# Harmonic Growth in 2D via Biorthogonal Polynomials

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CNLS/T-13

November 15, 2007

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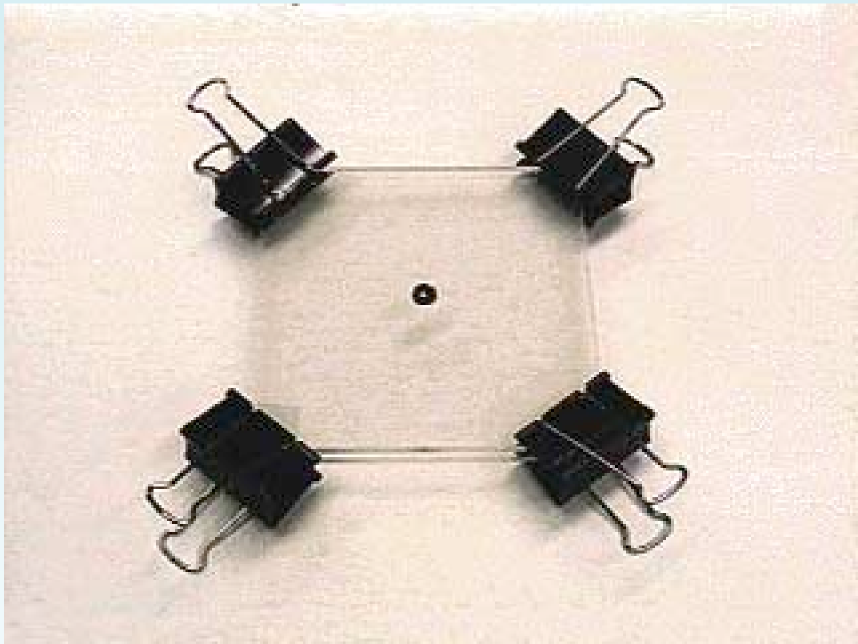

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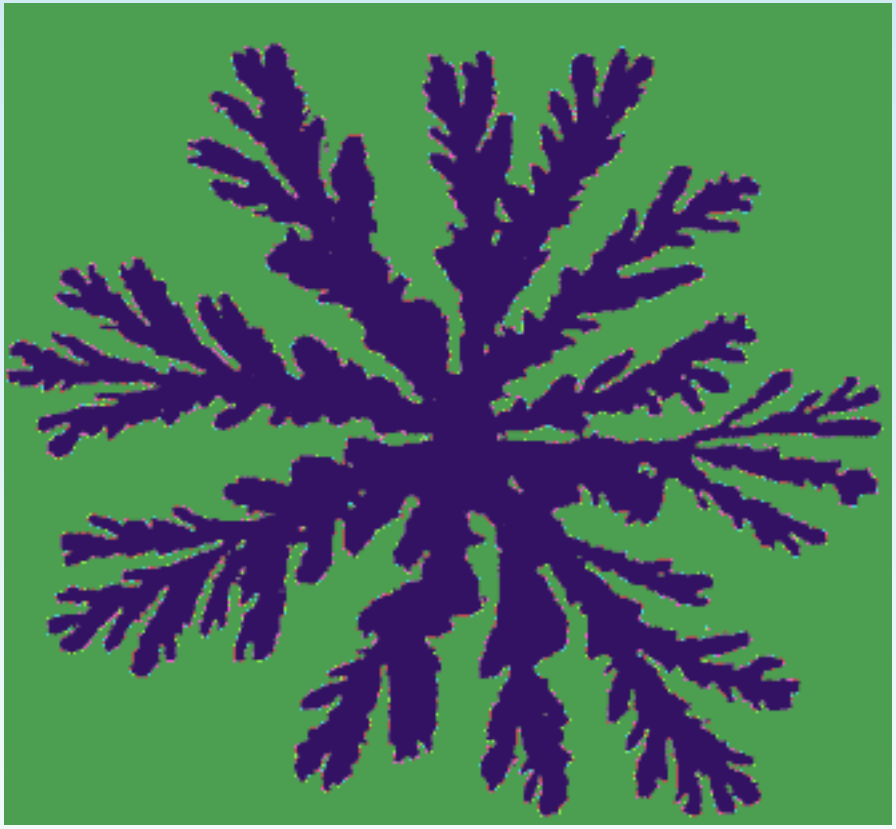
$$V_n \sim -\nabla_n p, \quad \Delta p = 0 \ \text{ on } D_-, \quad p \sim -\log|z|, \ z \to \infty, \quad p = 0 \ \text{ on } D_+$$

