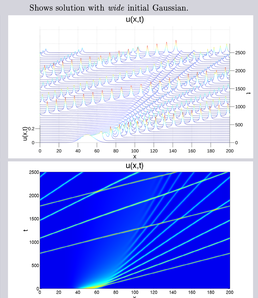
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = 2$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-30}{10}\right)^2\right]$

T=2500; 16384 zones

Shows solution with *wide* initial Gaussian.



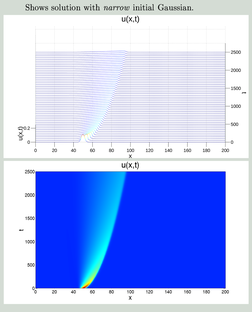
Burgers-$\alpha\beta$, inverse Helmholtz filter

$\alpha = 1;\ \beta = 0.5,\ \nu = 0.005$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-30}{2.5}\right)^2\right]$

T=2500; 16384 zones

Shows solution with *narrow* initial Gaussian.



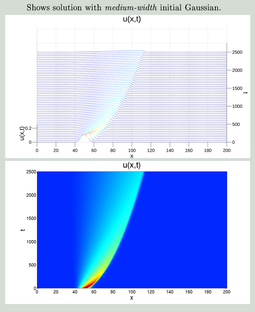
Burgers-$\alpha\beta$, inverse Helmholtz filter

$\alpha = 1;\ \beta = 0.5,\ \nu = 0.005$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-30}{5}\right)^2\right]$

T=2500; 16384 zones

Shows solution with *medium-width* initial Gaussian.



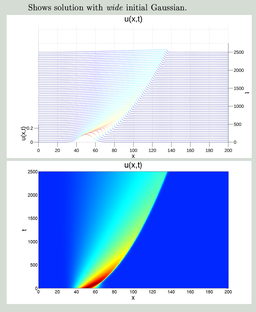
Burgers-$\alpha\beta$, inverse Helmholtz filter

$\alpha = 1;\ \beta = 0.5,\ \nu = 0.005$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-30}{10}\right)^2\right]$

T=2500; 16384 zones

Shows solution with *wide* initial Gaussian.



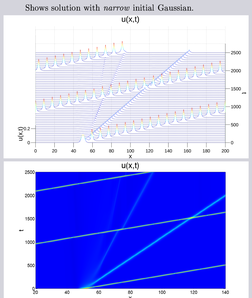
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = 3$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-30}{2.5}\right)^2\right]$

T=2500; 2048 zones

Shows solution with *narrow* initial Gaussian.



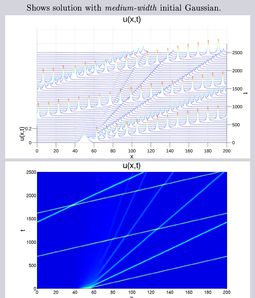
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = 3$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-30}{5}\right)^2\right]$

T=2500; 16384 zones

Shows solution with *medium-width* initial Gaussian.



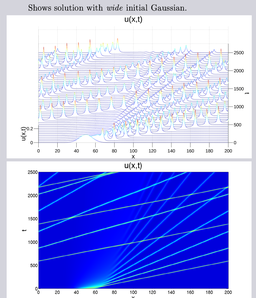
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = 3$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-30}{10}\right)^2\right]$

T=2500; 16384 zones

Shows solution with *wide* initial Gaussian.



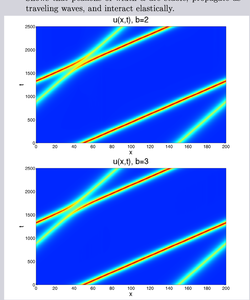
B Equation, inverse Helmholtz filter

$\alpha = 5;\ b$ as shown

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left(-\left|\frac{x-50}{5}\right|\right) + \frac{1}{10\sqrt{\pi}} \exp\left(-\left|\frac{x-150}{5}\right|\right)$

T=2500; 16384 zones

Shows that peakons of width α are stable, propagate as traveling waves, and interact elastically.



