Math 220, section 31
Spring 2001
Final Exam

Problem 1 Provide short computations, proofs or examples as required. (100 points)

(i) Evaluate

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{\cos x}{e^x-1}\right).$$

(Hint: expand all functions to the second order in x.)

(ii) For all complex numbers z satisfying $\mathrm{Re}\, z = 1$, prove that $1/z$ lies on a circle. (Hint: compute the magnitude of $\frac{1}{z}-\frac{1}{2}$ for all such z.)

(iii) Consider the system of equations

$$\begin{aligned} 2ax_1 + a^2x_2 &= c_1 \\ -x_1-(a+b)x_2 &= c_2 \end{aligned}$$

What condition(s) should a, b, c_1 and c_2 satisfy for this system to have a unique solution for x_1 and x_2? What is the solution?

(iv) A system of coupled oscillators are governed by the equations

$$\begin{aligned} \ddot{x}_1 &= -3x_1+6x_2 \\ \ddot{x}_2 &= 2x_1-7x_2 \end{aligned}$$

Write this equation in the matrix form $\ddot{\mathbf{x}} = A\mathbf{x}$. Guessing a solution with an exponential time dependence $\mathbf{x}(t) = \mathbf{v}_0 e^{\omega t}$, find the allowed values of ω.

(v) $u^2+\sin(v) = e^{yv}+y^3$ and

$$x = \int_u^{\cos(v)} e^{-t^2}\,dt.$$

Find $(\partial y/\partial u)_x$.

(vi) If $z = f(x^3 + 3xy^2 - y^3)$, show that

$$(6xy - 3y^2)\frac{\partial z}{\partial x} - (3x^2 + 3y^2)\frac{\partial z}{\partial y} = 0.$$

(vii) Make the change of variables $u = y/x, v = x + y$ and evaluate the integral

$$\int_0^1 \left[\int_0^x \frac{(x+y)\sin(x+y)}{x^2} dy \right] dx,$$

(viii) The mass density of a disk $r \leq a$ is given by $\rho = r(1 + \cos\theta)$. Find the mass of the disk, and its center of mass.

(ix) If D is a domain in the plane, and ∂D is its boundary, show that

$$\oint_{\partial D} xy^2 dy = \int \int_D y^2 dx dy.$$

Use this formula, along with symmetry, to calculate the moments of inertia about the x and y axes for a square with uniform density ρ whose vertices are $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$.

(x) Find a function $w(x,y,z)$ such that the vector field $(z+y^2)\mathbf{i} + (2xy + z^2)\mathbf{j} + w\mathbf{k}$ represents a magnetostatic field.

Problem 2 A model for the temperature T of a body that is heated by radiation of intensity $I(t)$, and that cools by radiating into space is the differential equation

$$C\frac{dT}{dt} + \alpha T = I(t)$$

where C, the heat capacity of the object, and α, the cooling rate, are constants.

If $I(t) = I_0 + I_1\cos(\omega t)$, find a solution for the temperature $T(t)$. What is the maximum temperature of the object? At what time does the maximum in the temperature occur? (Hint: Solve the equation for I_0 and $I_1\cos(\omega t)$ separately and then use linearity of the equation. Also, $\cos(\omega t)$ is the real part of $e^{i\omega t}$.) (25 points)

Problem 3 A particle of mass m is moving under the influence of gravity on the surface $z = 5(x^2 + y^2) + 3e^{2xy}$.

(i) What is the gravitational potential energy of the particle?

(ii) Find an equilibrium point (x_0, y_0) for the particle. Find an approximation to the gravitational potential by expanding to the second order in $x - x_0$ and $y - y_0$.

(iii) Assuming that the kinetic energy is $K = m(\dot{x}^2 + \dot{y}^2)/2$, and using the approximate potential energy from part (ii) find the equations of motion for the particle.

(iv) Write the equations in the matrix form $\ddot{\mathbf{v}} = A\mathbf{v}$, where $\mathbf{v} = (x\, y)^T$ and A is a constant matrix.

(v) Guessing a solution with an exponential time dependence $\mathbf{v}_0 e^{\beta t}$, find the general solution for the motion of the particle. (30 points)

Problem 4 An electric field is given by $\mathbf{E} = x\mathbf{i} + y\mathbf{j} + (2z + z^2)\mathbf{k}$. The surface S is given by $z = x^2 + y^2$, $x^2 + y^2 \leq 4$.

(i) Find the electrostatic potential ϕ for this field $\mathbf{E}$.

(ii) Find the electric flux through the surface S. (Hint: The flux is given by $\int\int_S \mathbf{E} \cdot \hat{\mathbf{n}} dA$.)

(iii) Find the total charge inside the hemisphere $x^2 + y^2 + z^2 \leq 1, z \geq 0$. (Hint: Spherical polar coordinates. Volume element is $r^2 \sin\theta d\phi d\theta dr$) (25 points)

Problem 5 $f(x + 2\pi) = f(x)$ for all x and

$$f(x) = \begin{cases} -1 & -\pi \leq x < 0 \\ 1 & 0 \leq x < \pi \end{cases}$$

(i) Find the Fourier series for $f(x)$ in temrs of sines and cosines.

(ii) Express $f(x)$ as a series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

(Hint: use the result from part (i)) (20 points).