# Patterns: fingers, fjords, cusps



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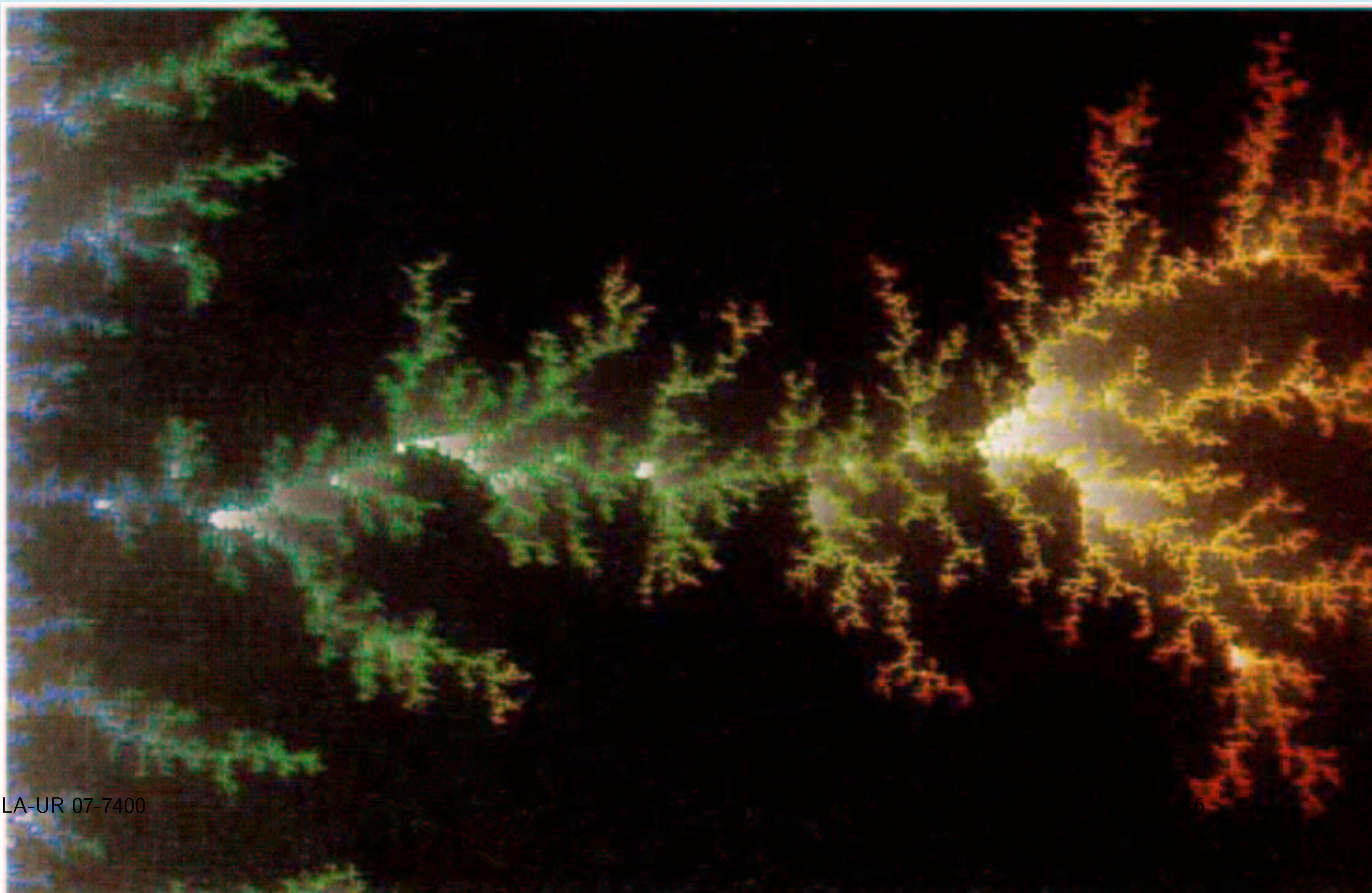
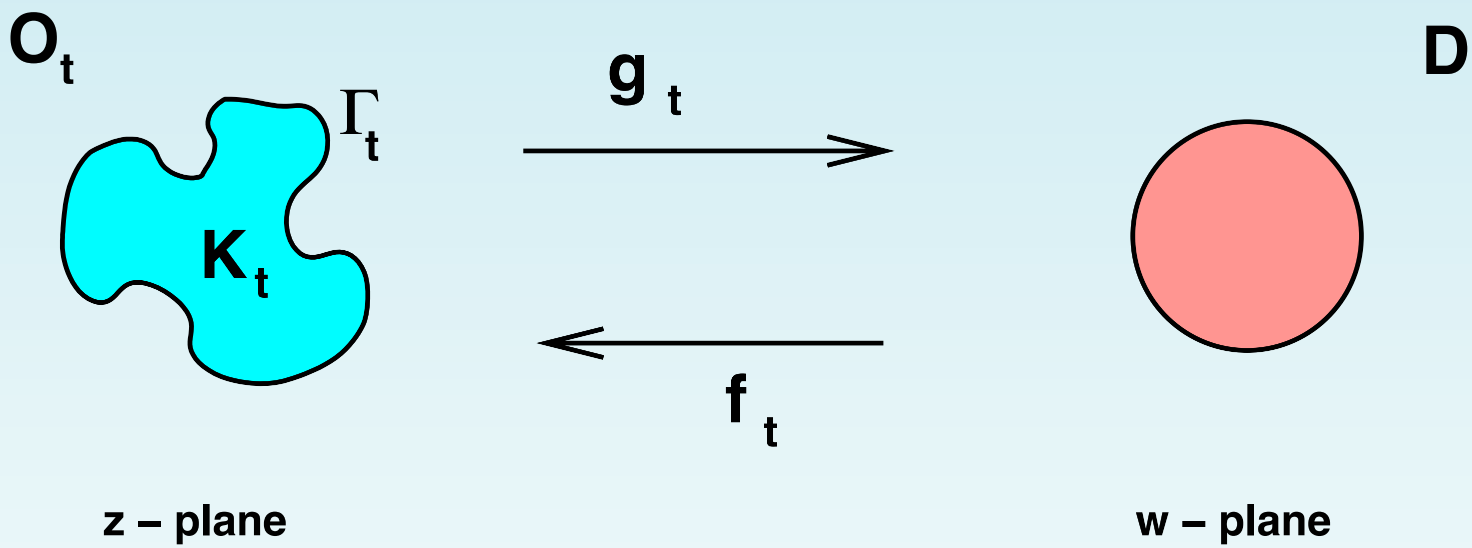