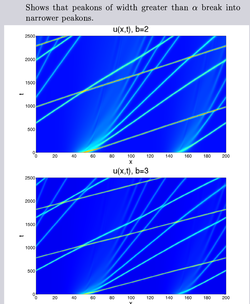
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b$ as shown

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left(-\left|\frac{x-50}{5}\right|\right) + \frac{1}{10\sqrt{\pi}} \exp\left(-\left|\frac{x-150}{5}\right|\right)$

T=2500; 16384 zones

Shows that peakons of width greater than α break into narrower peakons.



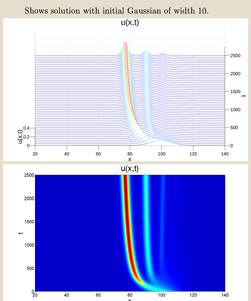
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = -2$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-100}{10}\right)^2\right]$

T=2500; 8192 zones

Shows solution with initial Gaussian of width 10.



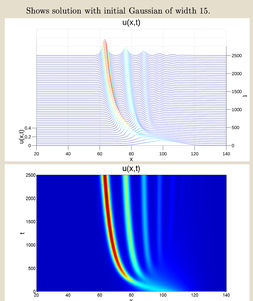
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = -2$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-100}{15}\right)^2\right]$

T=2500; 8192 zones

Shows solution with initial Gaussian of width 15.



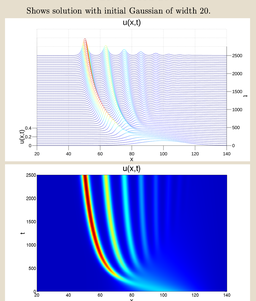
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = -2$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-100}{20}\right)^2\right]$

T=2500; 8192 zones

Shows solution with initial Gaussian of width 20.



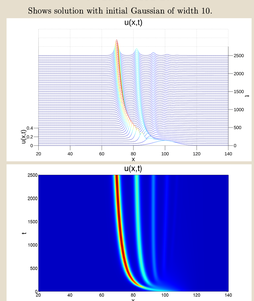
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = -3$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-100}{10}\right)^2\right]$

T=2500; 8192 zones

Shows solution with initial Gaussian of width 10.



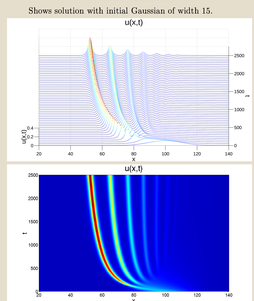
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = -3$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-100}{15}\right)^2\right]$

T=2500; 8192 zones

Shows solution with initial Gaussian of width 15.



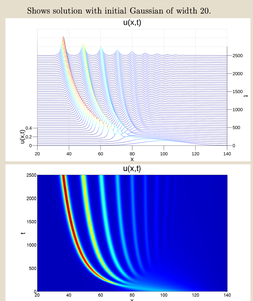
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = -3$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-100}{20}\right)^2\right]$

T=2500; 8192 zones

Shows solution with initial Gaussian of width 20.



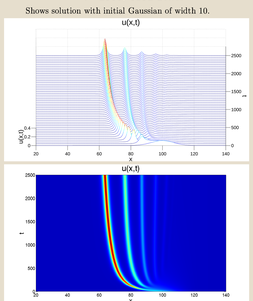
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = -4$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-100}{10}\right)^2\right]$

T=2500; 8192 zones

Shows solution with initial Gaussian of width 10.



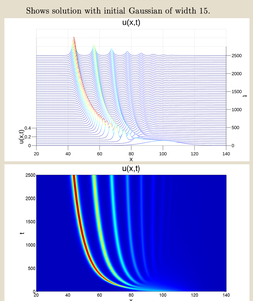
B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = -4$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-100}{15}\right)^2\right]$

T=2500; 8192 zones

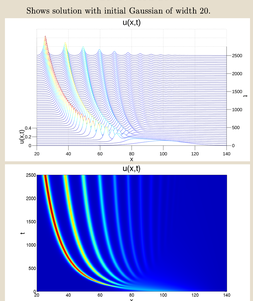
Shows solution with initial Gaussian of width 15.



B Equation, inverse Helmholtz filter

$\alpha = 1;\ b = -4$

$u(x,0) = \frac{1}{5\sqrt{\pi}} \exp\left[-\left(\frac{x-100}{20}\right)^2\right]$

T=2500; 8192 zones

Shows solution with initial Gaussian of width 20.