# The solution: conformal transformations 101




- 1947: Polubarinova-Kochina, Galin and Kufarev

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$$\text{Poisson bracket} : \{z, \bar{z}\}_{PB} \equiv w\left(\frac{\partial z}{\partial w}\frac{\partial \bar{z}}{\partial t_0} - \frac{\partial \bar{z}}{\partial w}\frac{\partial z}{\partial t_0}\right) = 1$$

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- 1D case: A.A. Markov (L-problem of moments)

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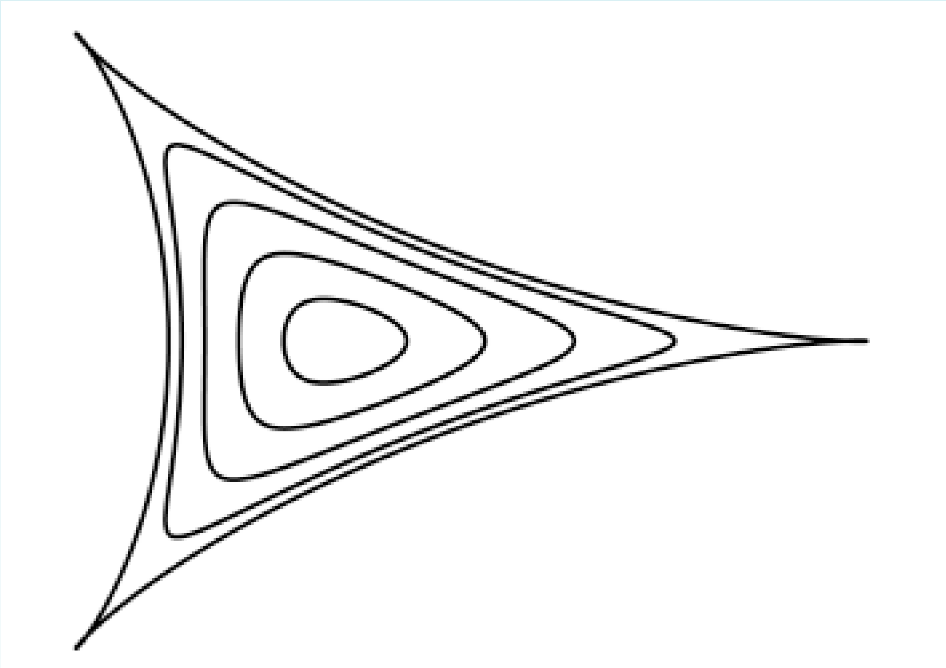

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$$\boxed{w'(z) \to \infty, \quad z \in \partial D}$$

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Regularization schemes ... singular perturbation problem!

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- M. Sakai, H. Shapiro, B. Gustaffson, M. Putinar

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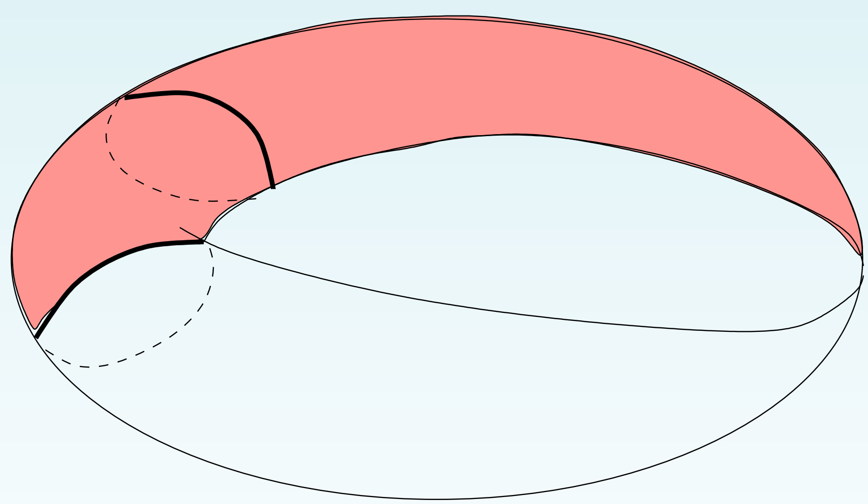
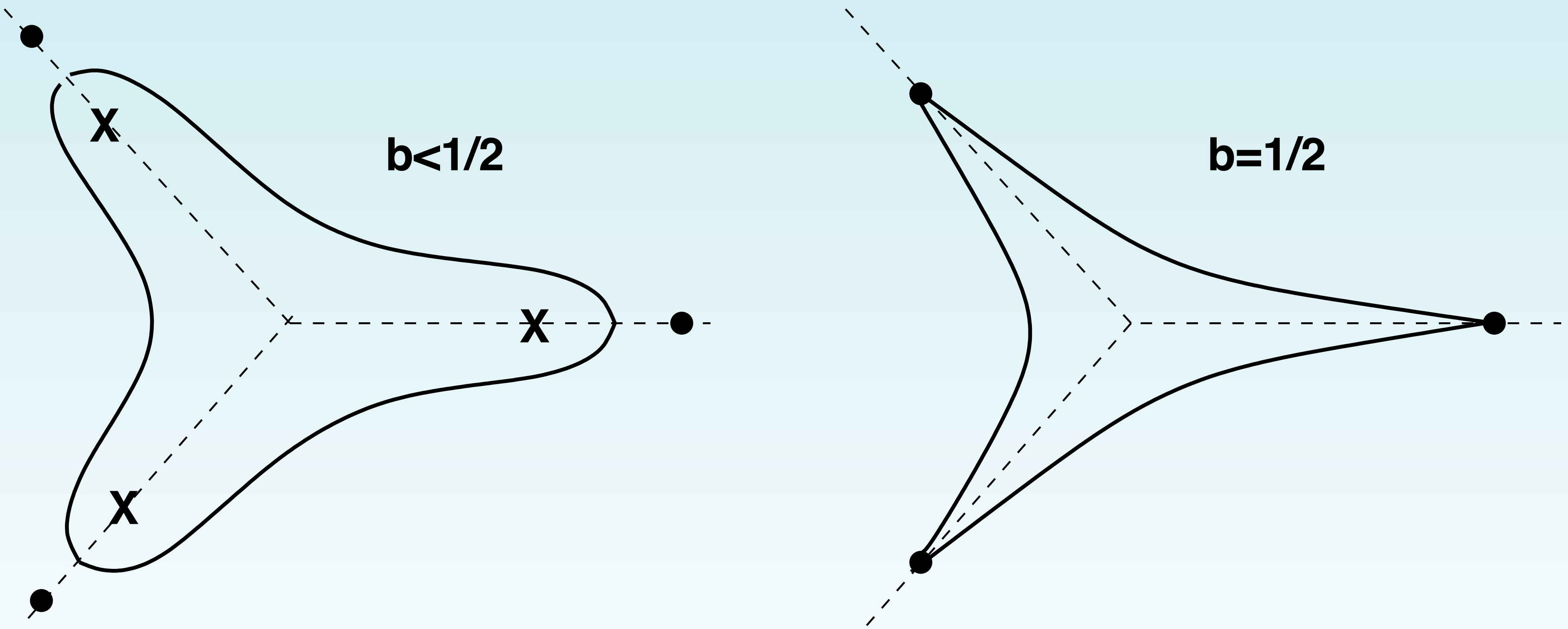
- Cusp: branch point $S'(z) \to \infty$ meets double point $S_1(z) = S_2(z)$



# Cusps as critical points



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$$\int f(z)\rho(z)\mathrm{d}^2z = \int f(z)\rho_s(z)\mathrm{d}^2z, \quad f(z) \text{ integrable}$$

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- Distribution of zeros of polynomials (branch cut of Schwarz function): $z \in [-a_n, a_n], a_n = \sqrt{2|t_2| r_n r_{n+1}}$

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- Large $N$ limit - becomes continuous growth law

$$|\psi_N(z)|^2 e^{-N|z|^2} \longrightarrow \delta_{\partial D}(z)$$

- Work with Ed. Saff (Vanderbilt) and N. Makarov (Caltech)

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- Becomes the Boutroux-Krichever condition for complex curve

# Isomonodromic deformations and weak solutions of harmonic growth

- Commutation relation $[\bar{z}, z] = \frac{1}{N}$ may be understood as isomonodromic deformation condition, using position of branch point $u(N)$ as scaling function.

- Becomes the Boutroux-Krichever condition for complex curve

$$\Re \oint y(z, N)\mathrm{d}z = 0.$$

- Preserves critical points of equations solved by wavefunctions

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